KAM Theory and Applications in Celestial Mechanics – Second and Third Lectures: from the Original KAM Theorem to Explicit Constructive Computational Algorithms

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KAM theorem (stated according to Kolmogorov's version)

Theorem

Consider a **general problem of dynamics** described by a Hamiltonian

$$H(\mathbf{p},\mathbf{q}) = \boldsymbol{\omega}\cdot\mathbf{p} + h(\mathbf{p}) + \varepsilon f(\mathbf{p},\mathbf{q}) \ .$$

If H is analytic in $\mathcal{B}_{\varrho}(\mathbf{0})\times\mathbb{T}^n$ (with $\mathcal{B}_{\varrho}(\mathbf{0})\subset\mathbb{R}^n$) and

- $\boldsymbol{\omega}$ is Diophantine, this means there are $\gamma>0$ and $\tau\ge n-1$ such that $|\mathbf{k}\cdot\boldsymbol{\omega}|>\gamma/|\mathbf{k}|^{\tau}\ \forall\ \mathbf{k}\in\mathbb{Z}^n\setminus\{\mathbf{0}\}$,
- $h(\mathbf{p})=\mathcal{O}(\|\mathbf{p}\|^2)$ is non-degenerate, i.e. $\det\left(\frac{\partial^2 h}{\partial p_i\partial p_j}\right)_{i,j}\neq 0$,
- ε is small enough, i.e $|\varepsilon|<\varepsilon^*$, being $\varepsilon^*>0$ a threshold value,

then there exists a canonical transformation $(\mathbf{p},\mathbf{q})=\Psi(\mathbf{P},\mathbf{Q})$ leading the system to the Kolmogorov's normal form Hamiltonian, i.e.

$$\mathcal{K}(\mathbf{P},\mathbf{Q}) = \boldsymbol{\omega}\cdot\mathbf{P} + \mathcal{R}(\mathbf{P},\mathbf{Q}) \ ,$$

where $\mathcal{K}=H\circ\Psi$ and $\mathcal{R}=\mathcal{O}(\|\mathbf{P}\|^2)$. Moreover, Ψ is close to the identity $\mathbf{I}$, i.e. $\Psi-\mathbf{I}=\mathcal{O}(\varepsilon)$.

KAM theorem: some of the main consequences

- The Cauchy's problem for the Hamilton equations of Kolmogorov's normal form $\mathcal{K}$ can be solved for initial conditions with $\mathbf{P}(0)=\mathbf{0}$:

$$\dot{P}_j=-\frac{\partial\,\mathcal{K}}{\partial Q_j}(\mathbf{0},\mathbf{Q})=0\,,\quad \dot{Q}_j=\frac{\partial\,\mathcal{K}}{\partial P_j}(\mathbf{0},\mathbf{Q})=\omega_j\,,\quad \forall\ j=1\,,\,\ldots\,,\,n\,.$$

- Since Ψ is canonical, a solution of the Cauchy's problem can be provided also in $(\mathbf{p},\mathbf{q})$ coordinates, according to the scheme

$$\begin{array}{ccc}
(\mathbf{p}(0),\mathbf{q}(0)) & \stackrel{(\Psi)^{-1}}{\longrightarrow} & (\mathbf{P}(0)=\mathbf{0}\,,\ \mathbf{Q}(0)) \\
 & & \Big\downarrow \Phi^{t}_{\omega\cdot\mathbf{P}} \\
(\mathbf{p}(t),\mathbf{q}(t)) & \stackrel{\Psi}{\longleftarrow} & (\mathbf{P}(t)=\mathbf{0}\,,\ \mathbf{Q}(t)=\mathbf{Q}(0)+\omega t)
\end{array}\ .$$

- The (KAM) nD torus $\{(\mathbf{p},\mathbf{q}):\ (\mathbf{p},\mathbf{q})=\Psi(\mathbf{0},\mathbf{Q})\ \forall\ \mathbf{Q}\in\mathbb{T}^n\}$ is invariant with respect to the flow Φ^t_H induced by H. That torus is a "sligth" distorsion of the unperturbed one, i.e. $\{\mathbf{0}\}\times\mathbb{T}^n$.

KAM theorem: from local to global

- **Almost all** the angular velocities ω are **Diophantine**!
  More formally, let us introduce the set of diophantine frequencies $\Omega_{\gamma\,,\,\tau} = \{\omega \;:\; |\mathbf{k}\cdot\boldsymbol{\omega}| > \gamma/|\mathbf{k}|^{\tau}\ \forall\ \mathbf{k}\in\mathbb{Z}^n\setminus\{\mathbf{0}\}\}$; it is easy to prove that the Lebesgue measure of $\mathcal{B}_R(\mathbf{0})\setminus\Omega_{\gamma\,,\,\tau}$ shrinks to zero when $\gamma\to 0^+$, for all fixed values of $R>0$ and $\tau>n-1$.
- The **volume** of the phase space that is **not occupied by KAM invariant tori shrinks to zero** when $\varepsilon\to 0$.
  Actually, it has been proved that the Lebesgue measure of the complementary set of the KAM tori is $\mathcal{O}(\sqrt{\varepsilon})$ (see *Neishtadt, 1982*).
- Purely analytical estimates of the threshold value $\varepsilon^* = \varepsilon^*(\gamma,\tau,h,f)$ are often complicate and *ridicolously* small.
  Nevertheless, in some cases, **computer–assisted** rigorous proofs can **ensure the existence of KAM tori** for values of ε **very close to the real (numerical) breakdown threshold of a fixed invariant torus** (see *Calleja, Celletti & de la Llave, 2013*).
  Moreover, in concrete applications we are often interested in **using the constructive method that is the basis of a proof**.

The Hamiltonian of the Sun–Jupiter–Saturn system

The Hamiltonian of a three–body planetary system writes as

$$F(\mathbf{r},\tilde{\mathbf{r}}) = T^{(0)}(\tilde{\mathbf{r}}) + U^{(0)}(\mathbf{r}) + T^{(1)}(\tilde{\mathbf{r}}) + U^{(1)}(\mathbf{r}) \, ,$$

where $\mathbf{r}$ are the heliocentric coordinates and $\tilde{\mathbf{r}}$ the conjugated momenta, while the functional terms appearing above are such that

$$T^{(0)}(\tilde{\mathbf{r}}) = \frac{1}{2}\sum_{j=1}^{2} \|\tilde{\mathbf{r}}_j\|^2 \left(\frac{1}{m_0} + \frac{1}{m_j}\right) \, ,$$

$$U^{(0)}(\mathbf{r}) = -\mathcal{G}\sum_{j=1}^{2} \frac{m_0 m_j}{\|\mathbf{r}_j\|} \, ,$$

$$T^{(1)}(\tilde{\mathbf{r}}) = \frac{\tilde{\mathbf{r}}_1 \cdot \tilde{\mathbf{r}}_2}{m_0} \, ,$$

$$U^{(1)}(\mathbf{r}) = -\mathcal{G}\frac{m_1 m_2}{\|\mathbf{r}_1 - \mathbf{r}_2\|} \, ,$$

being $\mathcal{G}$, m_0 , m_1 and m_2 the gravitational constant, the mass of the "star" and the masses of the two "planets", respectively.

