Celestial and Spaceflight Mechanics Laboratory

Celestial Mechanics of Asteroid Systems

D.J. Scheeres

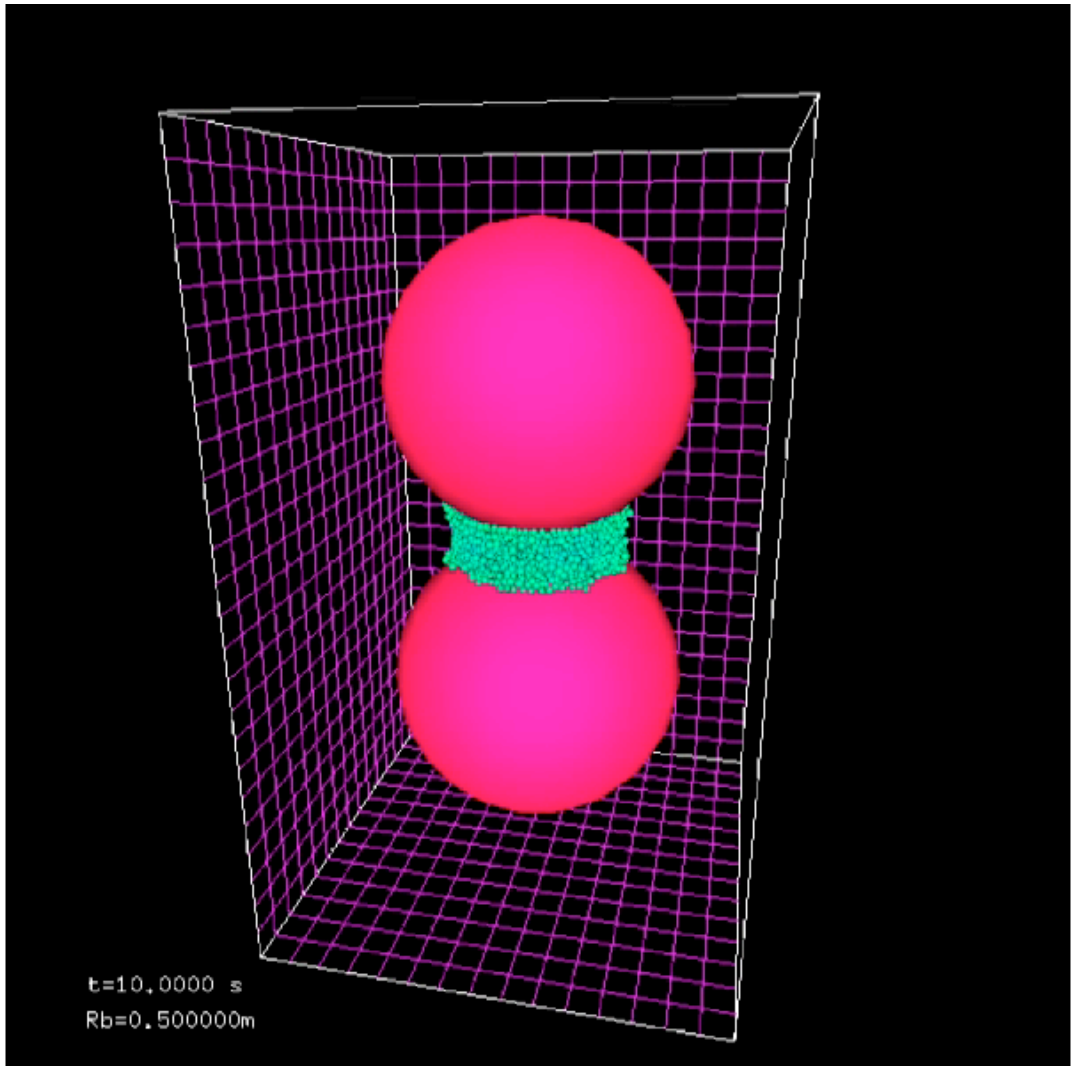
Department of Aerospace Engineering Sciences

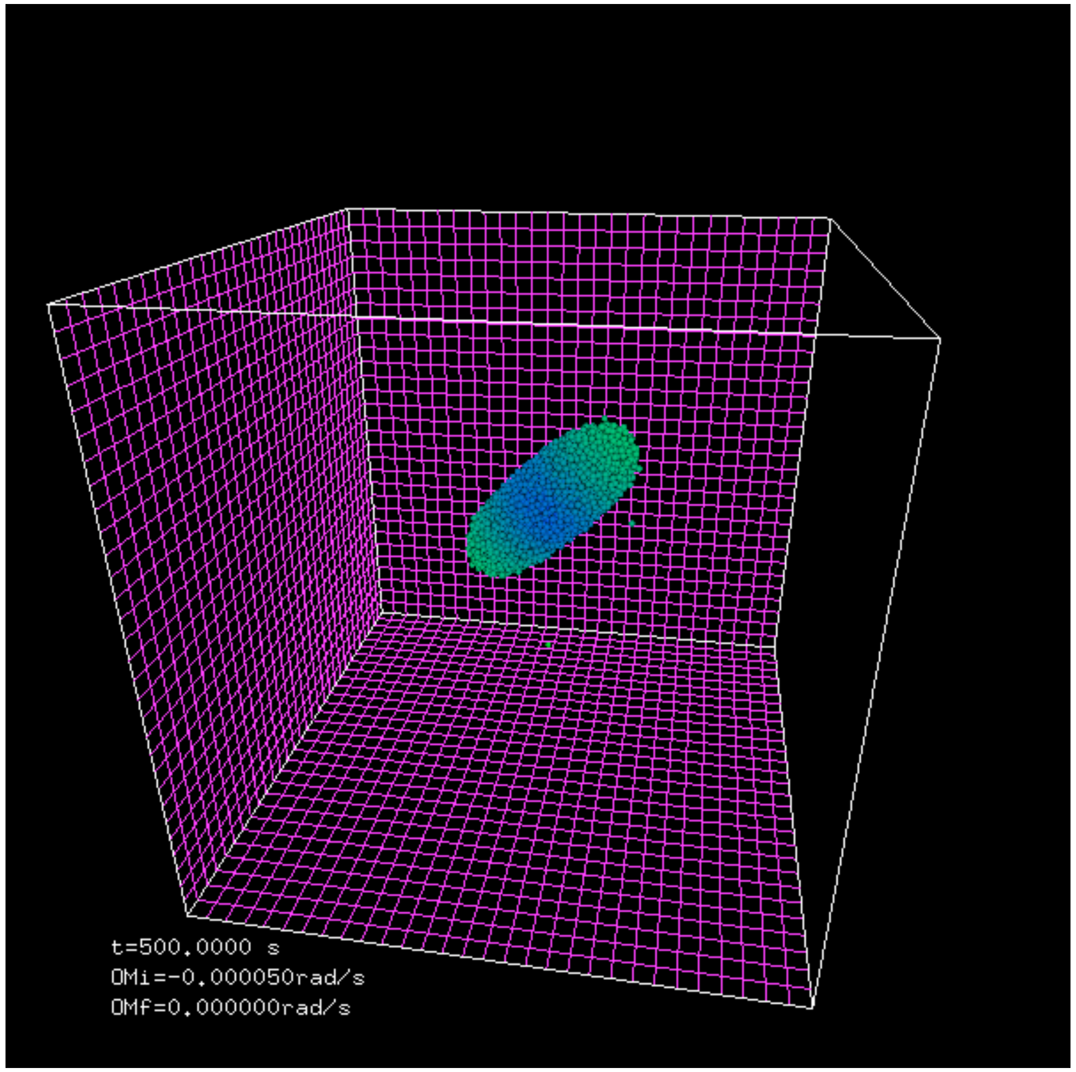
The University of Colorado

scheeres@colorado.edu

Granular Mechanics and Asteroids

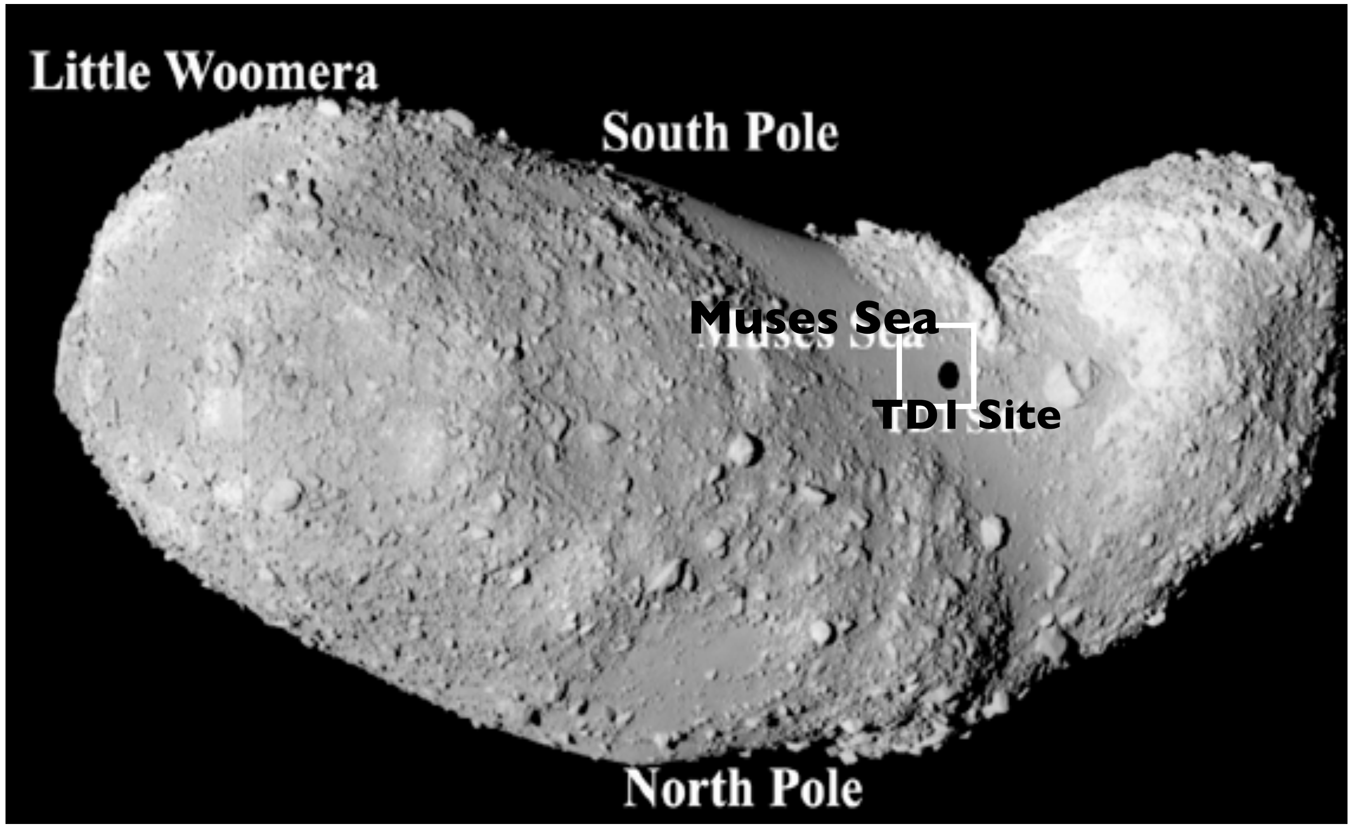
- Asteroid systems are best modeled as self-gravitating granular systems under self-attraction and cohesion