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Orbital elements (a brief sketch)

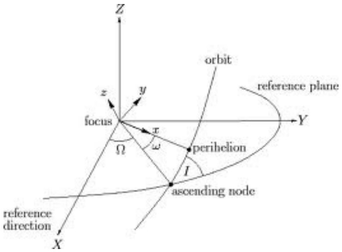

Figure: Schematic representation of the orbital angles I, Ω and ω. The elliptic osculating Keplerian orbit lies in the xy plane, while XYZ is a fixed reference.

In Celestial Mechanics position–velocity coordinates are represented in terms of the **osculating** Keplerian orbit. The classical orbital elements locating a Keplerian ellipse (and a point on it) are $(a\,,\,e\,,\,I\,,\,M\,,\,\omega\,,\,\Omega)$, where **inclination** I, **perihelion argument** ω and **longitude of node** Ω are represented in figure above. a and e are the **semimajor-axis** and the **eccentricity** of the ellipse, respectively. The position on the Keplerian ellipse is given by the **true anomaly** v, i.e. an angle starting form the perihelion. M is related to v as it will be recalled in the next slide.

Reduction of the angular momentum (a brief sketch)

Starting from the **true anomaly** v, one can determine the **eccentric anomaly** E and the **mean anomaly** M, by the well known equations:

$$\tan\frac{v}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}\ , \qquad M = E - e\sin E\ .$$

- **Remark**: starting from the heliocentric coordinates, we automatically reduced the linear momentum; thus, the original three–body problem (having 9 **d**egrees **o**f **f**reedom) can be represented by 6 pairs of canonical coordinates.
- **Remark**: for both the planets, we can use the three pairs of Delaunay's action–angle coordinates, where the angles are just $M_j\,,\ \omega_j\,,\ \Omega_j$ with $j = 1\,,\,2$.
- **Remark**: if a XYZ reference frame with the axis Z **parallel to the total angular momentum J** is chosen, it is easy to prove that the Hamiltonian does not depend on $\Omega_1 + \Omega_2$ and $\Omega_1 - \Omega_2 = \pi$.
  This allows one to perform the Jacobi's reduction of the nodes: the three–body problem is represented by a **4 d.o.f.** Hamiltonian, where $\Omega_1\ ,\ \Omega_2$ and the related actions do not appear.

The Poincaré canonical variables for the reduced three–body Hamiltonian (having 4 d.o.f.)

$$\Lambda_j = \frac{m_0 m_j}{m_0 + m_j}\sqrt{\mathcal{G}(m_0 + m_j)a_j} \qquad \lambda_j = M_j + \omega_j$$

$$\xi_j = \sqrt{2\Lambda_j}\sqrt{1-\sqrt{1-e_j^2}}\cos(\omega_j) \quad \eta_j = -\sqrt{2\Lambda_j}\sqrt{1-\sqrt{1-e_j^2}}\sin(\omega_j)$$

where a_j, e_j, M_j and ω_j are the semi-major axis, the eccentricity, the mean anomaly and perihelion argument of the j–th planet, respectively.

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Introduction of the secular Hamiltonian

- **Remark**: it is well known that the main (Keplerian) part of the Hamiltonian is integrable, because it depends only on the actions:
$$T^{(0)} + U^{(0)} = -\frac{\mathcal{G}^2\, m_0^3}{2}\left(\frac{m_1^3}{m_0+m_1}\frac{1}{\Lambda_1^2} + \frac{m_2^3}{m_0+m_2}\frac{1}{\Lambda_2^2}\right) .$$
- **Remark**: in order to have information about the "final fate" of the orbits, it is natural to **average the model over the fast angles**:
$$\langle F\rangle_{\boldsymbol{\lambda}} = \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} F\, \mathrm{d}\lambda_1 \mathrm{d}\lambda_2 .$$
- **Remark**: since Λ_1 and Λ_2 are integrals of motion for $\langle F\rangle_{\boldsymbol{\lambda}}$, they can be evaluated (with respect to a numerical average value of the semi-major axes) and the dependence of the Hamiltonian on them can be disregarded. Thus, we **"dropped" two d.o.f.**
- **Remark**: since $\langle T^{(1)}\rangle_{\boldsymbol{\lambda}} = 0$, **we are lead to study the 2 d.o.f. Hamiltonian $H^{(sec)} = H^{(sec)}(\xi_1, \xi_2, \eta_1, \eta_2)$, such that**
$$H^{(sec)} = \langle U^{(1)}\rangle_{\boldsymbol{\lambda}} = -\mathcal{G}\frac{m_1 m_2}{(2\pi)^2}\int_{\mathbb{T}^2}\frac{\mathrm{d}\lambda_1\mathrm{d}\lambda_2}{\|\mathbf{r}_1 - \mathbf{r}_2\|} .$$

About the secular Hamiltonian

- From the D'Alembert rules, it follows that

$$H^{(sec)} = H_0^{(sec)} + H_2^{(sec)} + H_4^{(sec)} + \ldots \,,$$

where $H_{2j}^{(sec)}$ is a hom. pol. of degree $(2j+2)$ in $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, $\forall\ j \in \mathbb{N}$.

- $\boldsymbol{\xi} = \boldsymbol{\eta} = 0$ is an elliptic equilibrium point (for a general proof, one could adapt that reported in *Biasco, Chierchia & Valdinoci, 2006*).
- The complete expansion of $H_0^{(sec)} + H_2^{(sec)}$ is in *Robutel, 1995*.
- Lagrange and Laplace produced $H_0^{(sec)}$ with all the known planets to **explain the oscillations of eccentricities and inclinations**.
- The quadratic term $H_0^{(sec)}$ can be made diagonal by a linear canonical transformation $\mathcal{D}$. The new Hamiltonian is then given by $H^{(\mathcal{D})} = H^{(sec)} \circ \mathcal{D}$, being $H^{(\mathcal{D})} = H_0^{(\mathcal{D})} + H_2^{(\mathcal{D})} + H_4^{(\mathcal{D})} + \ldots$ its decomposition in even homogeneous polynomials and

$$H_0^{(\mathcal{D})} = \frac{\nu_1}{2}\left(\xi_1^2 + \eta_1^2\right) + \frac{\nu_2}{2}\left(\xi_2^2 + \eta_2^2\right) \ .$$

- $H^{(\mathcal{D})}$ has the same structure as the Hénon–Heiles Hamiltonian (except the parity property).