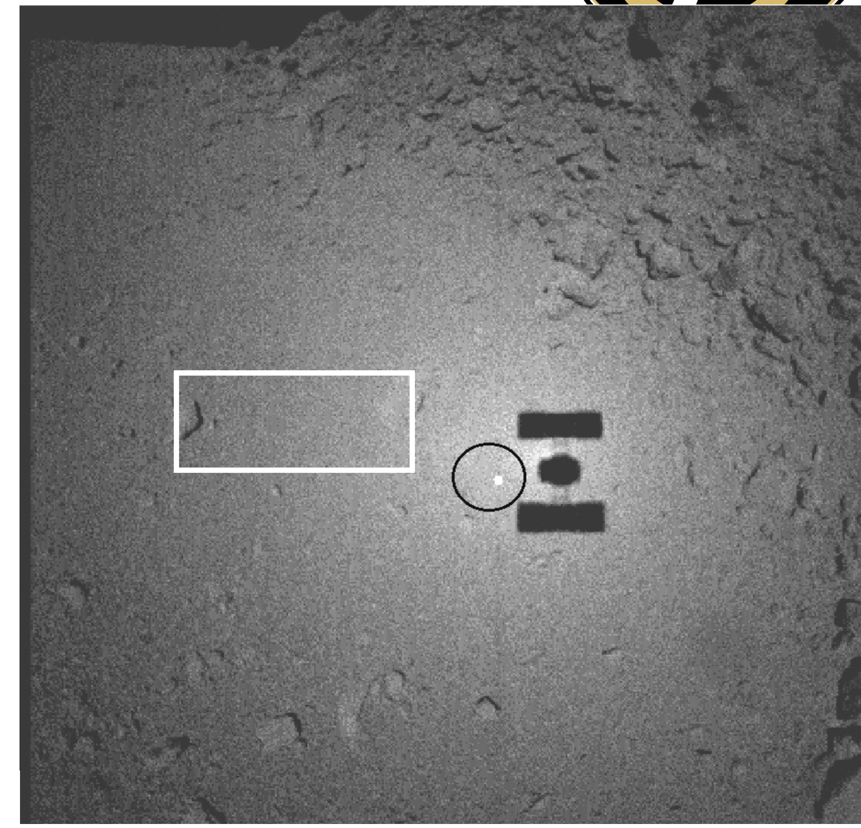




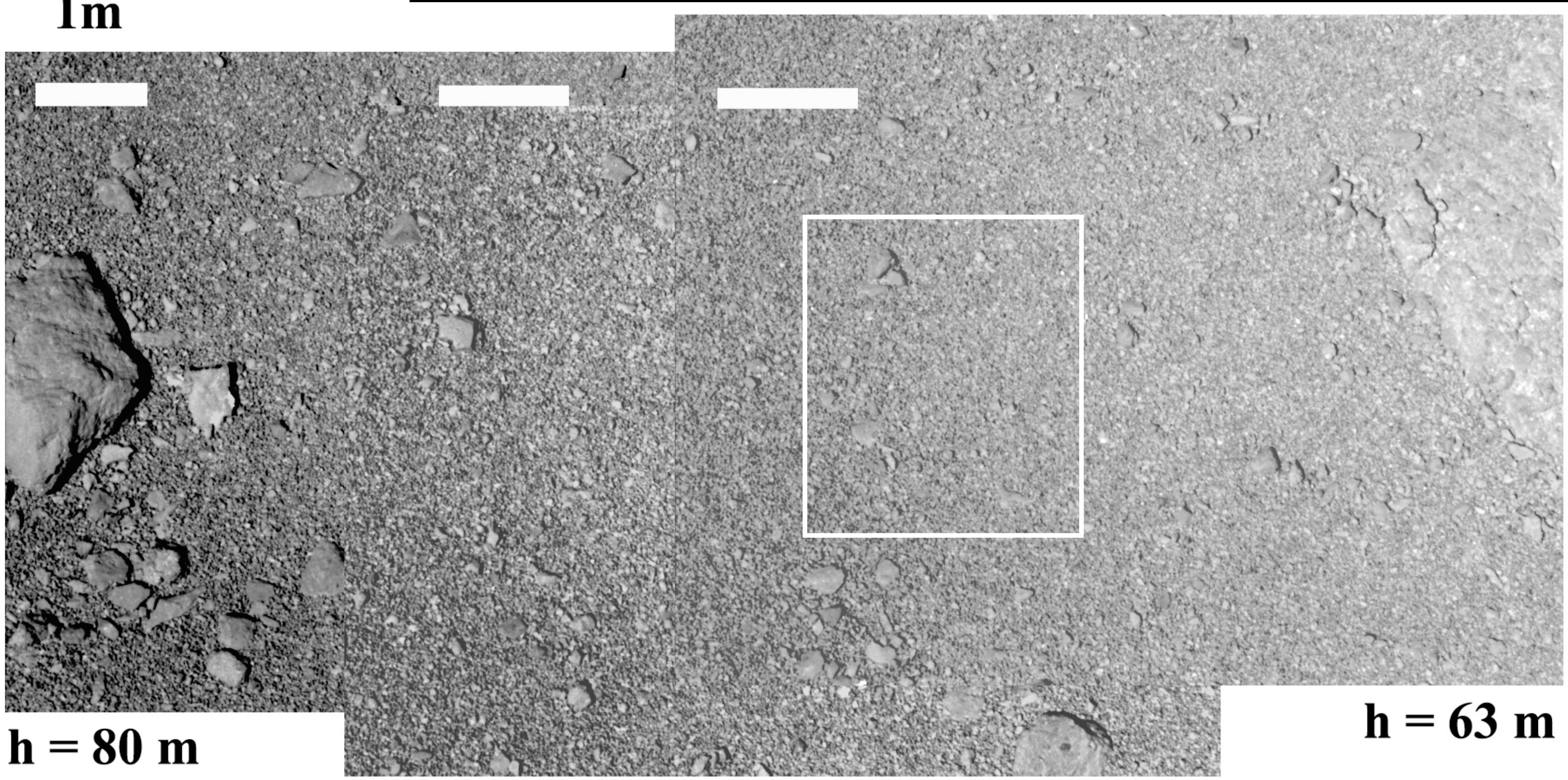
Motivated by Observations

All images courtesy JAXA/ISAS





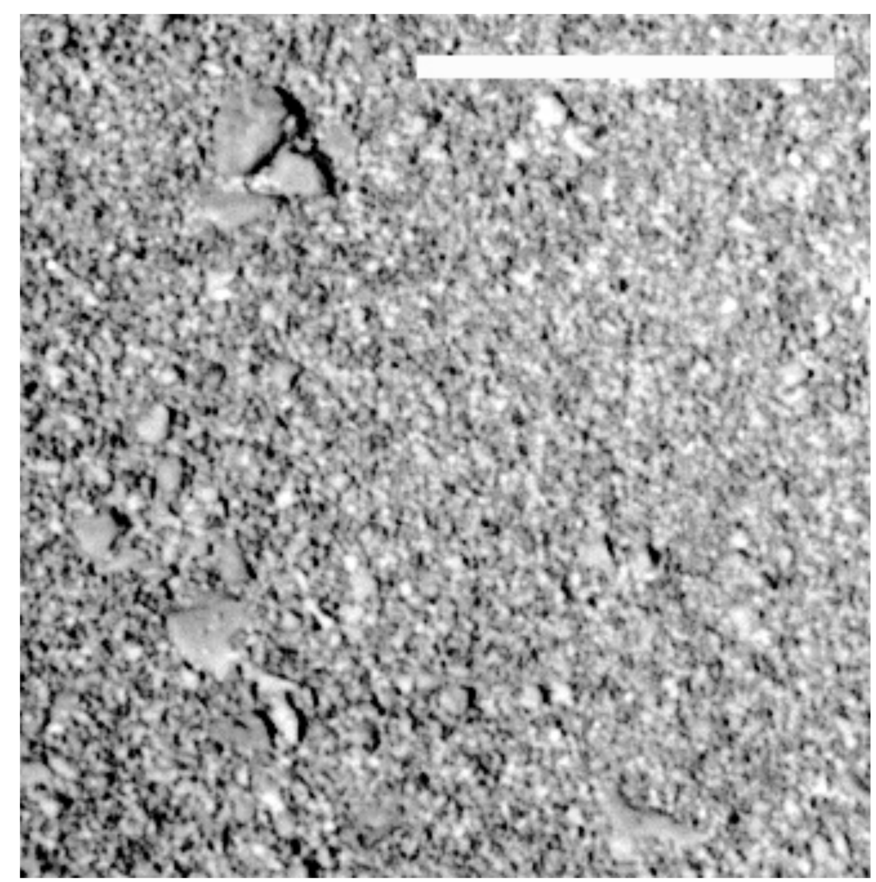
1m



h = 80 m

h = 68 m

h = 63 m



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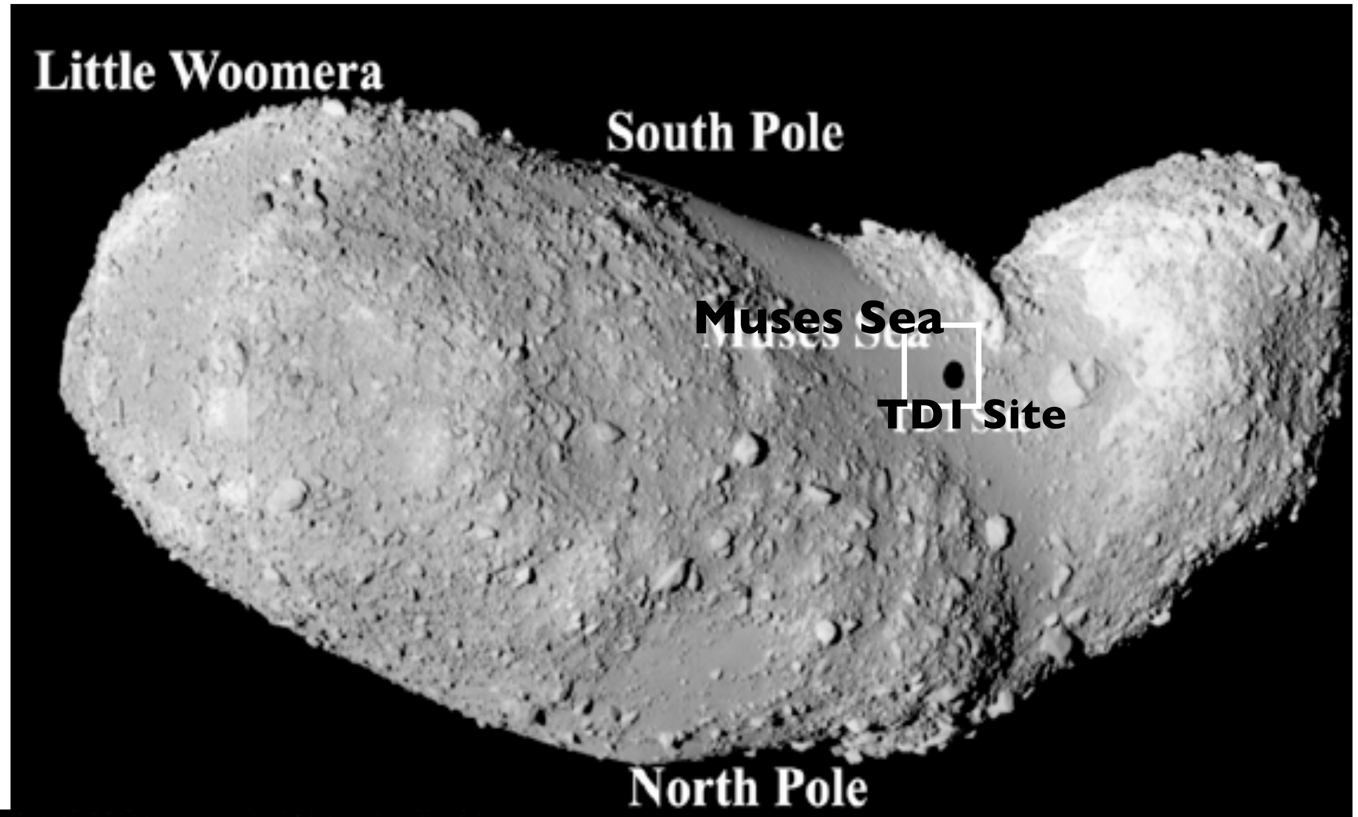
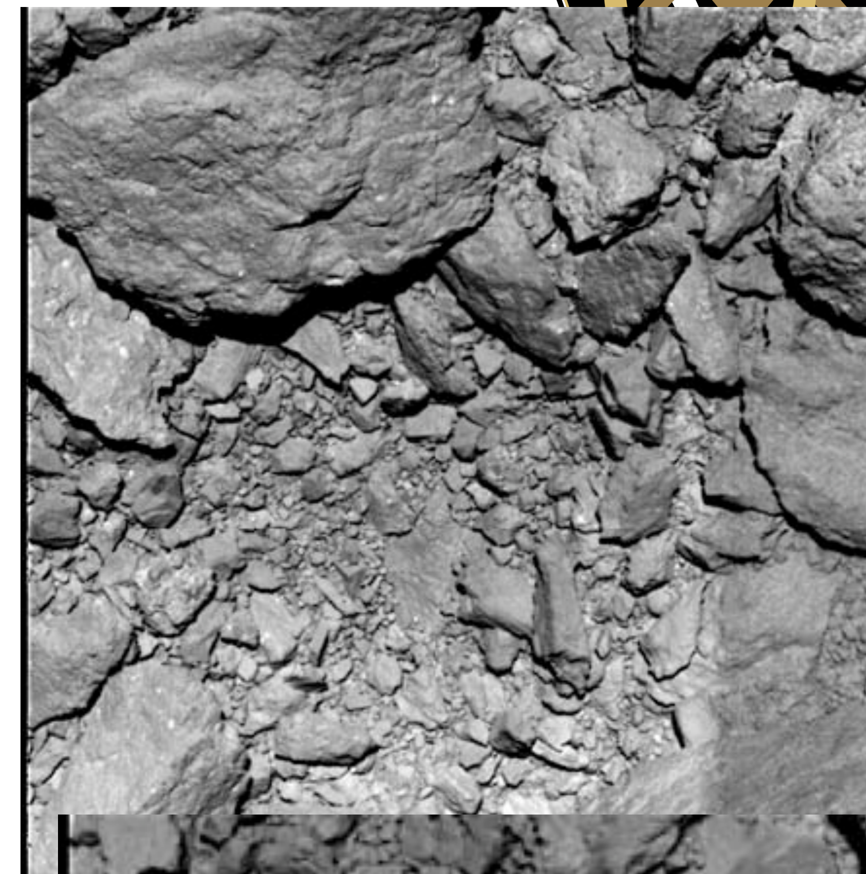
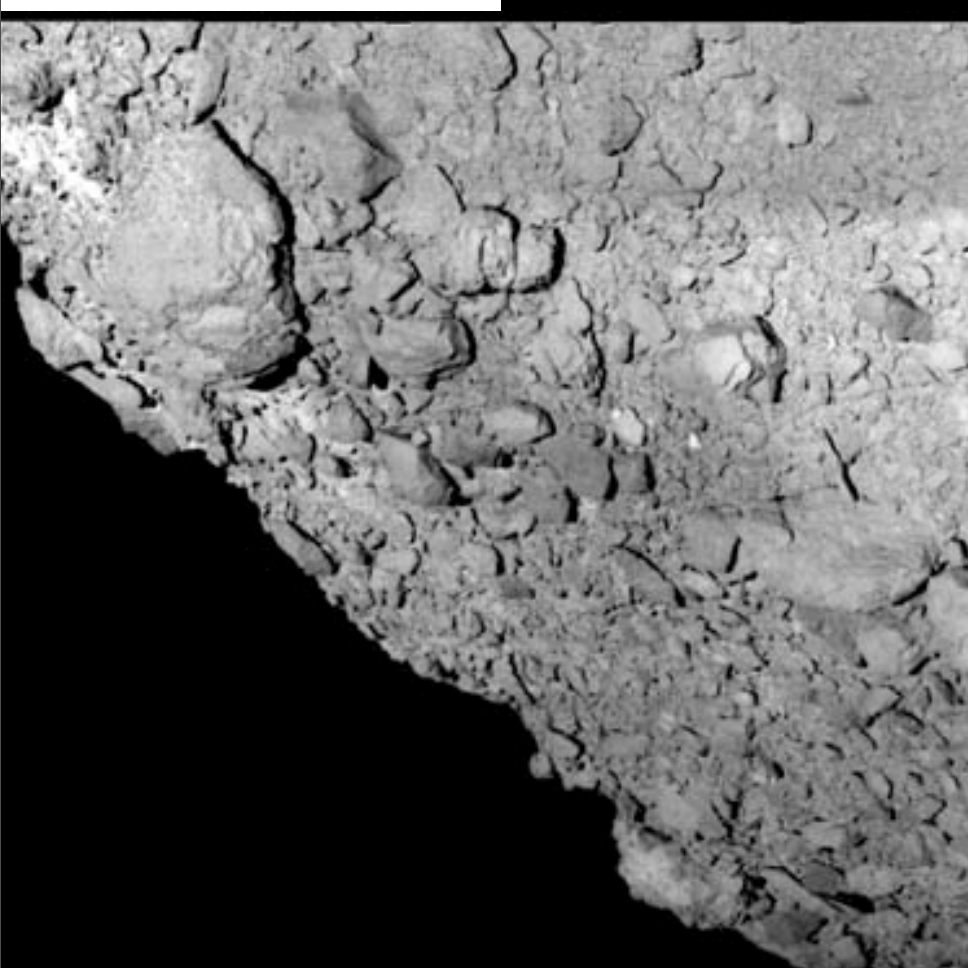
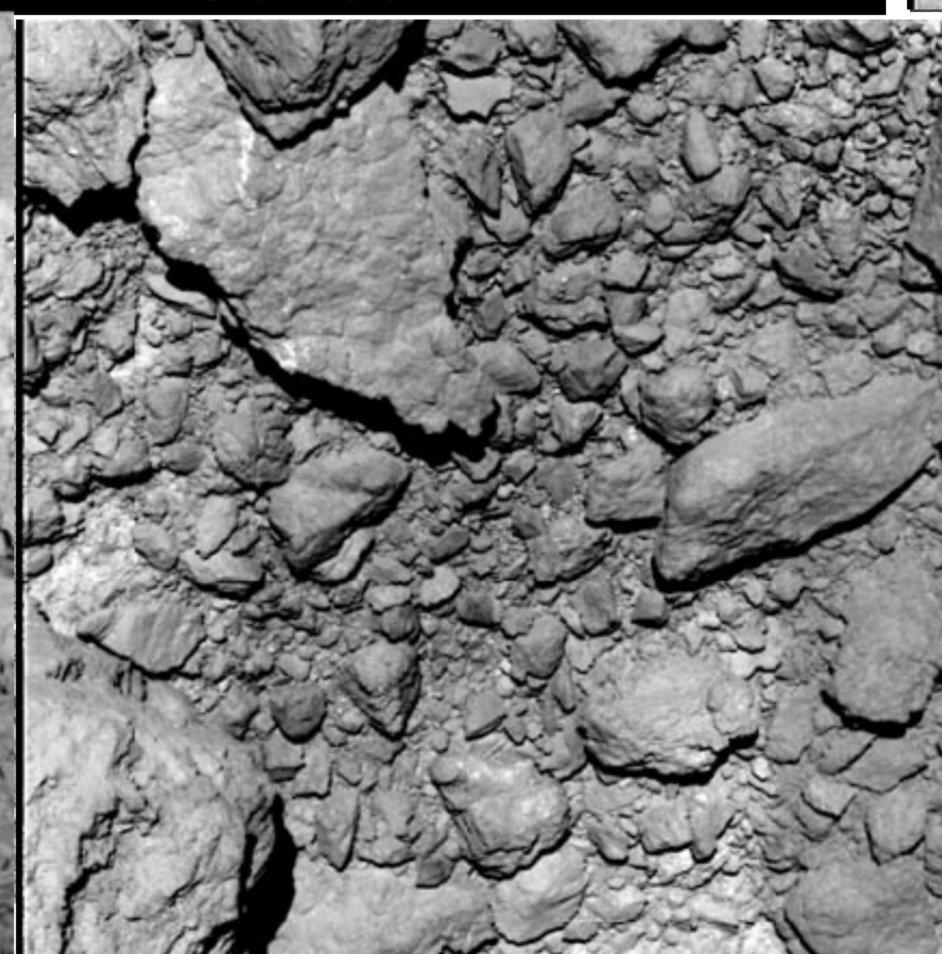