SJS problem: secular part up to order 2 in the masses

$$
\begin{aligned}
\mathcal{H}^{(\mathrm{I})}_{Sec}(\xi_1,\xi_2,\eta_1,\eta_2) = & -7.9817980076760655\times 10^{-5}\,\xi_1^2 \quad -3.8147366852664142\times 10^{-5}\,\xi_1\xi_2 \\
& -1.0268297004552840\times 10^{-4}\,\xi_2^2 \quad -7.7243404322933866\times 10^{-5}\,\eta_1^2 \\
& -3.3707608458413656\times 10^{-5}\,\eta_1\eta_2 \quad -9.9999984803151030\times 10^{-5}\,\eta_2^2 \\
& -5.6436223750712493\times 10^{-4}\,\xi_1^4 \quad -1.6416392742006527\times 10^{-3}\,\xi_1^3\xi_2 \\
& -6.2973632765649690\times 10^{-3}\,\xi_1^2\xi_2^2 \quad -2.2899781169560516\times 10^{-3}\,\xi_1^2\eta_1^2 \\
& -3.4760626376493233\times 10^{-3}\,\xi_1^2\eta_1\eta_2 \quad -4.2185659067704027\times 10^{-3}\,\xi_1^2\eta_2^2 \\
& -8.3857323933716650\times 10^{-3}\,\xi_1\xi_2^3 \quad -1.3039354499285056\times 10^{-3}\,\xi_1\xi_2\eta_1^2 \\
& -4.7425253805301244\times 10^{-3}\,\xi_1\xi_2\eta_1\eta_2 \quad -6.8531825574944021\times 10^{-3}\,\xi_1\xi_2\eta_2^2 \\
& -6.2286577646952779\times 10^{-3}\,\xi_2^4 \quad -3.9500271580332256\times 10^{-3}\,\xi_2^2\eta_1^2 \\
& -8.3527962865484184\times 10^{-3}\,\xi_2^2\eta_1\eta_2 \quad -1.1373126532783053\times 10^{-2}\,\xi_2^2\eta_2^2 \\
& -1.7268965474147260\times 10^{-3}\,\eta_1^4 \quad -3.1475592574520577\times 10^{-3}\,\eta_1^3\eta_2 \\
& -6.6349860468623900\times 10^{-3}\,\eta_1^2\eta_2^2 \quad -6.8332272847022573\times 10^{-3}\,\eta_1\eta_2^3 \\
& -5.1387074709303473\times 10^{-3}\,\eta_2^4 \quad -6.0701804574696823\times 10^{-3}\,\xi_1^6 \\
& -3.2119532714927534\times 10^{-2}\,\xi_1^5\xi_2 \quad -2.7549740027179186\times 10^{-1}\,\xi_1^4\xi_2^2 \\
& -7.4777792652719661\times 10^{-2}\,\xi_1^4\eta_1^2 \quad -2.0878989193600397\times 10^{-1}\,\xi_1^4\eta_1\eta_2 \\
& -2.2556116819663772\times 10^{-1}\,\xi_1^4\eta_2^2 \quad -1.2799874137338532\times 10^{0}\,\xi_1^3\xi_2^3 \\
& -2.1957803068778765\times 10^{-1}\,\xi_1^3\xi_2\eta_1^2 \quad -1.0754609127577874\times 10^{0}\,\xi_1^3\xi_2\eta_1\eta_2 \\
& -1.2525556241525955\times 10^{0}\,\xi_1^3\xi_2\eta_2^2 \quad -3.0996153271501390\times 10^{0}\,\xi_1^2\xi_2^4 \\
& -6.4320374599537216\times 10^{-1}\,\xi_1^2\xi_2^2\eta_1^2 \quad -3.9977642968441485\times 10^{0}\,\xi_1^2\xi_2^2\eta_1\eta_2 \\
& -4.7878280093164562\times 10^{0}\,\xi_1^2\xi_2^2\eta_2^2 \quad -1.3017334854810814\times 10^{-1}\,\xi_1^2\eta_1^4 \\
& -5.6443027855889856\times 10^{-1}\,\xi_1^2\eta_1^3\eta_2 \quad -1.6091206979341728\times 10^{0}\,\xi_1^2\eta_1^2\eta_2^2 \\
& -2.4623339524089656\times 10^{0}\,\xi_1^2\eta_1\eta_2^3 \quad -1.6707157626697120\times 10^{0}\,\xi_1^2\eta_2^4 \\
& -3.2771819750467199\times 10^{0}\,\xi_1\xi_2^5 \quad -6.5831703615139023\times 10^{-1}\,\xi_1\xi_2^3\eta_1^2 \\
& -6.1109646323834843\times 10^{0}\,\xi_1\xi_2^3\eta_1\eta_2 \quad -7.3380945672001685\times 10^{0}\,\xi_1\xi_2^3\eta_2^2 \\
& -1.8774424112239909\times 10^{-1}\,\xi_1\xi_2\eta_1^4 \quad -1.0570482388082987\times 10^{0}\,\xi_1\xi_2\eta_1^3\eta_2 \\
& -3.6262703950516264\times 10^{0}\,\xi_1\xi_2\eta_1^2\eta_2^2 \quad -6.1600785219363594\times 10^{0}\,\xi_1\xi_2\eta_1\eta_2^3 \\
& -4.0619119716320187\times 10^{0}\,\xi_1\xi_2\eta_2^4 \quad -1.2962091387801093\times 10^{0}\,\xi_2^6 \\
& -6.0253186540222901\times 10^{-1}\,\xi_2^4\eta_1^2 \quad -4.3130729937971006\times 10^{0}\,\xi_2^4\eta_1\eta_2 \\
& -5.0726955487757852\times 10^{0}\,\xi_2^4\eta_2^2 \quad -3.7694653469144916\times 10^{-1}\,\xi_2^2\eta_1^4 \\
& -1.8736854171280113\times 10^{0}\,\xi_2^2\eta_1^3\eta_2 \quad -5.8658678966225395\times 10^{0}\,\xi_2^2\eta_1^2\eta_2^2 \\
& -9.4014451177163672\times 10^{0}\,\xi_2^2\eta_1\eta_2^3 \quad -6.2640076184402940\times 10^{0}\,\xi_2^2\eta_2^4 \\
& -6.1464575048410168\times 10^{-2}\,\eta_1^6 \quad -3.5591582332419240\times 10^{-1}\,\eta_1^5\eta_2 \\
& -1.3743527471116275\times 10^{0}\,\eta_1^4\eta_2^2 \quad -3.3335872837233516\times 10^{0}\,\eta_1^3\eta_2^3 \\
& -5.2949133058695930\times 10^{0}\,\eta_1^2\eta_2^4 \quad -5.0893561220459285\times 10^{0}\,\eta_1\eta_2^5 \\
& -2.4875190197883796\times 10^{0}\,\eta_2^6
\end{aligned}
$$