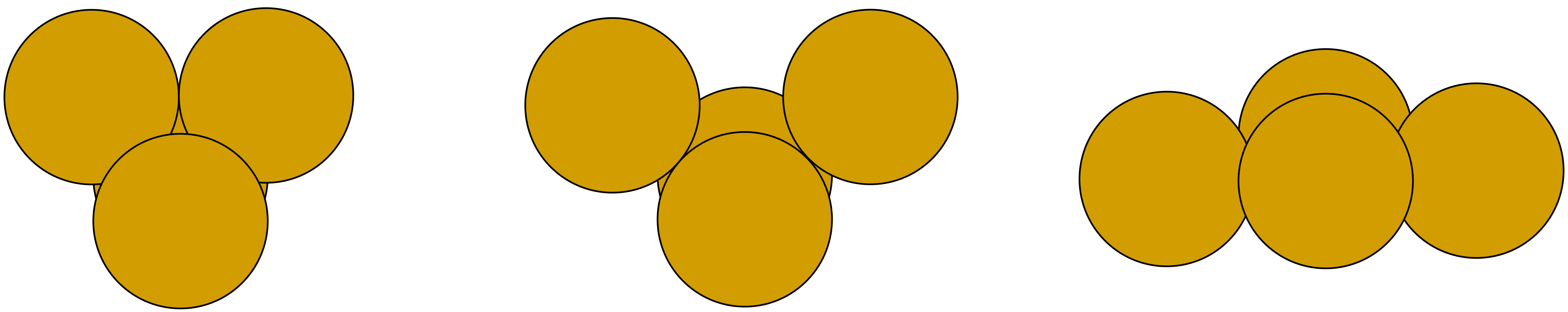






Discrete Granular Mechanics and Celestial Mechanics

- The theoretical mechanics of few-body systems can shed light on more complex aggregations and their evolution



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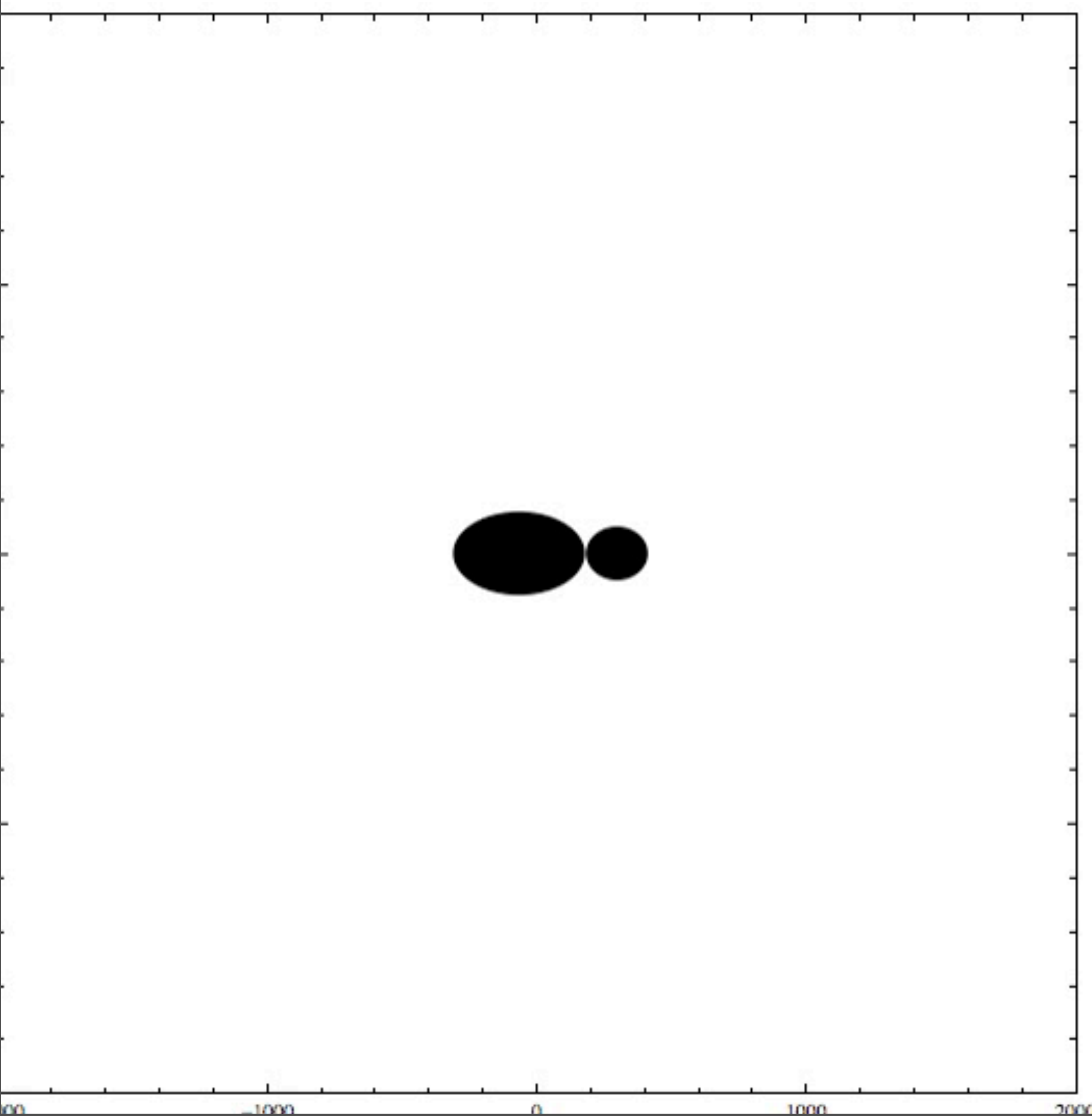
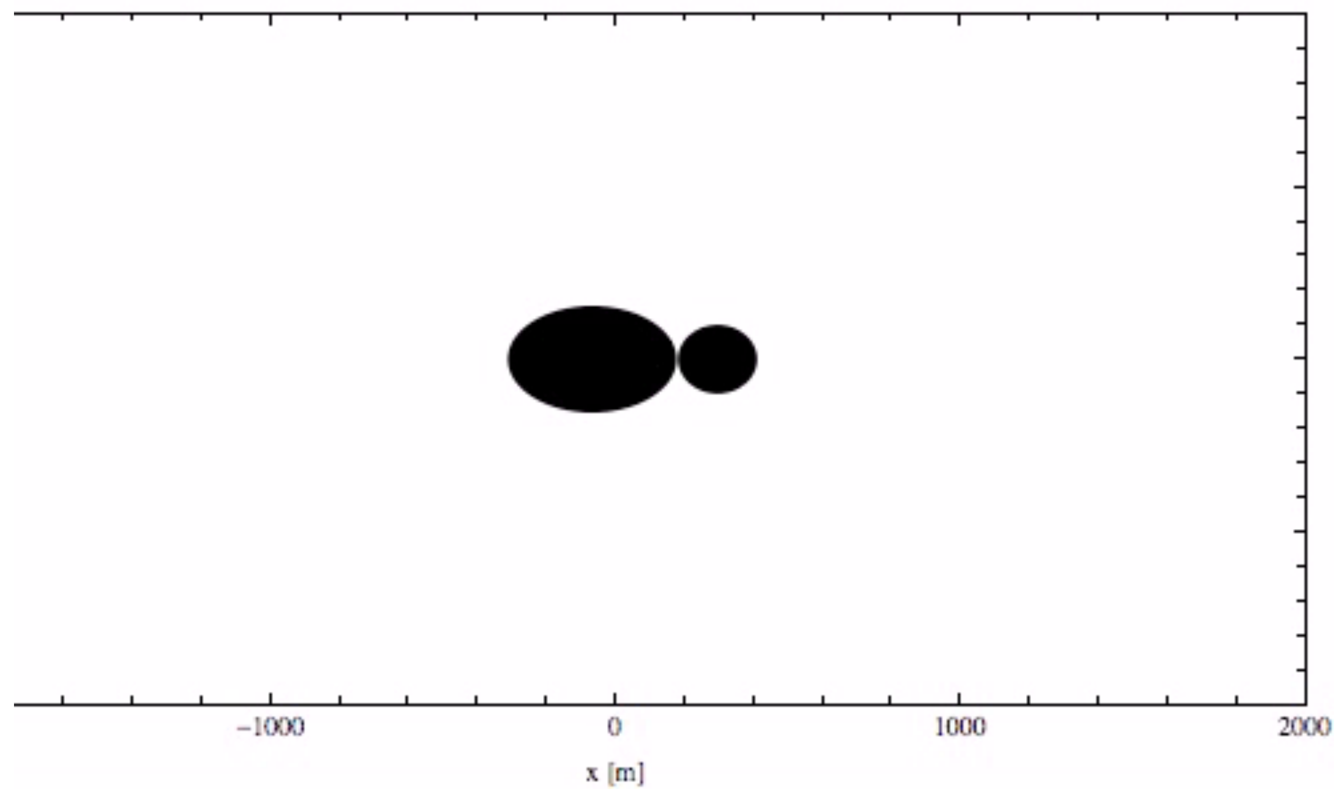
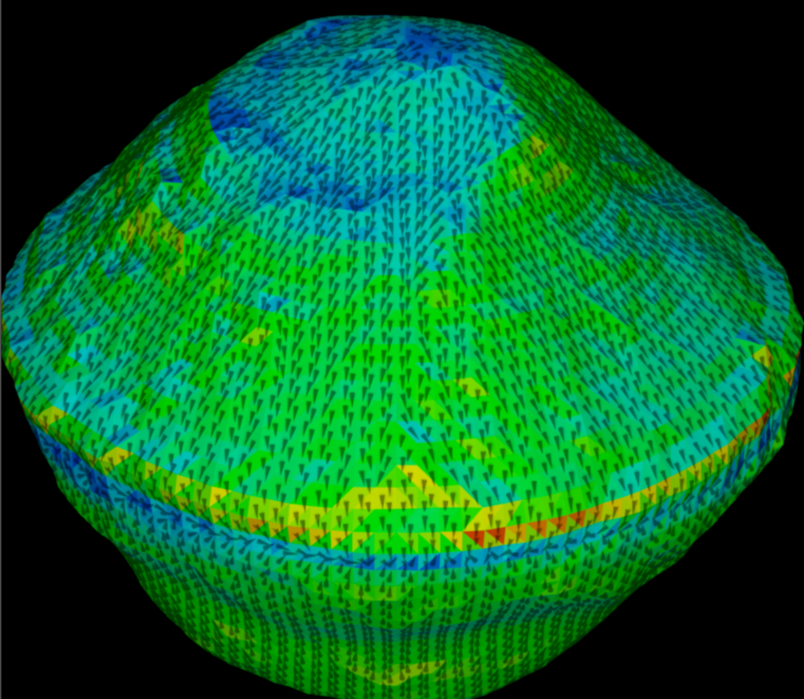


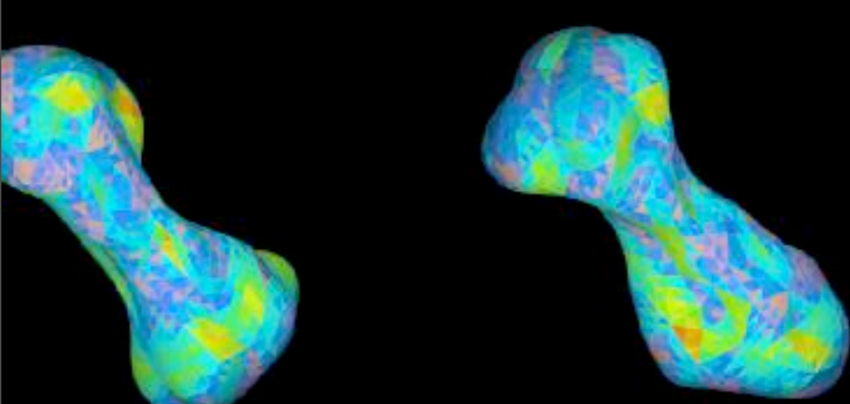


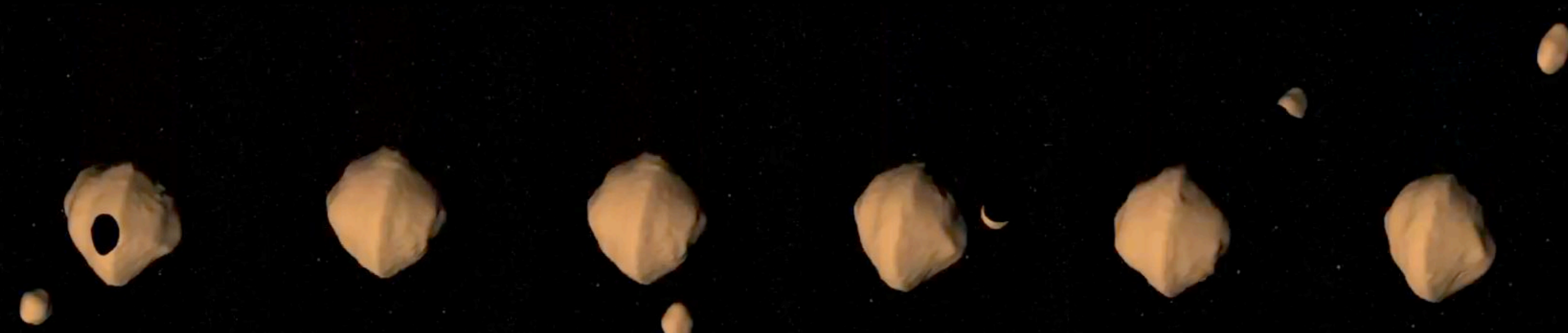
Movies by S.A. Jacobson

Fundamental and Simple Question:

What are the expected configurations for collections of self-gravitating grains?







Fundamental Concepts:

- The N-body problem:

$$m_i \ddot{\mathbf{r}}_i = -\frac{\partial U}{\partial \mathbf{r}_i}$$

$$i = 1, 2, \ldots N$$

$$U = -\frac{\mathcal{G}}{2} \sum_{j=1}^{N} \sum_{k=1, \neq i}^{N} \frac{m_j m_k}{r_{jk}} \qquad \mathbf{r}_{jk} = \mathbf{r}_k - \mathbf{r}_j \qquad r_{jk} = \left| \mathbf{r}_k - \mathbf{r}_j \right|$$

$$\mathbf{0} = \sum_{j=1}^{N} m_j \mathbf{r}_j$$

- Mass:
  - In the Newtonian N-Body Problem each particle has a total mass m_i modeled as a point mass of infinite density

Fundamental Concepts:

- Angular Momentum:

$$\mathbf{H} = \sum_{j=1}^{N} m_j \mathbf{r}_j \times \dot{\mathbf{r}}_j$$

$$= \frac{1}{2M} \sum_{j=1}^{N} \sum_{k=1}^{N} m_j m_k \mathbf{r}_{jk} \times \dot{\mathbf{r}}_{jk} \qquad M = \sum_{j=1}^{N} m_j$$

  - Mechanical angular momentum is conserved for a closed system, independent of internal physical processes.
  - The most fundamental conservation principle in Celestial Mechanics.