Basics of Lie series in action–angle coordinates

- Let $\mathcal{A}$ be the can. transf. introducing action–angle variables $(\mathbf{I},\boldsymbol{\varphi})$ so that $\xi_j=\sqrt{2I_j}\cos\varphi_j\,,\ \eta_j=\sqrt{2I_j}\sin\varphi_j\,,\ \forall\ j=1,\,2\,.$ Let us define $H^{(\mathrm{I})}=H^{(\mathcal{D})}\circ\mathcal{A}\,.$ Thus, $H^{(\mathrm{I})}=H^{(\mathrm{I})}(\mathbf{I},\boldsymbol{\varphi})\,.$
- Let χ and f be a Hamiltonian and a dynamical function, resp. Let us recall that $\dot{f}=\mathcal{L}_{\chi}f$, where the **Lie derivative operator $\mathcal{L}_{\chi}\cdot$ is given by the Poisson bracket, so that $\mathcal{L}_{\chi}f=\{f,\chi\}$**, being $\{f,\chi\}=\sum_{j=1}^{n}\left(\frac{\partial f}{\partial\varphi_j}\frac{\partial\chi}{\partial I_j}-\frac{\partial f}{\partial I_j}\frac{\partial\chi}{\partial\varphi_j}\right)$ (for a generic number n of d.o.f.).
- Since $\ddot{f}=\mathcal{L}_{\chi}\dot{f}=\mathcal{L}_{\chi}^2 f$ and the same can be done for each derivative, **formally we can write the effect of the flow along χ as**

$$\Phi_{\chi}^{t}f=\sum_{j\ge0}\frac{1}{j!}\mathcal{L}_{\chi}^{j}f=\exp\mathcal{L}_{\chi}f\ ,$$

  where the operator $\exp\mathcal{L}_{\chi}\cdot$ is known as the **Lie series**.
- We can apply the Lie series to each canonical coordinate, then $\Phi_{\chi}^{t}(\mathbf{I},\boldsymbol{\varphi})=(\exp\mathcal{L}_{\chi}I_1\,,\ \ldots\ ,\ \exp\mathcal{L}_{\chi}I_n\,,\ \exp\mathcal{L}_{\chi}\varphi_1\,,\ \ldots\ ,\ \exp\mathcal{L}_{\chi}\varphi_n)\,.$ Since the flow Φ_{χ}^{t} along a Hamiltonian χ is known to be symplectic, the **Lie series induce a canonical transformation**, if (some suitable norm of) χ is small enough.

Basics of normal form theory in action–angle coordinates

- Because of the **"exchange" theorem for Lie series** (see *Gröbner, 1960*), after having performed a canonical transformation related to the flow induced by the **generating function** χ, a Hamiltonian in the new coordinates can be calculated as:

  $$H(\exp\mathcal{L}_{\chi} I_1\,,\,\ldots\,,\,\exp\mathcal{L}_{\chi} I_n\,,\,\exp\mathcal{L}_{\chi}\varphi_1\,,\,\ldots\,,\,\exp\mathcal{L}_{\chi}\varphi_n) = \exp\mathcal{L}_{\chi} H\ .$$

  In words, it means: "just compute the Lie series of the Hamiltonian and, eventually, change the symbols of the coordinates".
- Liouville-Arnold-Jost theorem suggests us to determine a generating function related to a canonical transformation removing the **angular dependence**. Let us write the Fourier expansion of a "perturbing" function so that $f(\mathbf{I},\varphi) = \sum_{\mathbf{k}\in\mathbb{Z}^n} c_{\mathbf{k}}(\mathbf{I})\exp(i\mathbf{k}\cdot\varphi)$; close to a system of harmonic oscillators $\boldsymbol{\nu}\cdot\mathbf{I}$ we often solve a **"homological" equation of type** $\mathcal{L}_{\chi}\boldsymbol{\nu}\cdot\mathbf{I} + f(\mathbf{I},\varphi) = Z(\mathbf{I})$, where the generating function χ and the **normal form part** Z are determined so that

  $$\chi(\mathbf{I},\varphi) = \sum_{\mathbf{k}\in\mathbb{Z}^n\setminus\{\mathbf{0}\}} \frac{c_{\mathbf{k}}(\mathbf{I})}{i\mathbf{k}\cdot\boldsymbol{\nu}}\exp(i\mathbf{k}\cdot\varphi)\ , \qquad Z = \langle f\rangle_{\varphi}\ ,$$

  if $\mathbf{k}\cdot\boldsymbol{\nu}\neq 0\ \forall\ \mathbf{k}\neq\mathbf{0}$ (small divisors problem!).

Partial Birkhoff normalization of the secular Hamiltonian

Let us focus on the Hamiltonian in action–angle coordinates:

$$H^{(\mathrm{I})}(\mathbf{I},\boldsymbol{\varphi}) = \boldsymbol{\nu}\cdot\mathbf{I} + \sum_{s=2}^{\infty}\mathcal{P}_{2s}^{(\mathrm{I})}(\mathbf{I},\boldsymbol{\varphi}) \ ,$$

where $\mathcal{P}_{2s}^{(\mathrm{I})}$ is an hom. pol. of degree $2s$ in the square roots of actions $\mathbf{I}$ and a trigonometric pol. of degree $2s$ in angles $\boldsymbol{\varphi}$.
The following way to expand the Hamiltonian highlights both the size of the perturbation (horizontally) and the degree in action (vertically):

$$\begin{array}{llll} & & & \ddots \\ & & & +\mathcal{P}_{8}^{(\mathrm{I})}(\mathbf{I},\boldsymbol{\varphi}) \\ & & +\mathcal{P}_{6}^{(\mathrm{I})}(\mathbf{I},\boldsymbol{\varphi}) & \\ & +\mathcal{P}_{4}^{(\mathrm{I})}(\mathbf{I},\boldsymbol{\varphi}) & & \\ H^{(\mathrm{I})}(\mathbf{I},\boldsymbol{\varphi}) = \boldsymbol{\nu}\cdot\mathbf{I} & & & \end{array}$$

Partial Birkhoff normalization of the secular Hamiltonian

Let us solve the equation for the generating function $\mathcal{B}^{(\mathrm{II})}$:

$$\left\{\mathcal{B}^{(\mathrm{II})}\,,\,\boldsymbol{\nu}\cdot\mathbf{I}\right\}+\mathcal{P}_4^{(\mathrm{I})}(\mathbf{I},\boldsymbol{\varphi})=Z_4(\mathbf{I})\ ,$$

where $\{\cdot,\cdot\}$ is a Poisson bracket, Z_4 is the angular average of $\mathcal{P}_4^{(\mathrm{I})}$. That equation can be solved if $|\mathbf{k}\cdot\boldsymbol{\nu}|\neq 0\ \forall\ \mathbf{k}\in\mathbb{Z}^2$ such that $0<|\mathbf{k}|\le 4$. Therefore, we calculate the new Hamiltonian $H^{(\mathrm{II})}=\exp\mathcal{L}_{\mathcal{B}^{(\mathrm{II})}}H^{(\mathrm{I})}$, the expansion of which can be written as follows (with new terms $\mathcal{P}_{2s}^{(\mathrm{II})}$ sharing the same properties with $\mathcal{P}_{2s}^{(\mathrm{I})}\ \forall\ s\ge 2$):

$$
\begin{aligned}
&&&&\ddots\\
&&&&+\mathcal{P}_8^{(\mathrm{II})}(\mathbf{I},\boldsymbol{\varphi})\\
&&&+\mathcal{P}_6^{(\mathrm{II})}(\mathbf{I},\boldsymbol{\varphi})&\\
&&+Z_4(\mathbf{I})&&\\
H^{(\mathrm{II})}(\mathbf{I},\boldsymbol{\varphi})=&\ \boldsymbol{\nu}\cdot\mathbf{I}&&&
\end{aligned}
$$

Partial Birkhoff normalization of the secular Hamiltonian

Let us solve the equation for the generating function $\mathcal{B}^{(\mathrm{III})}$:

$$\left\{\mathcal{B}^{(\mathrm{III})}\,,\,\boldsymbol{\nu}\cdot\mathbf{I}\right\}+\mathcal{P}_6^{(\mathrm{II})}(\mathbf{I},\boldsymbol{\varphi})=Z_6(\mathbf{I})\ ,$$

where $\{\cdot,\cdot\}$ is a Poisson bracket, Z_6 is the angular average of $\mathcal{P}_6^{(\mathrm{II})}$. That equation can be solved if $|\mathbf{k}\cdot\boldsymbol{\nu}|\neq 0\ \forall\ \mathbf{k}\in\mathbb{Z}^2$ such that $0<|\mathbf{k}|\leq 6$. Therefore, we calculate the new Hamiltonian $H^{(\mathrm{III})}=\exp\mathcal{L}_{\mathcal{B}^{(\mathrm{III})}}H^{(\mathrm{II})}$, the expansion of which can be written as follows (with new terms $\mathcal{P}_{2s}^{(\mathrm{III})}$ sharing the same properties with $\mathcal{P}_{2s}^{(\mathrm{I})}\ \forall\ s\geq 3$):

$$\begin{aligned}H^{(\mathrm{III})}(\mathbf{I},\boldsymbol{\varphi})=\ &\boldsymbol{\nu}\cdot\mathbf{I}\\&+Z_4(\mathbf{I})\\&\quad+Z_6(\mathbf{I})\\&\qquad+\mathcal{P}_8^{(\mathrm{III})}(\mathbf{I},\boldsymbol{\varphi})\\&\qquad\quad\ddots\end{aligned}$$

Intermezzo: why *Frequency Analysis* is necessary?

- **Remark**: *if the Birkhoff normalization procedure would be infinitely iterated*, **the Taylor–Fourier series of the final Hamiltonian would not be convergent on any open set.**
- **Remark**: KAM theorem requires that, *in the integrable approximation*, **on the torus $\mathbf{p} = \mathbf{0}$ the frequency vector is equal to $\boldsymbol{\omega}$, that is fixed** *"a priori"*. In our previous case, $\mathbf{I} = \mathbf{0}$ corresponds to an equilibrium point and the limit frequency vector $\boldsymbol{\nu} \neq \boldsymbol{\omega}$.
- **Problem**: how can we determine the frequencies corresponding to some initial conditions? **Answer**: using *Frequency Analysis*.

Some detail about the procedure (see *Laskar, 1995, 1999*). First, make a numerical integration of the equations of motion for $t \in [0, T]$ and store the signals $t \to \xi_j(t) + \mathrm{i}\eta_j(t)$ at regular intervals of time. *Frequency analysis numerically determines the Fourier decomposition:*

$$\xi_j(t)+\mathrm{i}\eta_j(t) = \sum_{s\in\mathbb{N}} c_{j,s} \exp\left[\mathrm{i}\left(\nu_{j,s}^{(T)} t\right)\right] \quad \textit{with } c_{j,s} \in \mathbb{C}\ \forall\, j = 1\,,\,2\,,\ s \in \mathbb{N}\,.$$

On a KAM torus, (using the Hanning filter) **the convergence to the true result is very fast**: $|\nu_{j,s}^{(T)} - \mathbf{k}_{j,s} \cdot \boldsymbol{\omega}| = \mathcal{O}(1/T^4)$ for some $\mathbf{k}_{j,s} \in \mathbb{Z}^2$.

Initial translation of the actions

We shift the origin of the actions so to center the expansions about the torus related to a chosen frequency ω *in the integrable approximation*.

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We shift the origin of the actions so to center the expansions about the torus related to a chosen frequency $\boldsymbol{\omega}$ *in the integrable approximation*.

- Let us solve the following equations in the unknown vector $\mathbf{I}^*$:

$$\nu_j + \frac{\partial Z_4}{\partial I_j}(\mathbf{I}^*) + \frac{\partial Z_6}{\partial I_j}(\mathbf{I}^*) = \omega_j \qquad \forall\ j = 1,\, 2,\, 3\,;$$

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- Let the canonical transformation $\mathcal{T}$ be so that $\mathcal{T}(\mathbf{I}, \boldsymbol{\varphi}) = (\mathbf{p} + \mathbf{I}^*, \mathbf{q})$, introduce the new Hamiltonian $H^{(0)} = H^{(\mathrm{III})} \circ \mathcal{T}$ and expand it:

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$$H^{(0)}(\mathbf{p},\mathbf{q}) = \sum \quad \begin{array}{ccccc} \vdots & \vdots & \vdots & \vdots & \vdots \\ f_2^{(0,0)}(\mathbf{p}) & f_2^{(0,1)}(\mathbf{p},\mathbf{q}) & \ldots & f_2^{(0,s)}(\mathbf{p},\mathbf{q}) & \ldots \\ \boldsymbol{\omega}\cdot\mathbf{p} & f_1^{(0,1)}(\mathbf{p},\mathbf{q}) & \ldots & f_1^{(0,s)}(\mathbf{p},\mathbf{q}) & \ldots \\ 0 & f_0^{(0,1)}(\mathbf{q}) & \ldots & f_0^{(0,s)}(\mathbf{q}) & \ldots \end{array}$$

Some remarks about the algorithm constructing the Kolmogorov's normal form

- Since the set of Diophantine numbers has full Lebesgue measure, it is always possible to produce a completion of the numerical approximation of the frequency vector $\boldsymbol{\omega}$ so that it is Diophantine.
- The terms $f_j^{(0,s)}$ appearing in the formula giving $H^{(0)}$ are defined so to have particular functional properties:
  $f_j^{(0,s)}$ is a hom. pol. of degree j in actions $\mathbf{p}$ and a trigonometric pol. of degree $2s$ in $\mathbf{q}$. Thus, *the expansion of each $f_j^{(0,s)}$ is* **representable on a computer because it is finite**.
- The Kolmogorov's normalization algorithm requires to eliminate all the terms having degree equal to 0 or 1 in the actions, except $\boldsymbol{\omega} \cdot \mathbf{p}$.
- *Where is the* **small parameter?** *The size of the perturbation is ruled by the translation vector* $\mathbf{I}^*$. Moreover, going to the right in the expansion of $H^{(0)}$, the terms get smaller and smaller. In our problem, *from a physical point of view,* $\|\mathbf{I}^*\|$ **is of the same order of magnitude as either the square of the eccentricities or the square of the inclinations**.