Fundamental Concepts:

- Energy:

$$
\begin{aligned}
E &= T + U \\
T &= \frac{1}{2}\sum_{j=1}^{N} m_j \dot{\mathbf{r}}_j \cdot \dot{\mathbf{r}}_j \\
&= \frac{1}{4M}\sum_{j=1}^{N}\sum_{k=1}^{N} m_j m_k \dot{\mathbf{r}}_{jk} \cdot \dot{\mathbf{r}}_{jk}
\end{aligned}
$$

  - Not necessarily conserved for a closed system
  - Additional non-modeled physical effects internal to the system can lead to dissipation of energy (e.g., tidal forces, surface friction)
  - Physically occurs whenever relative motion exists within a system – *motivates the study of relative equilibria*

Leads to a more precise question ...

Q: *What are the minimum energy configurations for the Newtonian N-body problem at a fixed Angular Momentum?*

A: *There are none for N ≥3.*

A surprising and untenable result – all mechanical systems should have a minimum energy state...

Sundman's Inequality

- To investigate this we start with Sundman's Inequality
  - Apply Cauchy's Inequality to the Angular Momentum

$$H^2 = \frac{1}{4M^2}\left|\sum_{j,k=1}^{N} m_j m_k \mathbf{r}_{jk} \times \dot{\mathbf{r}}_{jk}\right|^2 \leq \frac{1}{4M^2}\left(\sum_{j,k=1}^{N} m_j m_k r_{jk}^2\right)\left(\sum_{j,k=1}^{N} m_j m_k \dot{r}_{jk}^2\right) = 2IT$$

- Sundman's Inequality is:

$$H^2 \leq 2IT$$

$$I = \sum_{i=1}^{N} m_i r_i^2 = \frac{1}{2M}\sum_{j,k=1}^{N} m_j m_k r_{jk}^2$$ Polar Moment of Inertia

Minimum Energy Function and Relative Equilibrium

- Leads to a lower bound on the energy of an *N*-body system by defining the "minimum energy function" E_m *(also known as the Amended Potential)*.

$$H^2 \leq 2IT \qquad\qquad T = E - U$$

$$E_m(\mathbf{Q}) = \frac{H^2}{2I(\mathbf{Q})} + U(\mathbf{Q}) \leq E$$

$$\mathbf{Q} = \{\mathbf{r}_{ij} : i, j = 1, 2, \ldots, N\}$$

  - E_m is only a function of the relative configuration $\mathbf{Q}$ of an *N*-body system
- Theorem: *Stationary values of E_m are relative equilibria of the N-body problem at a fixed value of angular momentum (Smale, Arnold)*
  - Equality occurs at relative equilibrium
  - Can be used to find central configurations and determine energetic stability

Example: Point Mass 2-Body Minimum Energy Configurations

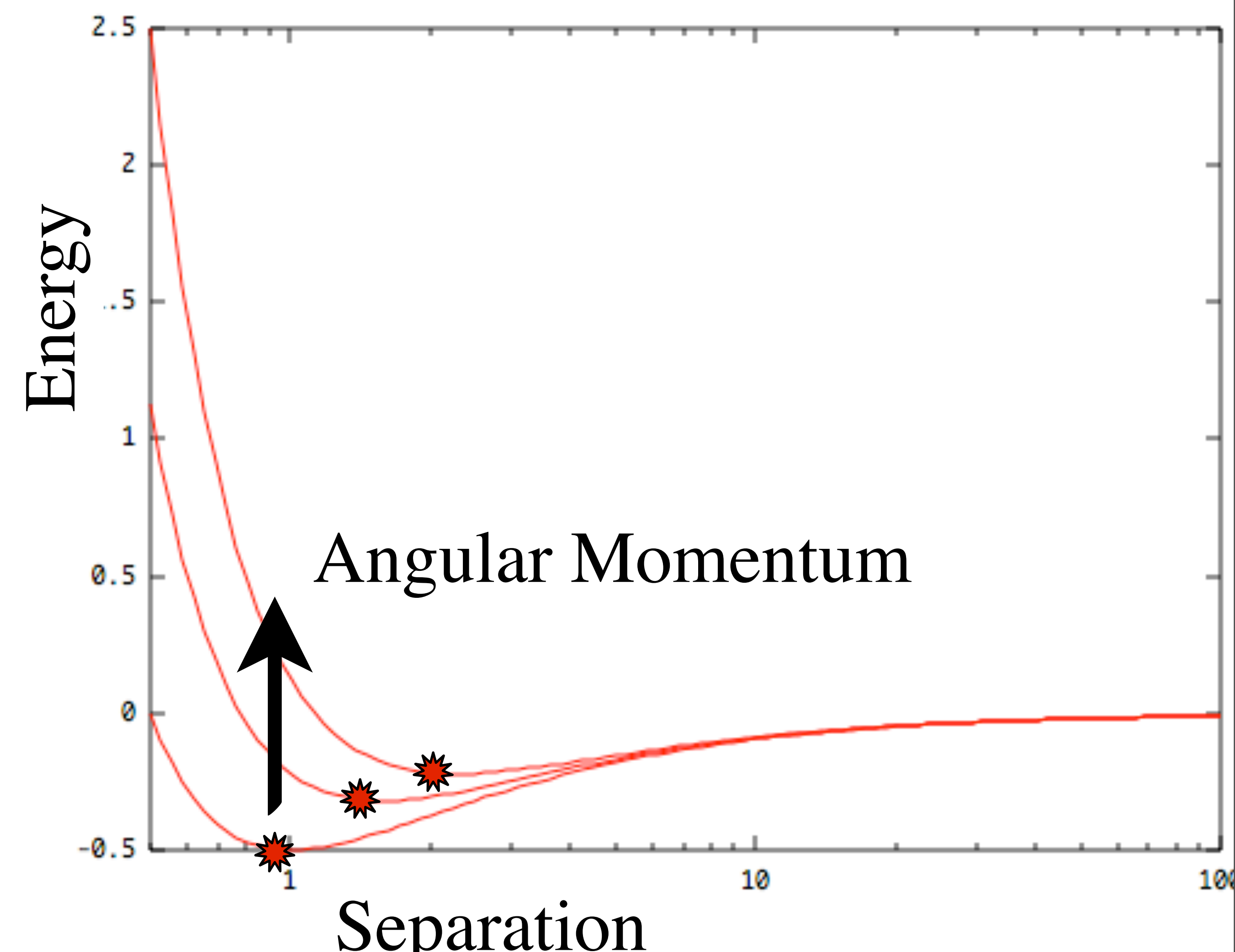
- Point Mass 2-Body Problem: Minimum is a circular orbit

$$E_m = \frac{h^2}{2d^2} - \frac{1}{d} \qquad \frac{\partial E_m}{\partial d} = -\frac{h^2}{d^3} + \frac{1}{d^2} = 0$$

$$d^* = h^2$$

$$E_m^* = -\frac{1}{2h^2}$$

$$\left.\frac{\partial^2 E_m}{\partial d^2}\right|_* = \frac{3h^2}{d^4} - \frac{2}{d^3} = \frac{1}{h^6} > 0$$



Point Mass *N*-Body Minimum Energy Configurations, $N \geq 3$

- Point Mass *3*-Body Problem:
  - Relative equilibria occur at the Lagrange and Euler Solutions
  - Euler solutions are always unstable ≠ minimum energy solutions
  - Lagrange solutions are *never* minimum energy solutions
- Point Mass *N*-Body Problem:
  - Central configurations are *never* minimum energy configurations, c.f. proof by R. Moeckel.
  - For any Point Mass $N \geq 3$ Problem, E_m can always –> -∞ while maintaining a constant level of angular momentum

*For the Point Mass N ≥ 3 Problem there are **no** non-singular minimum energy configurations*

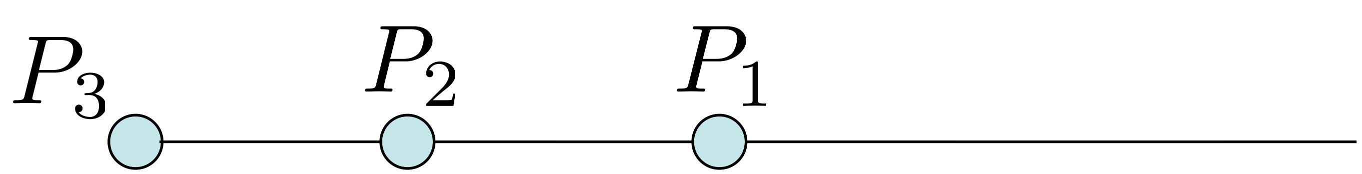
... does our original question even make sense?

Non-Definite Minimum of the Energy Function for $N \geq 3$

- Consider the minimum energy function for N=3:

$$E_m = \frac{H^2}{\frac{m}{3}\left[d_{12}^2 + d_{23}^2 + d_{31}^2\right]} - \mathcal{G}m^2\left[\frac{1}{d_{12}} + \frac{1}{d_{23}} + \frac{1}{d_{31}}\right]$$

  - Choose the distance and velocity between P_1 and (P_2 , P_3) to maintain a constant value of H.
  - Choose a zero-relative velocity between (P_2 , P_3) and let $d_{23} \to 0$, forcing $E_m \to -\infty$ while maintaining H.



  - Under energy dissipation, there is no lower limit on the system-level energy until the limits of Newtonian physics are reached.

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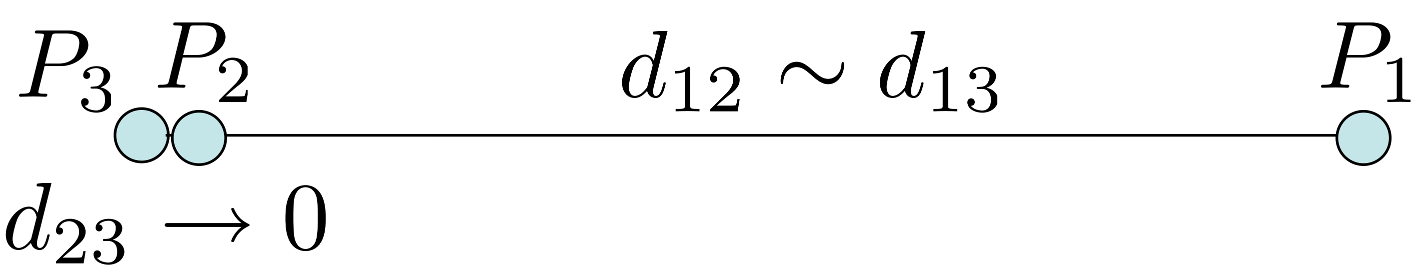

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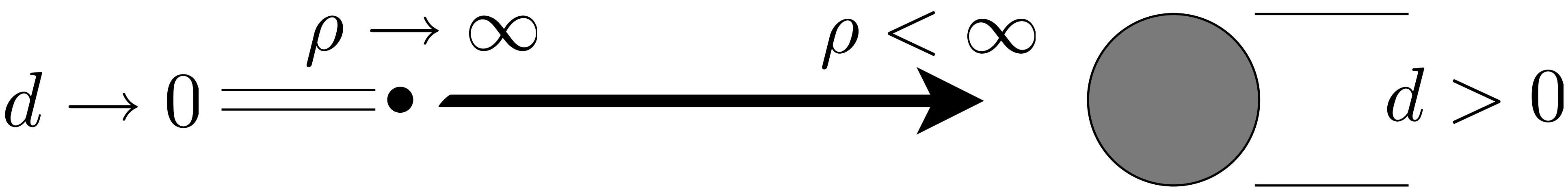
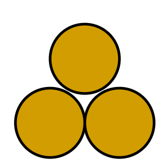
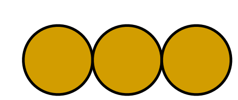
The Role of Density

- The lack of minimum energy configurations in the Point Mass *N*-body problem arises due to the infinite density of Point Masses
  - The resolution of this problem is simple and physically well motivated – *allow for finite density* – but has profound consequences:



  - Bodies with a given mass must now have finite size, when in contact we assume they exert surface normal forces and frictional forces
  - Moments of inertia, rotational angular momentum, rotational kinetic energy and mass distribution must now be tracked in *I*, *H*, *T* and *U*, even for spheres.
  - For low enough angular momentum the minimum energy configurations of an *N*-body problem has them resting on each other and spinning at a constant rate





Finite Density (Full-Body) Considerations

- Energy, angular momentum and polar moment of inertia all generalize to the case of finite density, along with the Sundman Inequality (*Scheeres, CMDA 2002*):

$$
\begin{aligned}
E &= T + U + \boxed{\frac{1}{2}\sum_{j=1}^{N} \Omega_j \cdot \mathbf{I}_j \cdot \Omega_j} \\
\mathbf{H} &= \frac{1}{2M}\sum_{j,k=1}^{N} m_j m_k \mathbf{r}_{jk} \times \dot{\mathbf{r}}_{jk} + \boxed{\sum_{j=1}^{N} \mathbf{I}_j \cdot \Omega_j} \\
I &= \frac{1}{2M}\sum_{j,k=1}^{N} m_j m_k r_{jk}^2 + \boxed{\frac{1}{2}\mathrm{Trace}\left(\sum_{j=1}^{N} \mathbf{I}_j\right)}
\end{aligned}
$$

$$H^2 \leq 2IT \qquad\qquad E_m = \frac{H^2}{2I} + U \leq E$$

Modified Sundman Inequality

- A sharper version of the Sundman Inequality can be derived for finite body distributions (*Scheeres, CMDA 2012*):
  - Define the total Inertia Dyadic of the Finite Density *N*-Body Problem:

$$\mathbf{I} \quad = \quad \sum_{i=1}^{N} \left[m_i \left(r_i^2 \mathbf{U} - \mathbf{r}_i \mathbf{r}_i \right) + A_i \cdot \mathbf{I}_i \cdot A_i^T \right]$$

  - Define the angular momentum unit vector $\hat{\mathbf{H}}$

$$I_H = \hat{\mathbf{H}} \cdot \mathbf{I} \cdot \hat{\mathbf{H}}$$

  - The modified Sundman Inequality is sharper and defines an updated Minimum Energy Function

$$H^2 \leq 2 I_H T \leq 2 I T$$

$$E_m \leq \boxed{\mathcal{E}_m = \frac{H^2}{2 I_H} + U} \leq E$$

Minimum Energy Configurations

- Theorem: *For finite density distributions, all N-body problems have minimum energy configurations*.
- Proof (*Scheeres, CMDA 2012)*:
  - Stationary values of $\mathcal{E}_m$ are relative equilibria, and include (for finite densities) resting configurations.
  - For a finite value of angular momentum *H*, the function $\mathcal{E}_m$ is compact and bounded.
  - By the Extreme Value Theorem, the minimum energy function $\mathcal{E}_m$ has a Global Minimum.
- Resolves the problem associated with minimum energy configurations of the Newtonian (Point Mass) *N*-Body Problem.