A half of the first Kolmogorov's normalization step

We define a new Hamiltonian $\hat{H}^{(1)} = \exp \mathcal{L}_{\chi_1^{(1)}} H^{(0)}$ where the generating function $\chi_1^{(1)}(\mathbf{q}) = X^{(1)}(\mathbf{q}) + \boldsymbol{\xi}^{(1)} \cdot \mathbf{q}$ is such that

$$\left\{ X^{(1)} \, , \, \boldsymbol{\omega} \cdot \mathbf{p} \right\} + f_0^{(0,1)} = 0 \; , \qquad \left\{ \boldsymbol{\xi}^{(1)} \cdot \mathbf{q} \, , \, f_2^{(0,0)} \right\} + \left\langle f_1^{(0,1)} \right\rangle = 0 \; ,$$

being $\langle \cdot \rangle$ the average over $\mathbf{q}$, $X^{(1)}$ a trig. pol. of deg. 2 and $\boldsymbol{\xi}^{(1)} \in \mathbb{R}^2$.

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$$\begin{array}{ccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & f_2^{(0,0)}(\mathbf{p}) & f_2^{(0,1)}(\mathbf{p},\mathbf{q}) & f_2^{(0,2)}(\mathbf{p},\mathbf{q}) & \ldots & f_2^{(0,s)}(\mathbf{p},\mathbf{q}) & \ldots \\
\sum & \boldsymbol{\omega} \cdot \mathbf{p} & f_1^{(0,1)}(\mathbf{p},\mathbf{q}) & f_1^{(0,2)}(\mathbf{p},\mathbf{q}) & \ldots & f_1^{(0,s)}(\mathbf{p},\mathbf{q}) & \ldots \\
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$$\begin{array}{ccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
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 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
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$$\begin{array}{ccccccc}
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The expansion of the new Hamiltonian $\hat{H}^{(1)}$ is calculated by studying the functional properties of all the terms of $\exp \mathcal{L}_{\chi_1^{(1)}} H^{(0)}$; e.g., $\{\chi_1^{(1)}, f_2^{(0,0)}\}$ is really like $f_1^{(0,1)}$, then $\hat{f}_1^{(1,1)} = f_1^{(0,1)} + \mathcal{L}_{\chi_1^{(1)}} f_2^{(0,0)}$ with $\langle \hat{f}_1^{(1,1)} \rangle = 0\,.$

$$\sum \quad \begin{array}{cccccc} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ f_2^{(0,0)}(\mathbf{p}) & f_2^{(0,1)}(\mathbf{p},\mathbf{q}) & f_2^{(0,2)}(\mathbf{p},\mathbf{q}) & \ldots & f_2^{(0,s)}(\mathbf{p},\mathbf{q}) & \ldots \\ \boldsymbol{\omega} \cdot \mathbf{p} & \hat{f}_1^{(1,1)}(\mathbf{p},\mathbf{q}) & f_1^{(0,2)}(\mathbf{p},\mathbf{q}) & \ldots & f_1^{(0,s)}(\mathbf{p},\mathbf{q}) & \ldots \\ 0 & 0 & f_0^{(0,2)}(\mathbf{q}) & \ldots & f_0^{(0,s)}(\mathbf{q}) & \ldots \end{array}$$

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$$\begin{array}{cccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & \hat{f}_2^{(1,0)}(\mathbf{p}) & \hat{f}_2^{(1,1)}(\mathbf{p},\mathbf{q}) & \hat{f}_2^{(1,2)}(\mathbf{p},\mathbf{q}) & \ldots & \hat{f}_2^{(1,s)}(\mathbf{p},\mathbf{q}) & \ldots \\
\hat{H}^{(1)}(\mathbf{p},\mathbf{q}) = \sum & \boldsymbol{\omega} \cdot \mathbf{p} & \hat{f}_1^{(1,1)}(\mathbf{p},\mathbf{q}) & \hat{f}_1^{(1,2)}(\mathbf{p},\mathbf{q}) & \ldots & \hat{f}_1^{(1,s)}(\mathbf{p},\mathbf{q}) & \ldots \\
 & 0 & 0 & \hat{f}_0^{(1,2)}(\mathbf{q}) & \ldots & \hat{f}_0^{(1,s)}(\mathbf{q}) & \ldots
\end{array}$$

Completing the first Kolmogorov's normalization step

We define a new Hamiltonian $H^{(1)} = \exp \mathcal{L}_{\chi_2^{(1)}} \hat{H}^{(1)}$ where the generating function $\chi_2^{(1)}(\mathbf{p}, \mathbf{q})$ is such that

$$\left\{ \chi_2^{(1)} \, , \, \boldsymbol{\omega} \cdot \mathbf{p} \right\} + \hat{f}_1^{(1,1)} = 0 \, ,$$

since $\langle \hat{f}_1^{(1,1)} \rangle = 0$, $\chi_2^{(1)}$ is linear in $\mathbf{p}$ and a trig. pol. of deg. 2 in $\mathbf{q}$.

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$$\begin{array}{ccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & \hat{f}_2^{(1,0)}(\mathbf{p}) & \hat{f}_2^{(1,1)}(\mathbf{p},\mathbf{q}) & \hat{f}_2^{(1,2)}(\mathbf{p},\mathbf{q}) & \ldots & \hat{f}_2^{(1,s)}(\mathbf{p},\mathbf{q}) & \ldots \\
\sum & \boldsymbol{\omega} \cdot \mathbf{p} & \hat{f}_1^{(1,1)}(\mathbf{p},\mathbf{q}) & \hat{f}_1^{(1,2)}(\mathbf{p},\mathbf{q}) & \ldots & \hat{f}_1^{(1,s)}(\mathbf{p},\mathbf{q}) & \ldots \\
 & 0 & 0 & \hat{f}_0^{(1,2)}(\mathbf{q}) & \ldots & \hat{f}_0^{(1,s)}(\mathbf{q}) & \ldots
\end{array}$$

Completing the first Kolmogorov's normalization step

We define a new Hamiltonian $H^{(1)} = \exp \mathcal{L}_{\chi_2^{(1)}} \hat{H}^{(1)}$ where the generating function $\chi_2^{(1)}(\mathbf{p}, \mathbf{q})$ is such that

$$\left\{ \chi_2^{(1)} \, , \, \boldsymbol{\omega} \cdot \mathbf{p} \right\} + \hat{f}_1^{(1,1)} = 0 \ ,$$

since $\langle \hat{f}_1^{(1,1)} \rangle = 0$, $\chi_2^{(1)}$ is linear in $\mathbf{p}$ and a trig. pol. of deg. 2 in $\mathbf{q}$.
The expansion of the new Hamiltonian $H^{(1)}$ is calculated by studying the functional properties of all the terms of $\exp \mathcal{L}_{\chi_2^{(1)}} \hat{H}^{(1)}$; e.g., $\{\chi_2^{(1)}, \boldsymbol{\omega} \cdot \mathbf{p}\}$ shares the same properties with $\hat{f}_1^{(1,1)}$, then $f_1^{(1,1)} = \hat{f}_1^{(1,1)} + \mathcal{L}_{\chi_2^{(1)}} \boldsymbol{\omega} \cdot \mathbf{p}$.