... back to the original question

- **Question:** What is the Minimum Energy configuration of a finite density *N*-Body System at a specified value of Angular Momentum?
- **Answer:** The Minimum Value of $\mathcal{E}_m$ across all stationary configurations, both *resting* and *orbital*.

$$\mathcal{E}_m(\mathbf{Q}_F) = \frac{H^2}{2I(\mathbf{Q}_F)} + U(\mathbf{Q}_F) \leq E$$

For Simple Spheres...

$$\mathbf{Q}_F = \{\mathbf{r}_{ij} | r_{ij} \geq (d_i + d_j)/2, i, j = 1, 2, \ldots, N\}$$

Relative Equilibrium

$$\delta\mathcal{E} = \frac{\partial \mathcal{E}}{\partial \mathbf{Q}} \cdot \delta\mathbf{Q} \geq 0$$

$$\forall \text{ Admissible } \delta\mathbf{Q}$$

Stability

$$\delta^2\mathcal{E} = \delta\mathbf{Q} \cdot \frac{\partial^2 \mathcal{E}}{\partial \mathbf{Q}^2} \cdot \delta\mathbf{Q} > 0$$

Minimum Energy Configurations of the *Spherical Full Body Problem*

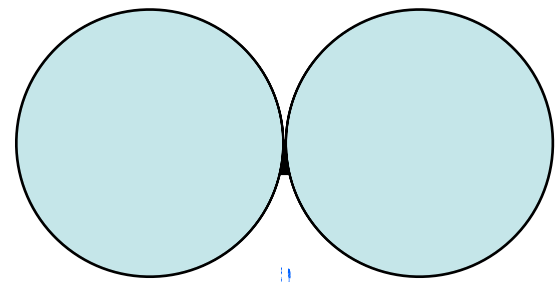
- For definiteness, consider the simplest change from point mass to finite spheres (then U is unchanged)
  - For a collection of N spheres of diameter d_i the only change in $\mathcal{E}_m$ is to I_H

$$I_H = \frac{1}{10}\sum_{i=1}^{N} m_i d_i^2 + \sum_{i=1}^{N} m_i r_i^2$$

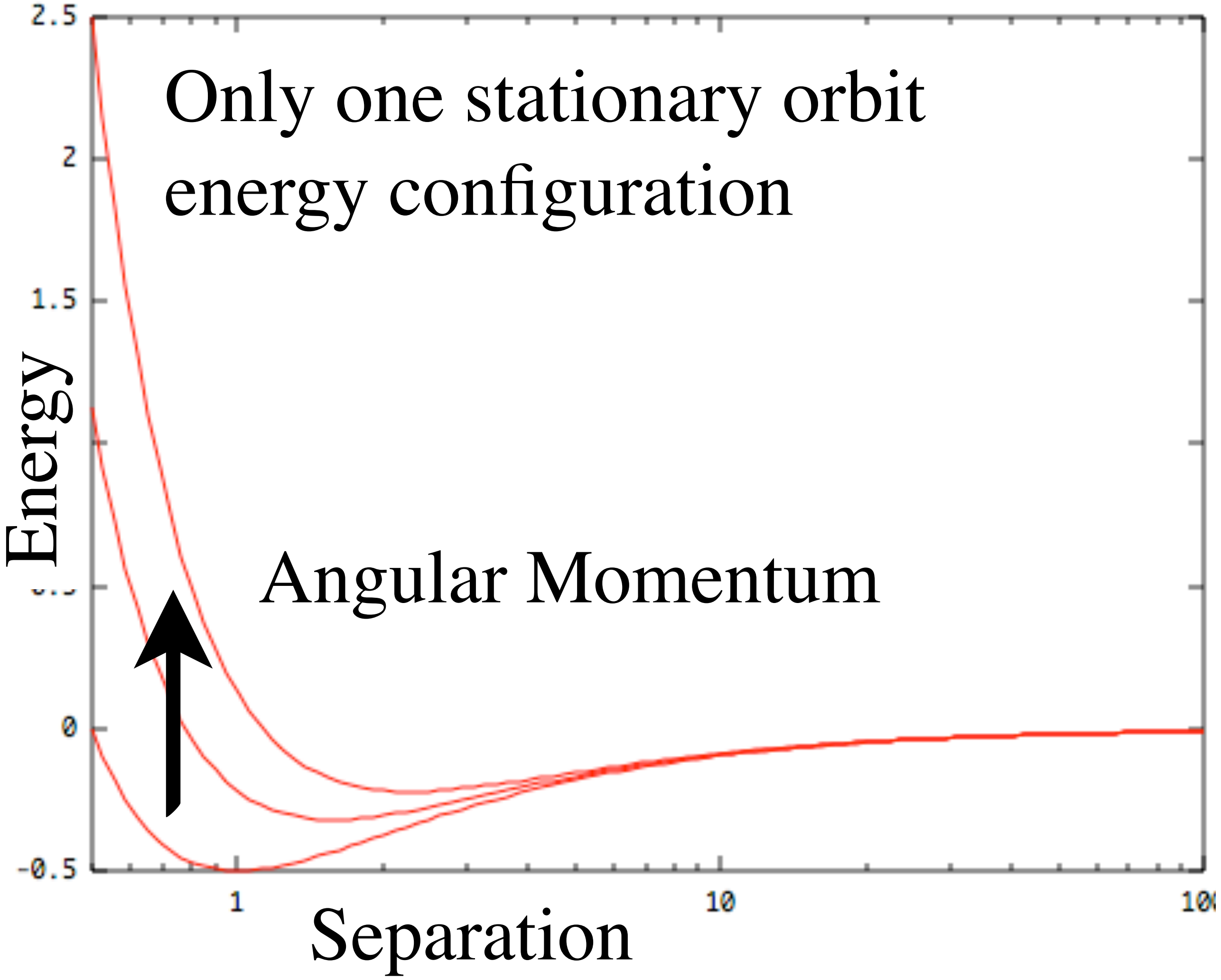
- But this dramatically changes the structure of the minimum energy configurations... take the 2-body problem for example with equal size spheres, normalized to unity radius

$$E_m = \frac{h^2}{2d^2} - \frac{1}{d} \qquad \textit{versus} \qquad \mathcal{E}_m = \frac{h^2}{2(0.4 + d^2)} - \frac{1}{d}$$

2-Body Problem

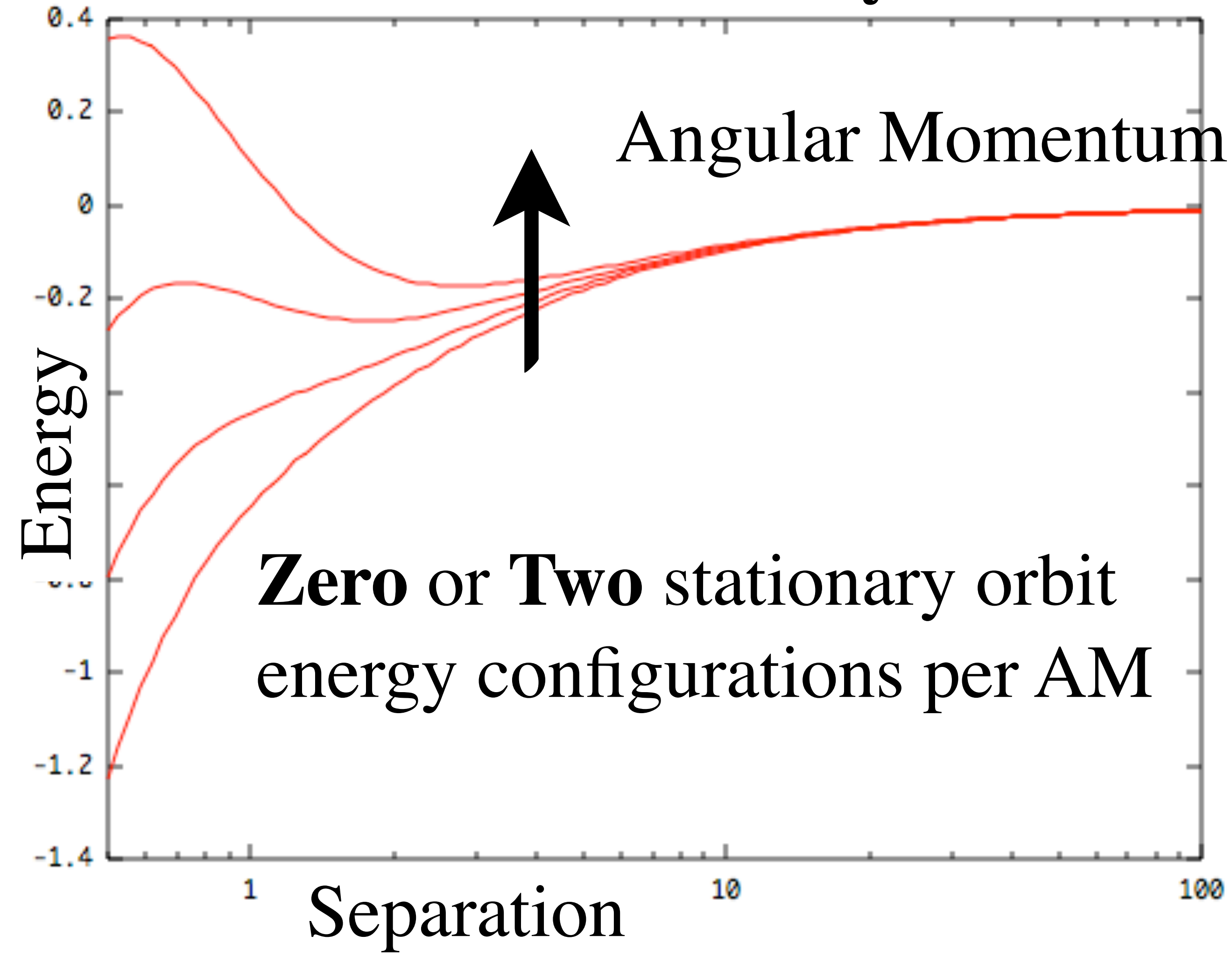


Point Mass Case



$$E_m = \frac{h^2}{2d^2} - \frac{1}{d}$$

Finite Density Case



$$\mathcal{E}_m = \frac{h^2}{2(0.4 + d^2)} - \frac{1}{d}$$

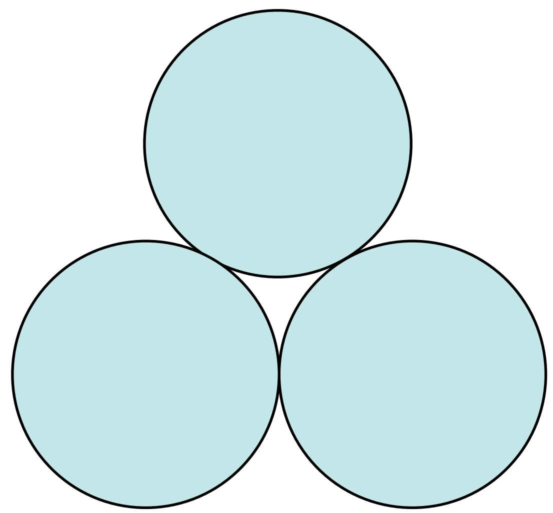
Reconfiguration and Fission

- As a system's AM is increased, there are two possible types of transitions between minimum energy states:
  - Reconfigurations, dynamically change the resting locations
  - Fissions, resting configurations split and enter orbit about each other

Reconfiguration: Occurs once the relative resting configuration becomes unstable.
For the 3BP occurs at a rotation rate beyond the Lagrange solution

Multiple resting configurations can exist at one angular momentum.
Resting and orbital stable configurations can exist at one angular momentum.

Reconfiguration and Fission



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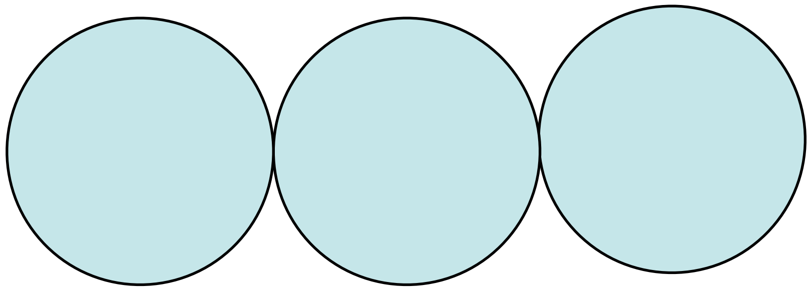

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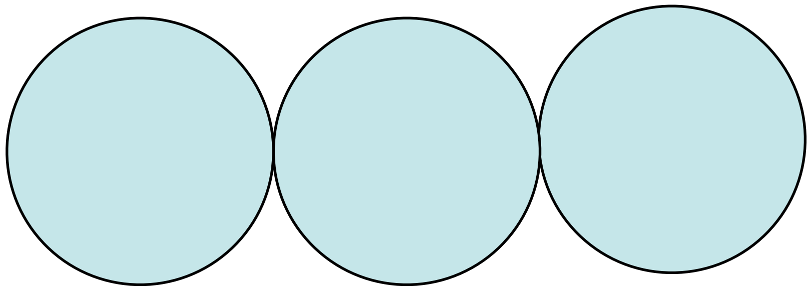

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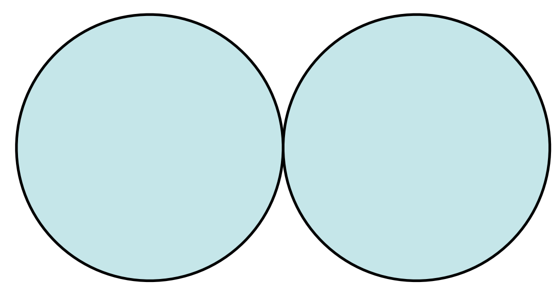
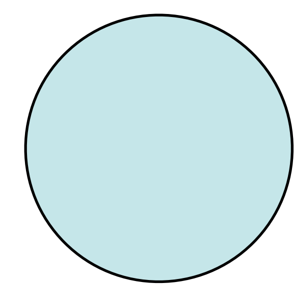
Reconfiguration and Fission





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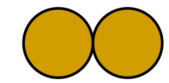
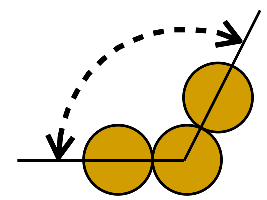
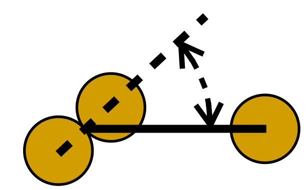
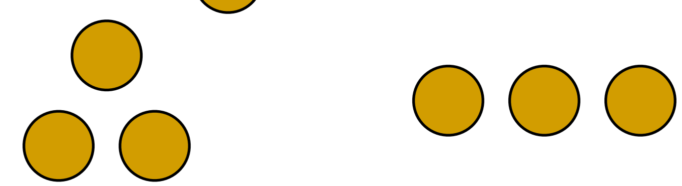
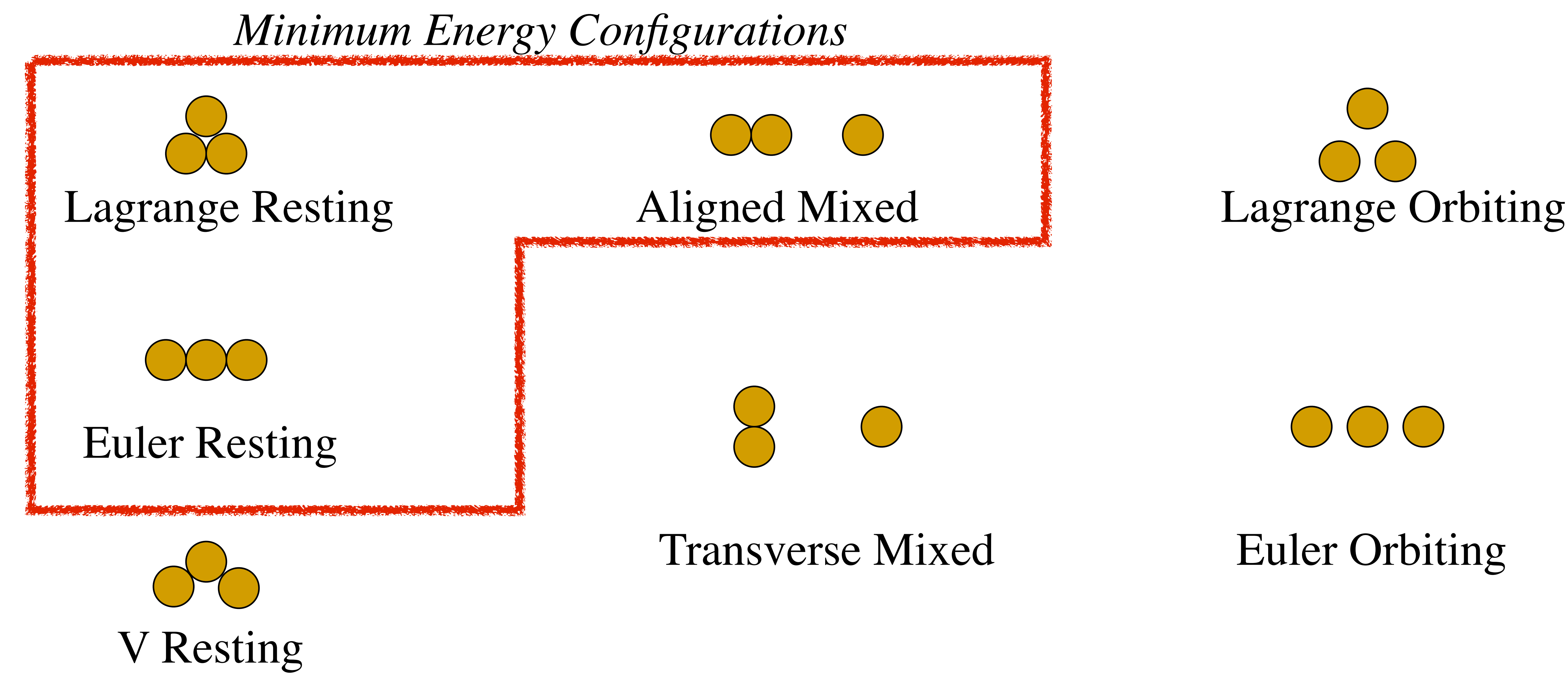
Resting and orbital stable configurations can exist at one angular momentum.

Internal Degrees of Freedom for Spherical Grains

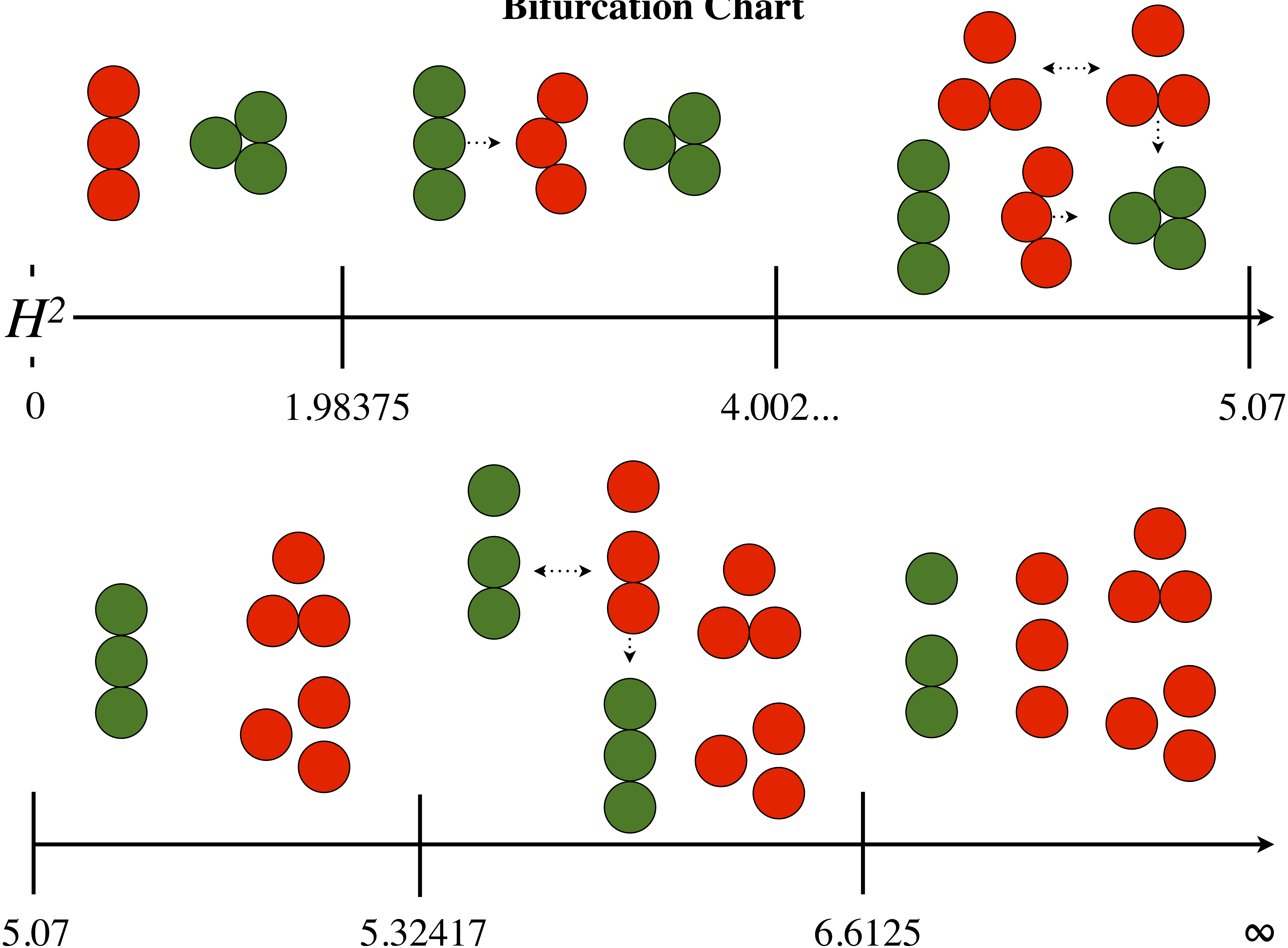
- 2-Body Results:
  - Contact case has 0 degrees of freedom 
  - Orbit case has 1 degree of freedom 
- 3-Body Results
  - Contact case has 1 degree of freedom 
  - Contact + Orbit case has 2 degrees of freedom 
  - Know all of the orbit configurations 
- 4-Body Results
  - Contact case has 2 degrees of freedom, multiple topologies
  - Many more possible Orbit + Contact configurations
  - 3-dimensional configurations
  - Don't even know precisely how many orbit configurations exist... but they are all energetically unstable (Moeckel)!

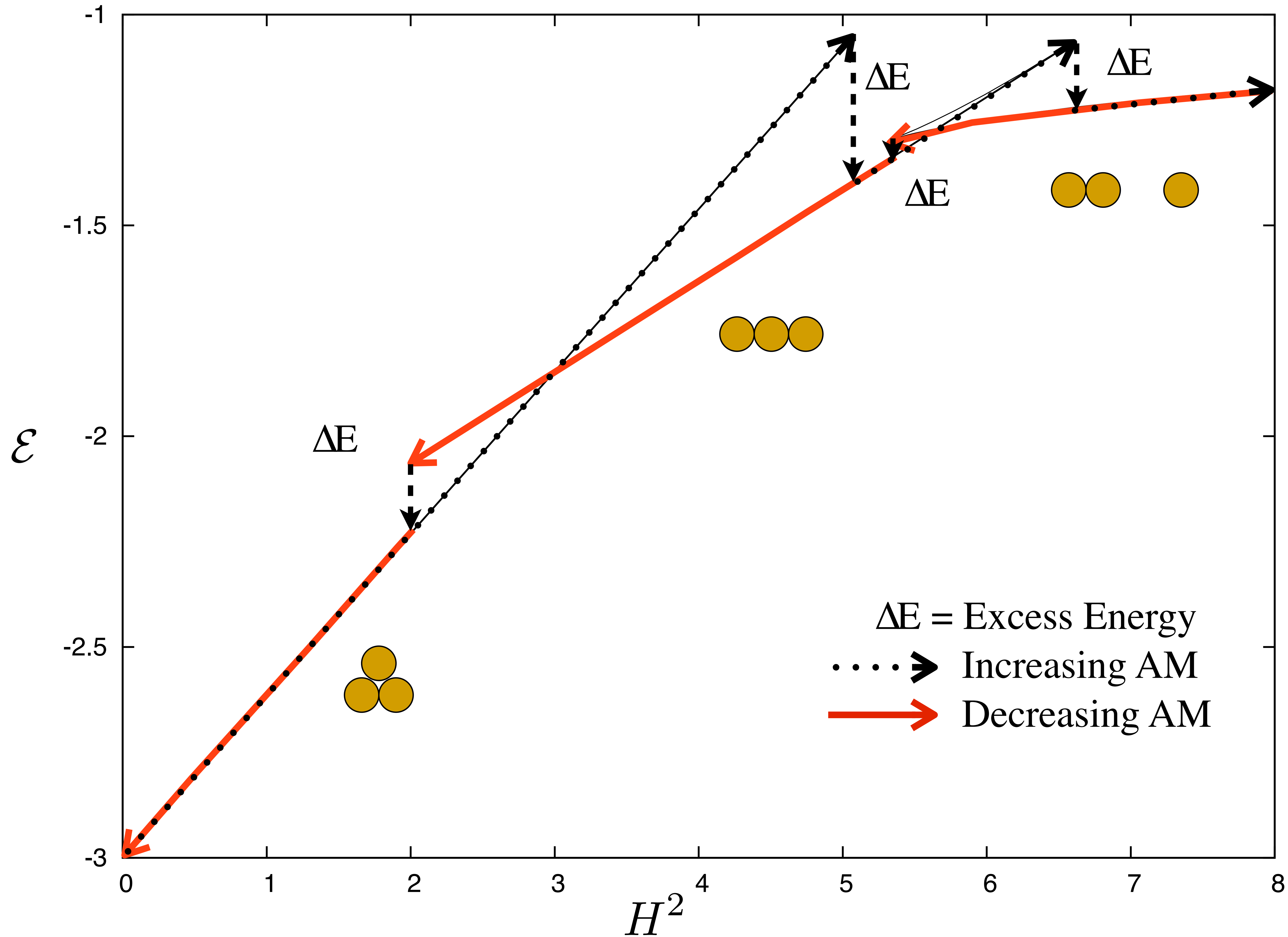
All minimum energy states can be uniquely identified in the finite density 3 Body Problem

Static & Variable Resting Configurations | Mixed Configurations | Orbiting Configurations



Bifurcation Chart

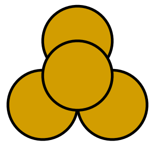


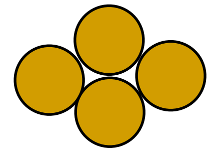


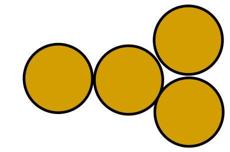
-1
-1.5
-2
-2.5
-3
0
1
2
3
4
5
6
7
8
$\mathcal{E}$
H^2
ΔE
ΔE
ΔE
ΔE
ΔE = Excess Energy
Increasing AM
Decreasing AM

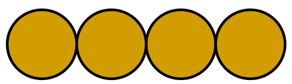
Finite Density 4 Body Problem

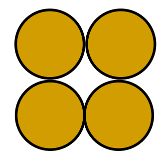
Static Resting Equilibrium Configurations