$$\begin{array}{ccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & \hat{f}_2^{(1,0)}(\mathbf{p}) & \hat{f}_2^{(1,1)}(\mathbf{p},\mathbf{q}) & \hat{f}_2^{(1,2)}(\mathbf{p},\mathbf{q}) & \ldots & \hat{f}_2^{(1,s)}(\mathbf{p},\mathbf{q}) & \ldots \\
\sum & \boldsymbol{\omega} \cdot \mathbf{p} & \hat{f}_1^{(1,1)}(\mathbf{p},\mathbf{q}) & \hat{f}_1^{(1,2)}(\mathbf{p},\mathbf{q}) & \ldots & \hat{f}_1^{(1,s)}(\mathbf{p},\mathbf{q}) & \ldots \\
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$$\sum\quad \begin{array}{cccccc} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hat{f}_2^{(1,0)}(\mathbf{p}) & \hat{f}_2^{(1,1)}(\mathbf{p},\mathbf{q}) & \hat{f}_2^{(1,2)}(\mathbf{p},\mathbf{q}) & \ldots & \hat{f}_2^{(1,s)}(\mathbf{p},\mathbf{q}) & \ldots \\ \boldsymbol{\omega}\cdot\mathbf{p} & 0 & \hat{f}_1^{(1,2)}(\mathbf{p},\mathbf{q}) & \ldots & \hat{f}_1^{(1,s)}(\mathbf{p},\mathbf{q}) & \ldots \\ 0 & 0 & \hat{f}_0^{(1,2)}(\mathbf{q}) & \ldots & \hat{f}_0^{(1,s)}(\mathbf{q}) & \ldots \end{array}$$

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$$\begin{array}{ccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & \hat{f}_2^{(1,0)}(\mathbf{p}) & \hat{f}_2^{(1,1)}(\mathbf{p},\mathbf{q}) & \hat{f}_2^{(1,2)}(\mathbf{p},\mathbf{q}) & \ldots & \hat{f}_2^{(1,s)}(\mathbf{p},\mathbf{q}) & \ldots \\
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$$\begin{array}{ccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & \hat{f}_2^{(1,0)}(\mathbf{p}) & \hat{f}_2^{(1,1)}(\mathbf{p},\mathbf{q}) & \hat{f}_2^{(1,2)}(\mathbf{p},\mathbf{q}) & \ldots & \hat{f}_2^{(1,s)}(\mathbf{p},\mathbf{q}) & \ldots \\
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$$\begin{array}{ccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & \hat{f}_2^{(1,0)}(\mathbf{p}) & f_2^{(1,1)}(\mathbf{p},\mathbf{q}) & \hat{f}_2^{(1,2)}(\mathbf{p},\mathbf{q}) & \ldots & \hat{f}_2^{(1,s)}(\mathbf{p},\mathbf{q}) & \ldots \\
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\end{array}$$

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since $\langle \hat{f}_1^{(1,1)} \rangle = 0$, $\chi_2^{(1)}$ is linear in $\mathbf{p}$ and a trig. pol. of deg. 2 in $\mathbf{q}$.
The expansion of the new Hamiltonian $H^{(1)}$ is calculated by studying the functional properties of all the terms of $\exp \mathcal{L}_{\chi_2^{(1)}} \hat{H}^{(1)}$. *Therefore, we can get the recursive expressions of all* $\hat{f}_j^{(1,s)}$ *in the following expansion*:

$$H^{(1)}(\mathbf{p},\mathbf{q}) = \sum \begin{array}{cccccc} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ f_2^{(1,0)}(\mathbf{p}) & f_2^{(1,1)}(\mathbf{p},\mathbf{q}) & f_2^{(1,2)}(\mathbf{p},\mathbf{q}) & \ldots & f_2^{(1,s)}(\mathbf{p},\mathbf{q}) & \ldots \\ \boldsymbol{\omega}\cdot\mathbf{p} & 0 & f_1^{(1,2)}(\mathbf{p},\mathbf{q}) & \ldots & f_1^{(1,s)}(\mathbf{p},\mathbf{q}) & \ldots \\ 0 & 0 & f_0^{(1,2)}(\mathbf{q}) & \ldots & f_0^{(1,s)}(\mathbf{q}) & \ldots \end{array}$$

Completing all the Kolmogorov's normalization algorithm

The Kolmogorov's normalization step can be "infinitely" iterated, if

- the *frequency vector* $\boldsymbol{\omega}$ is **non-resonant** enough (e.g., diophantine, i.e. $|\mathbf{k}\cdot\boldsymbol{\omega}| > \gamma/|\mathbf{k}|^{\tau}$ with some fixed $\gamma > 0$ and $\tau \geq n-1$);
- the *hessian* of $f_2^{(0,0)}(\mathbf{p})$ is **non-degenerate**;
- the *perturbation* (or, equivalently, $\|\mathbf{I}^*\|$) is **small** enough.

Therefore, the sequence of $H^{(r)}$ is convergent to a Hamiltonian $H^{(\infty)}$ in Kolmogorov's normal form, the expansions of which is written as

$$
H^{(\infty)}(\mathbf{p},\mathbf{q}) = \sum
\begin{array}{ccccc}
\vdots & \vdots & \vdots & \vdots & \vdots \\
f_2^{(\infty,0)}(\mathbf{p}) & f_2^{(\infty,1)}(\mathbf{p},\mathbf{q}) & \ldots & f_2^{(\infty,s)}(\mathbf{p},\mathbf{q}) & \ldots \\
\boldsymbol{\omega}\cdot\mathbf{p} & 0 & \ldots & 0 & \ldots \\
0 & 0 & \ldots & 0 & \ldots
\end{array}
$$

Some remarks about the Kolmogorov's normal form

- The (KAM) torus corresponding to $\mathbf{p} = \mathbf{0}$ is obviously invariant with respect to the flow $\Phi^t_{H^{(\infty)}}$ induced by the Hamiltonian in Kolmogorov's normal form. In fact, if $\mathbf{p} = \mathbf{0}$ then

$$\dot{p}_j = -\frac{\partial H^{(\infty)}}{\partial q_j}(\mathbf{0}, \mathbf{q}) = 0 \ , \qquad \dot{q}_j = \frac{\partial H^{(\infty)}}{\partial p_j}(\mathbf{0}, \mathbf{q}) = \omega_j \ .$$

- This approach, based on the construction of the Kolmogorov's normal form by a sequence of Lie series can be translated in a proof (see *Benettin & al., 1984*, *Giorgilli & Locatelli, 1997*). Moreover, by implementing interval arithmetics and estimating all the truncated terms, this approach can be used to produce a **computer–assisted proof** (as in *Locatelli & Giorgilli, 2000*).