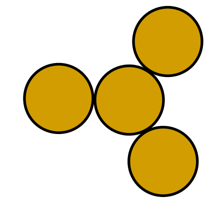
0 

1 

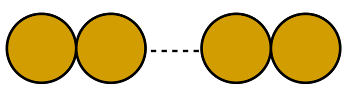
2 

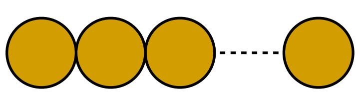
3 

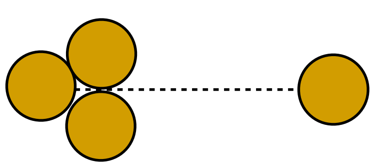
4 

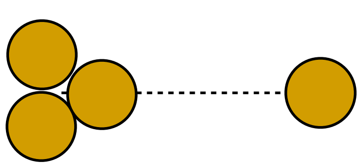
5 

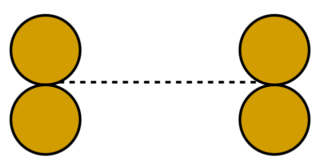
Mixed Equilibrium Configurations

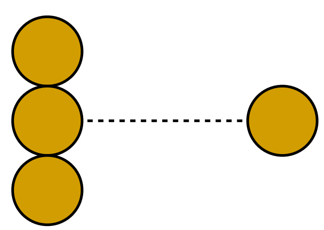
A 

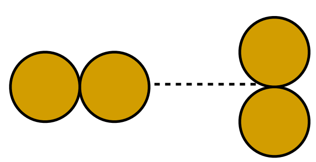
B 

C 

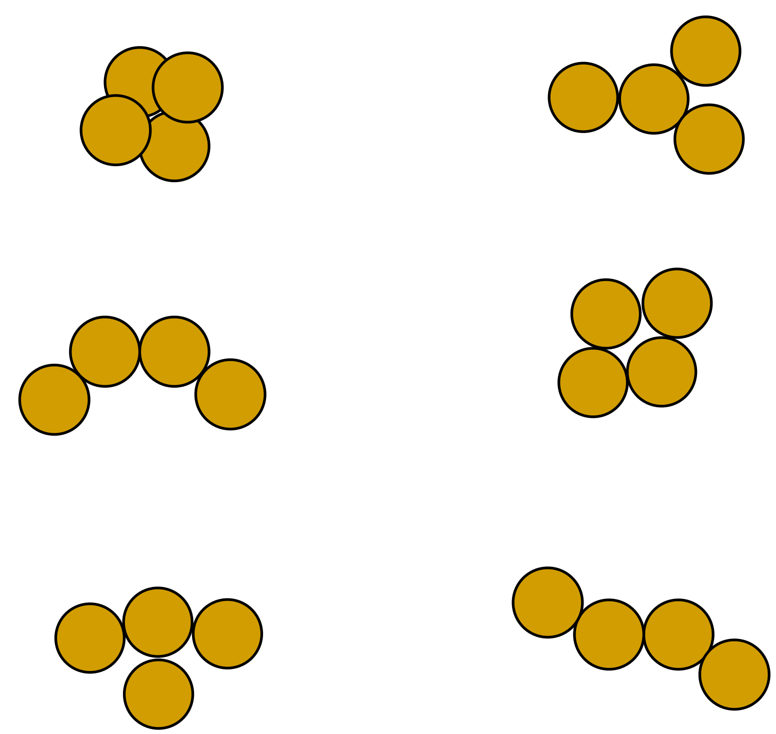
D 

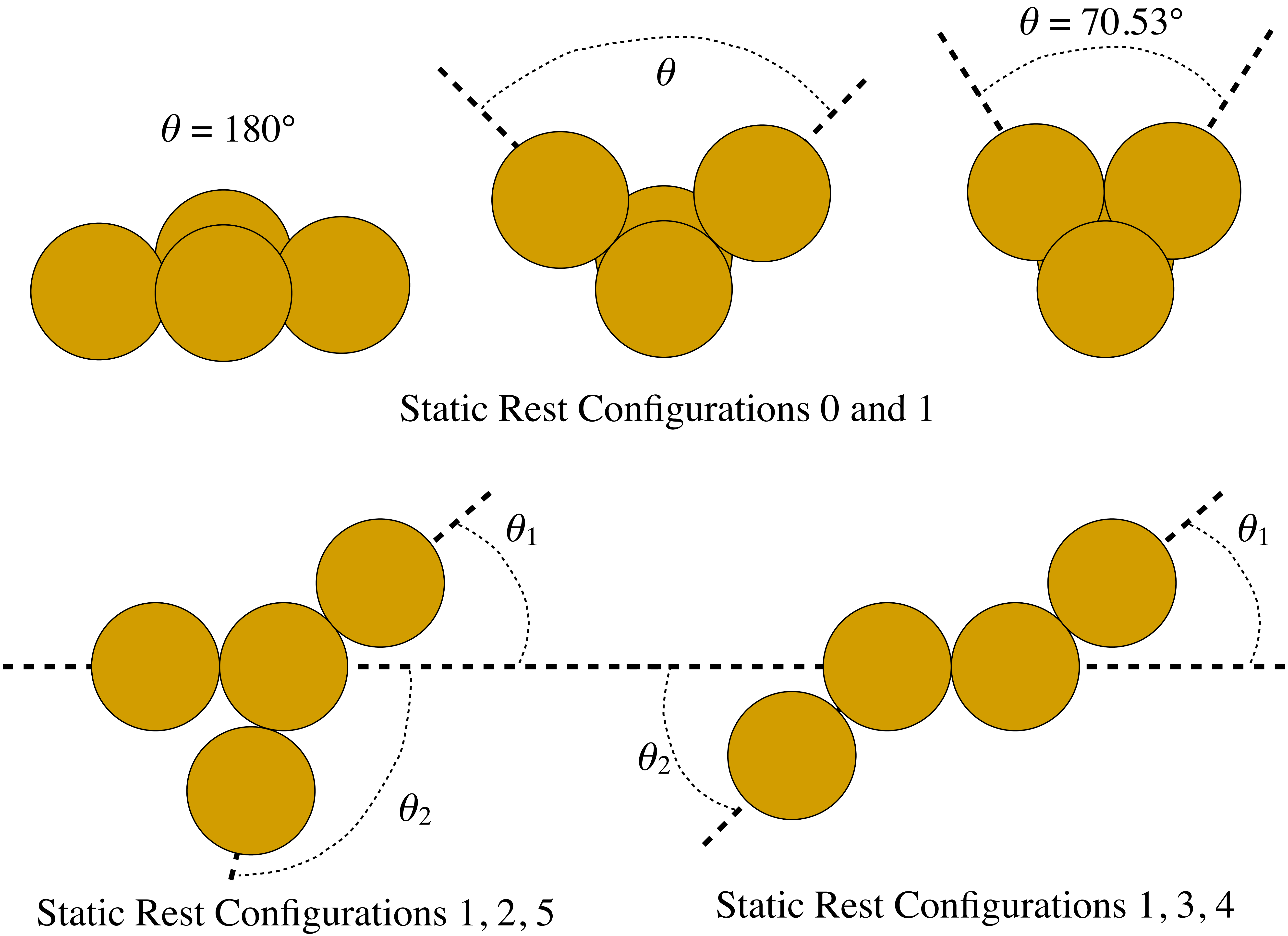
E 

F 

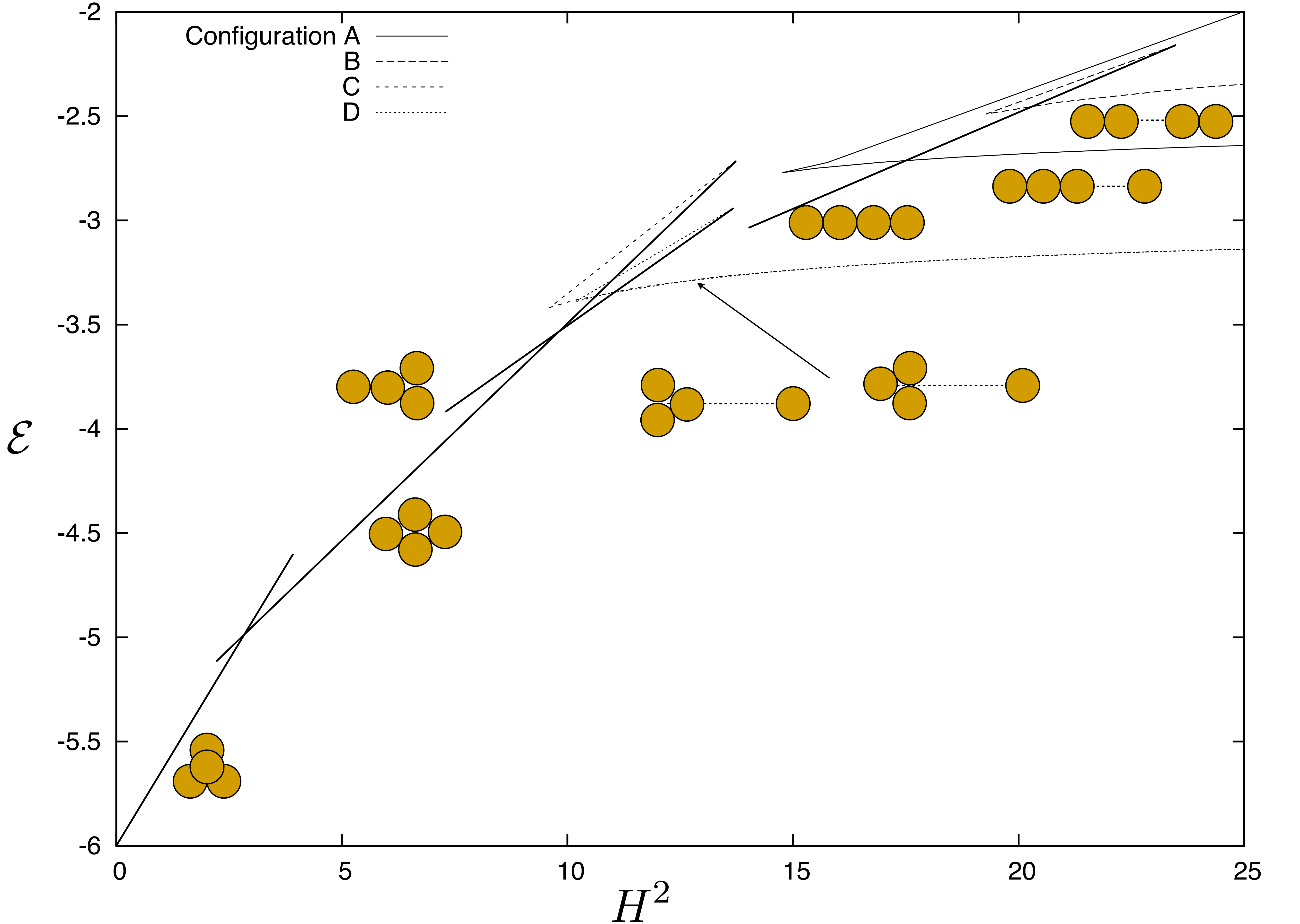
G 

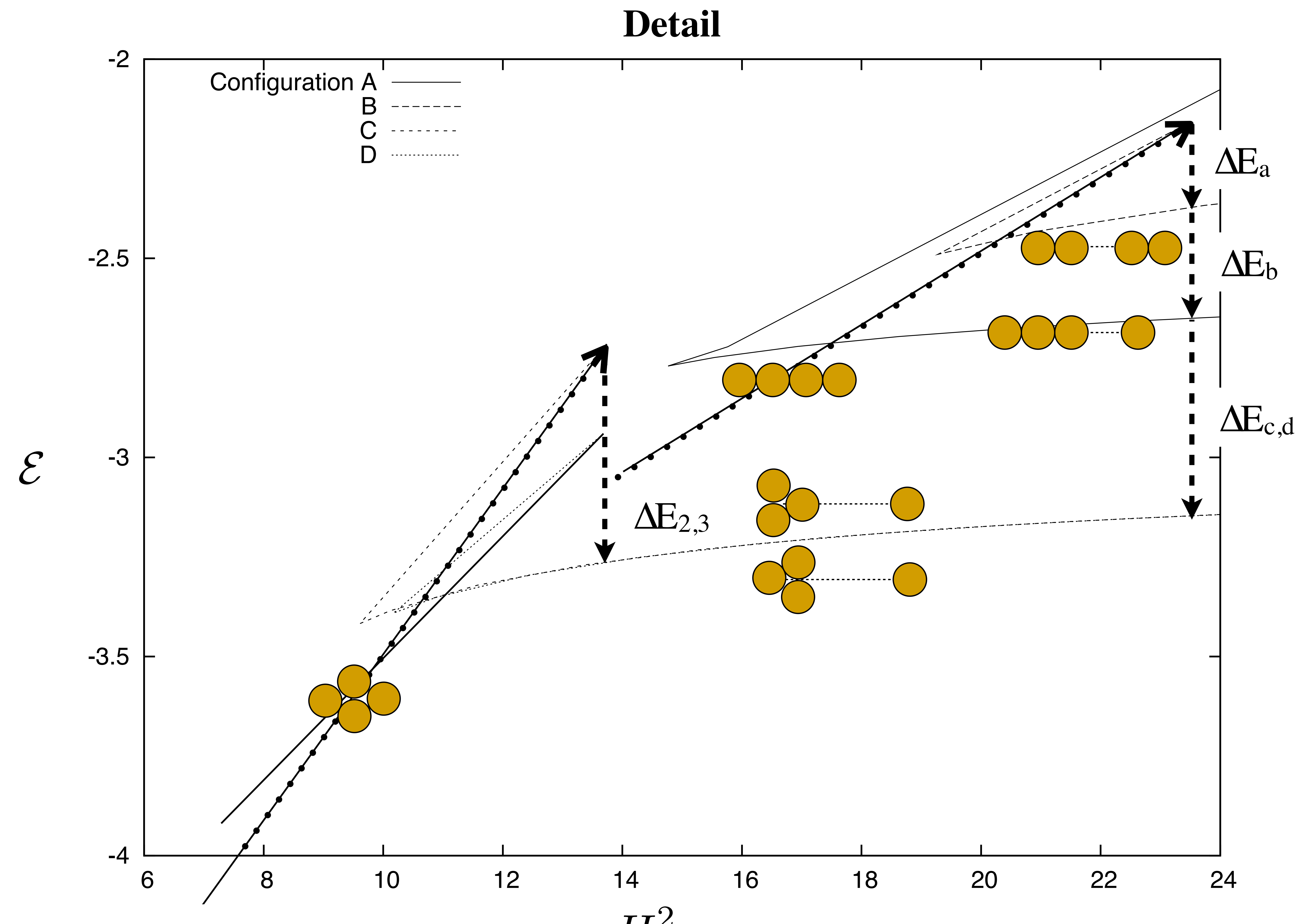
Variable Resting Equilibrium Configurations





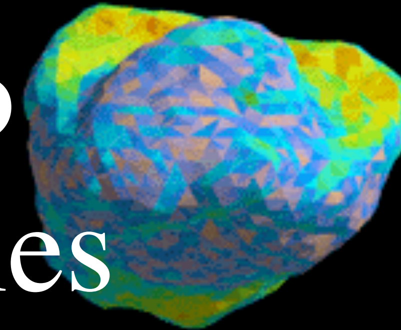
$\theta = 180°$
θ
$\theta = 70.53°$
Static Rest Configurations 0 and 1
θ_1
θ_2
θ_1
θ_2
Static Rest Configurations 1, 2, 5
Static Rest Configurations 1, 3, 4

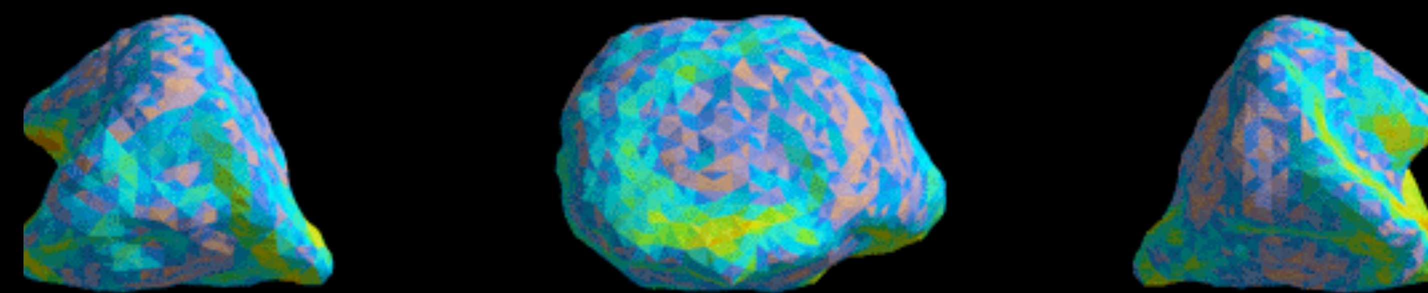

Configuration A
B
C
D
$\mathcal{E}$
H^2
-2
-2.5
-3
-3.5
-4
-4.5
-5
-5.5
-6
0
5
10
15
20
25