Testing the construction of the Kolmogorov's normal form

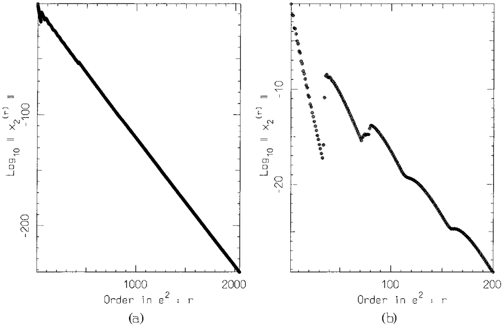

Figure: Decrease of the generating functions defined by the Kolmogorov's normalization algorithm. The plotted values are the uniform upper bounds of the norms of $\chi_2^{(r)}$. Box a has been enlarged in box b, where we can appreciate the change of the slope occurring when the calculation of the norms is no more made starting by the coefficients of the expansions, but only iterating the estimates (i.e., for $r = 33$).

While *the generating functions are explicitly calculated* according to the Kolmogorov normalization algorithm (i.e. when $1 \leq r \leq 33$), *their geometrical decrease is very evident*.

Some remarks about the Kolmogorov's normal form

Let $\mathcal{K}^{(r)}$ be the canonical transformation inducing the Kolmogorov's normalization up to the step r, i.e.,

$$\mathcal{K}^{(r)}(\mathbf{p},\mathbf{q}) = \exp\mathcal{L}_{\chi_2^{(r)}} \circ \exp\mathcal{L}_{\chi_1^{(r)}} \circ \ldots \circ \exp\mathcal{L}_{\chi_2^{(1)}} \circ \exp\mathcal{L}_{\chi_1^{(1)}} (\mathbf{p},\mathbf{q}) \ .$$

Let us now introduce $\mathcal{K}^{(\infty)} = \lim_{r\to\infty} \mathcal{K}^{(r)}$ and $\mathcal{C}^{(\infty)} = \mathcal{D} \circ \mathcal{A} \circ \exp\mathcal{L}_{\mathcal{B}^{(\mathrm{III})}} \circ \exp\mathcal{L}_{\mathcal{B}^{(\mathrm{II})}} \circ \exp\mathcal{L}_{\mathcal{B}^{(\mathrm{I})}} \circ \mathcal{T} \circ \mathcal{K}^{(\infty)}$, that is the composition of all the can. transf. of the algorithm. *The equations of motion can be integrated in a* **semi-analytic way** as follows:

$$\begin{array}{ccc}
\big(\boldsymbol{\xi}(0),\boldsymbol{\eta}(0)\big) & \xrightarrow{\left(\mathcal{C}^{(\infty)}\right)^{-1}} & \big(\mathbf{p}(0)=\mathbf{0}\,,\ \mathbf{q}(0)\big) \\
 & & \Big\downarrow \Phi^{t}_{\boldsymbol{\omega}\cdot\mathbf{p}} \\
\big(\boldsymbol{\xi}(t),\boldsymbol{\eta}(t)\big) & \xleftarrow{\mathcal{C}^{(\infty)}} & \big(\mathbf{p}(t)=\mathbf{0}\,,\ \mathbf{q}(t)=\mathbf{q}(0)+\boldsymbol{\omega}t\big)
\end{array} .$$

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Let us introduce $\mathcal{C}^{(r)} = \mathcal{D}\circ\mathcal{A}\circ\exp\mathcal{L}_{\mathcal{B}^{(\mathrm{III})}}\circ\exp\mathcal{L}_{\mathcal{B}^{(\mathrm{II})}}\circ\exp\mathcal{L}_{\mathcal{B}^{(\mathrm{I})}}\circ\mathcal{T}\circ\mathcal{K}^{(r)}$, that is the composition of all the can. transf. of the algorithm up to the step r. *The solution of the eqs. of motion can be* ***approximated in a semi-analytic way*** *as follows:*

$$\begin{array}{ccc}
\big(\boldsymbol{\xi}(0),\boldsymbol{\eta}(0)\big) & \overset{\left(\mathcal{C}^{(r)}\right)^{-1}}{\longrightarrow} & \big(\mathbf{p}(0)\simeq\mathbf{0}\,,\ \mathbf{q}(0)\big) \\
 & & \Big\downarrow \Phi^{t}_{\boldsymbol{\omega}\cdot\mathbf{p}} \\
\big(\boldsymbol{\xi}(t),\boldsymbol{\eta}(t)\big) & \overset{\mathcal{C}^{(r)}}{\longleftarrow} & \big(\mathbf{p}(t)\simeq\mathbf{p}(0)\,,\ \mathbf{q}(t)\simeq\mathbf{q}(0)+\boldsymbol{\omega}t\big)
\end{array}\ .$$

Testing the construction of the Kolmogorov's normal form

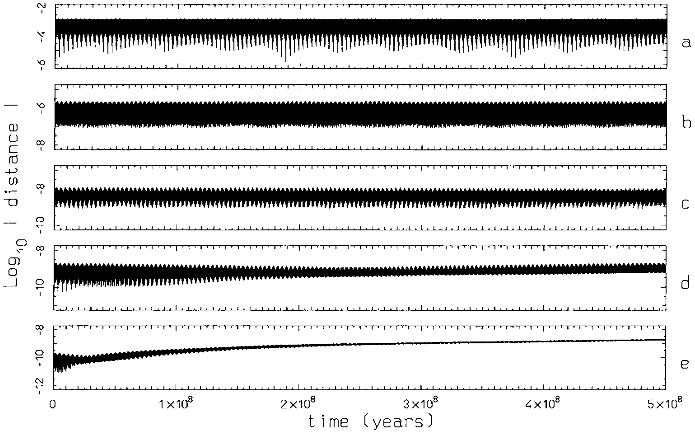

Figure: Study of the secular part up to order 2 in the masses of the SJS system. The distance $d = d(t)$ is between a *semi-analytic solution* and a numerical one. The curves a–e refer to the step r of Kolmogorov's algorithm with $r = 1$, 5, 9, 11, 13, resp. The convergence may be appreciated by looking at the vertical scale. The drift effect in box e is due to the numerical error on the frequency vector $\boldsymbol{\omega}$.

Rigorous proof of the topological confinement

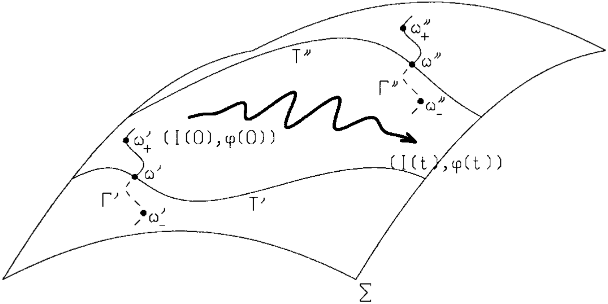

Figure: Scheme of the topological confinement of the orbit in a 4D phase space. The continuous curves Γ' and Γ'' represent two families of 2D invariant tori intersecting transversally the energy surface Σ. An orbit with initial datum in the gap between two tori will stay eternally trapped in that same region.

For the secular part up to order 2 in the masses of the SJS problem, a computer–assisted proof of stability has been produced.
In general, **there is no topological confinement for problems with more than two degrees of freedom.**