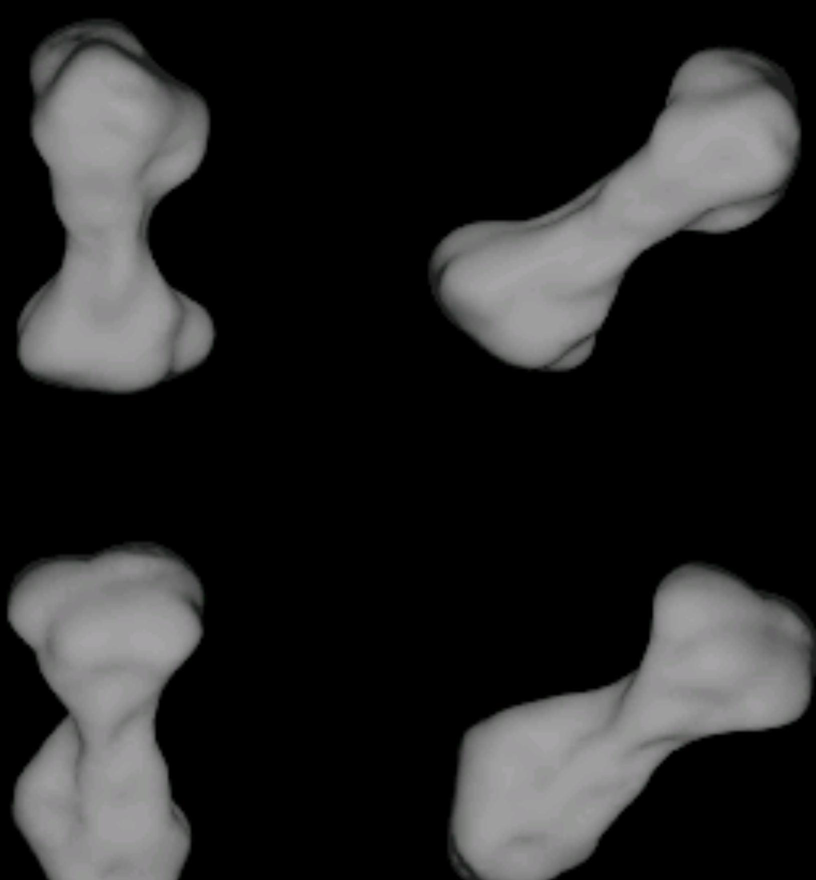


Tuesday, January 15, 2013

Generalizations to Non-Spherical Bodies

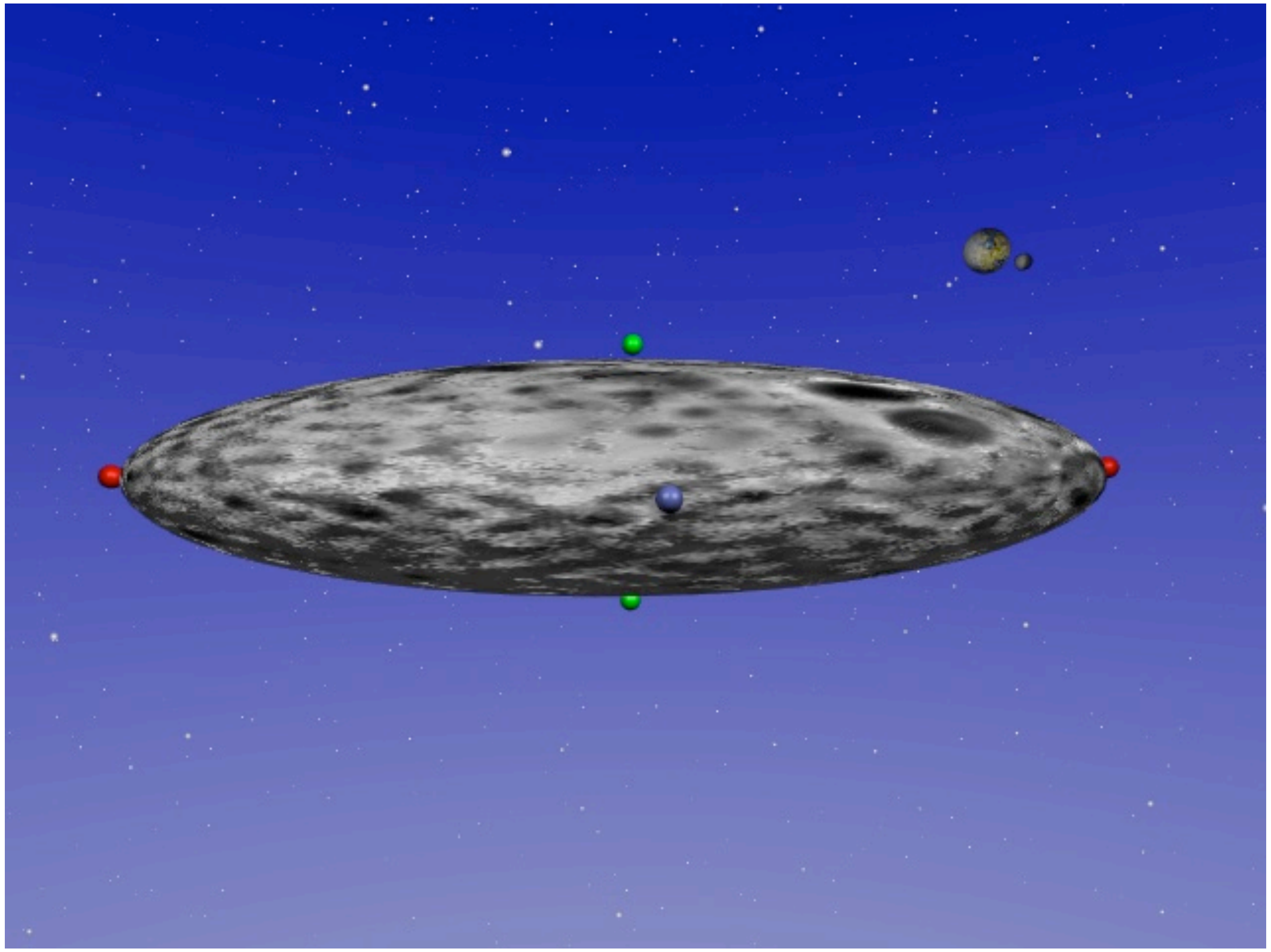






Migration of Surface Material

- If an asteroid's rotation rate changes the minimum energy resting points of particles will change (Guibout & Scheeres, *Celestial Mechanics 2003*)



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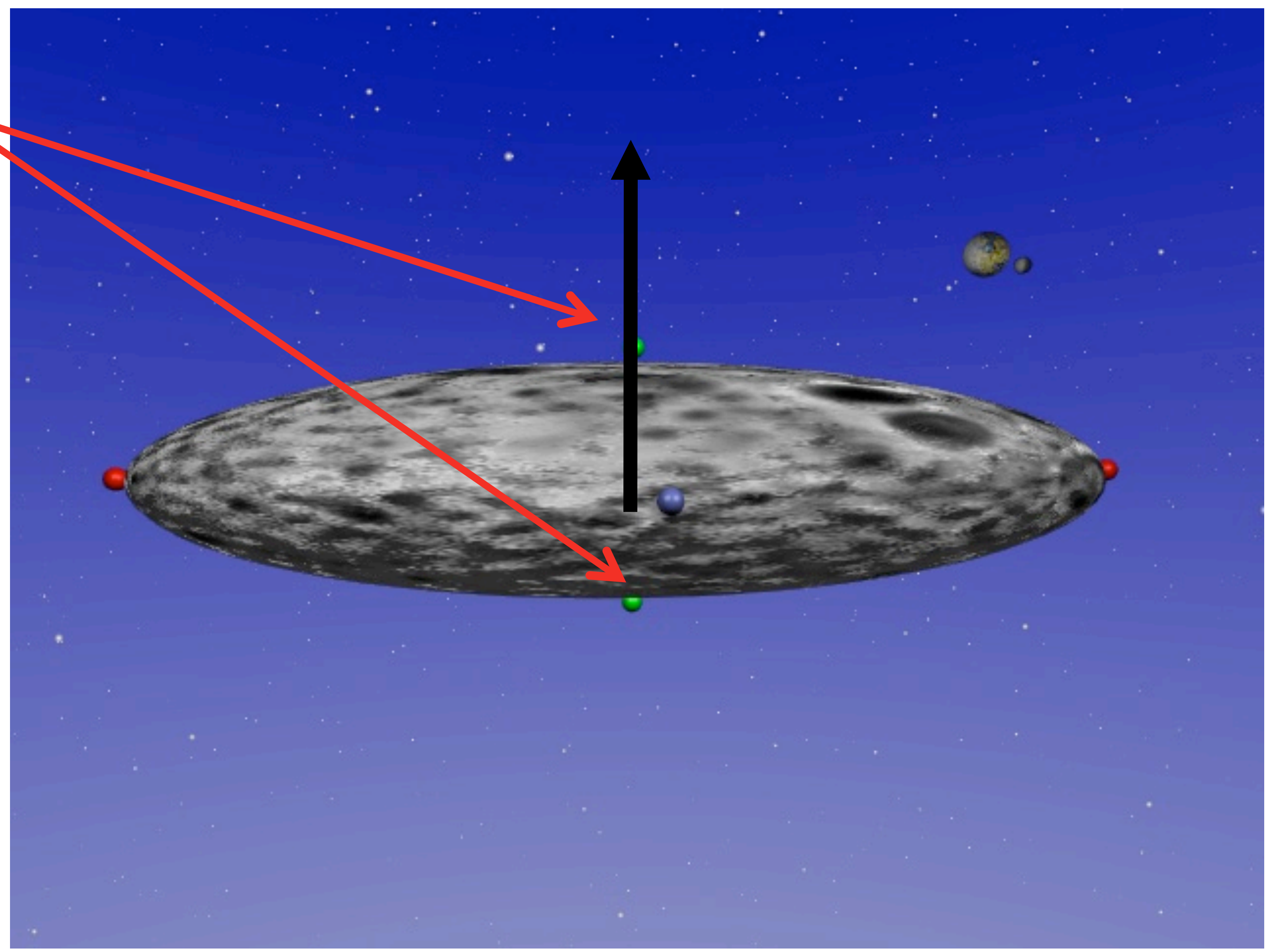
Slow rotation



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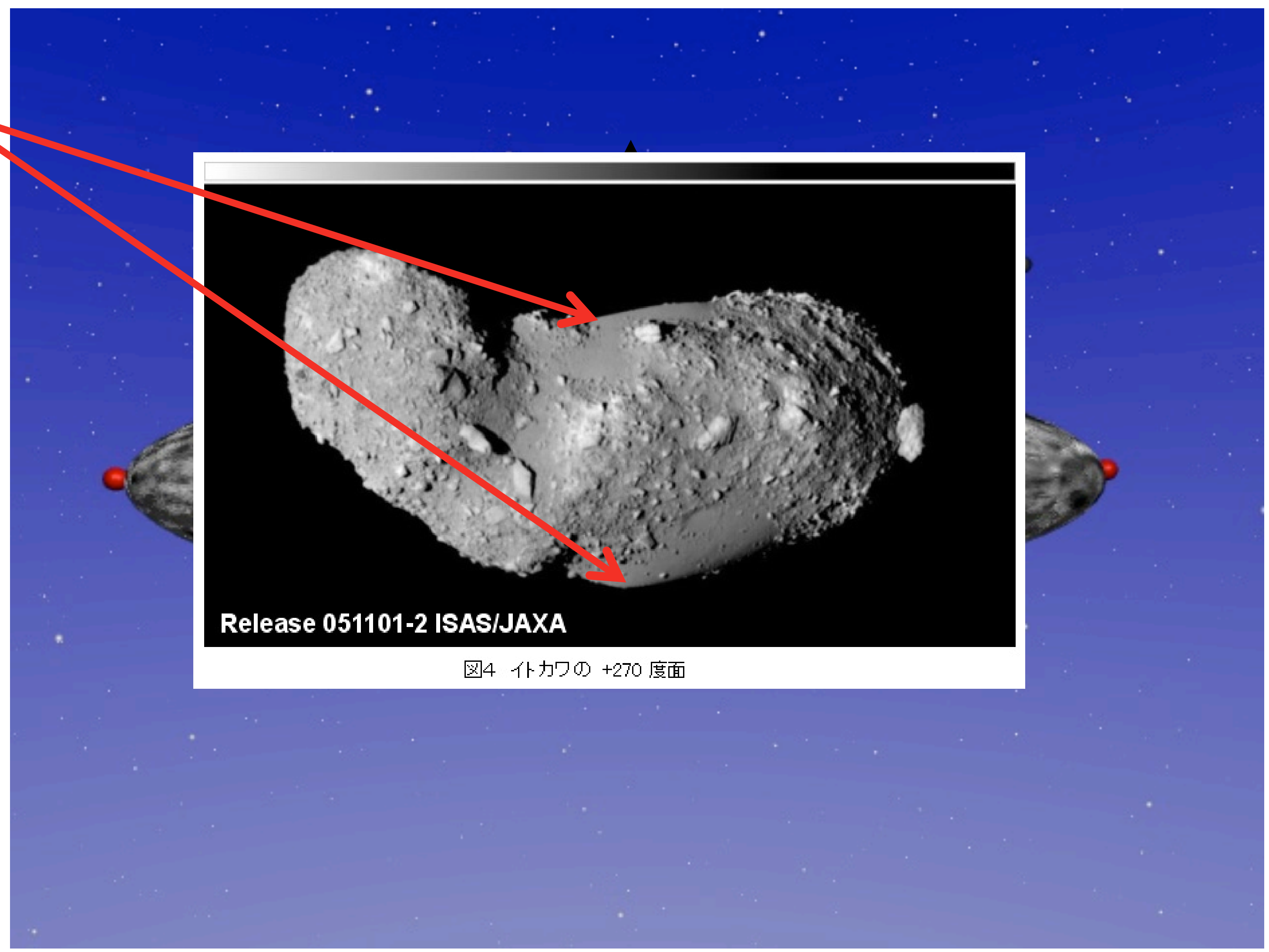

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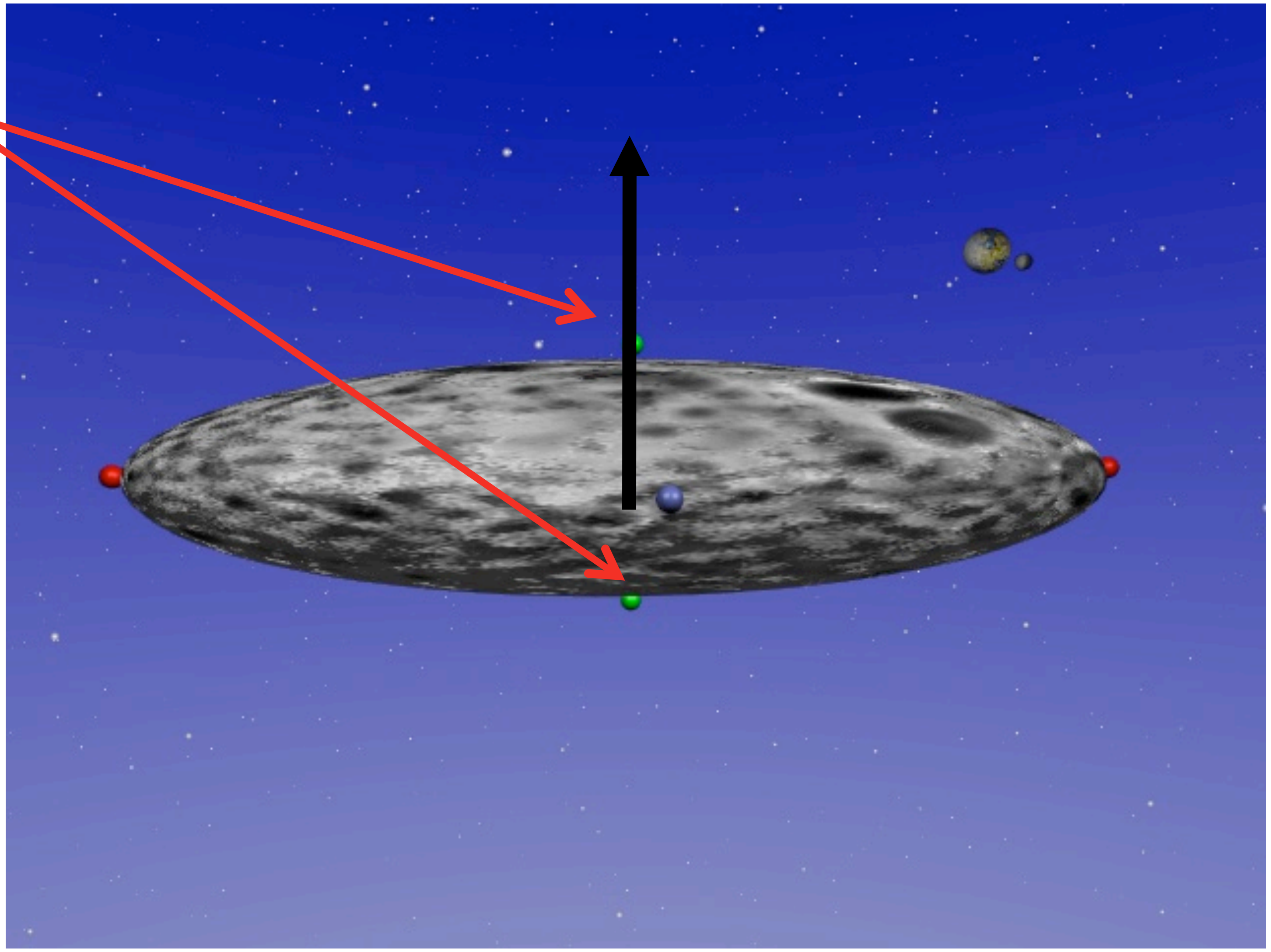

D.J. Scheeres, A. Richard Seebass Chair, University of Colorado at Boulder

Migration of Surface Material

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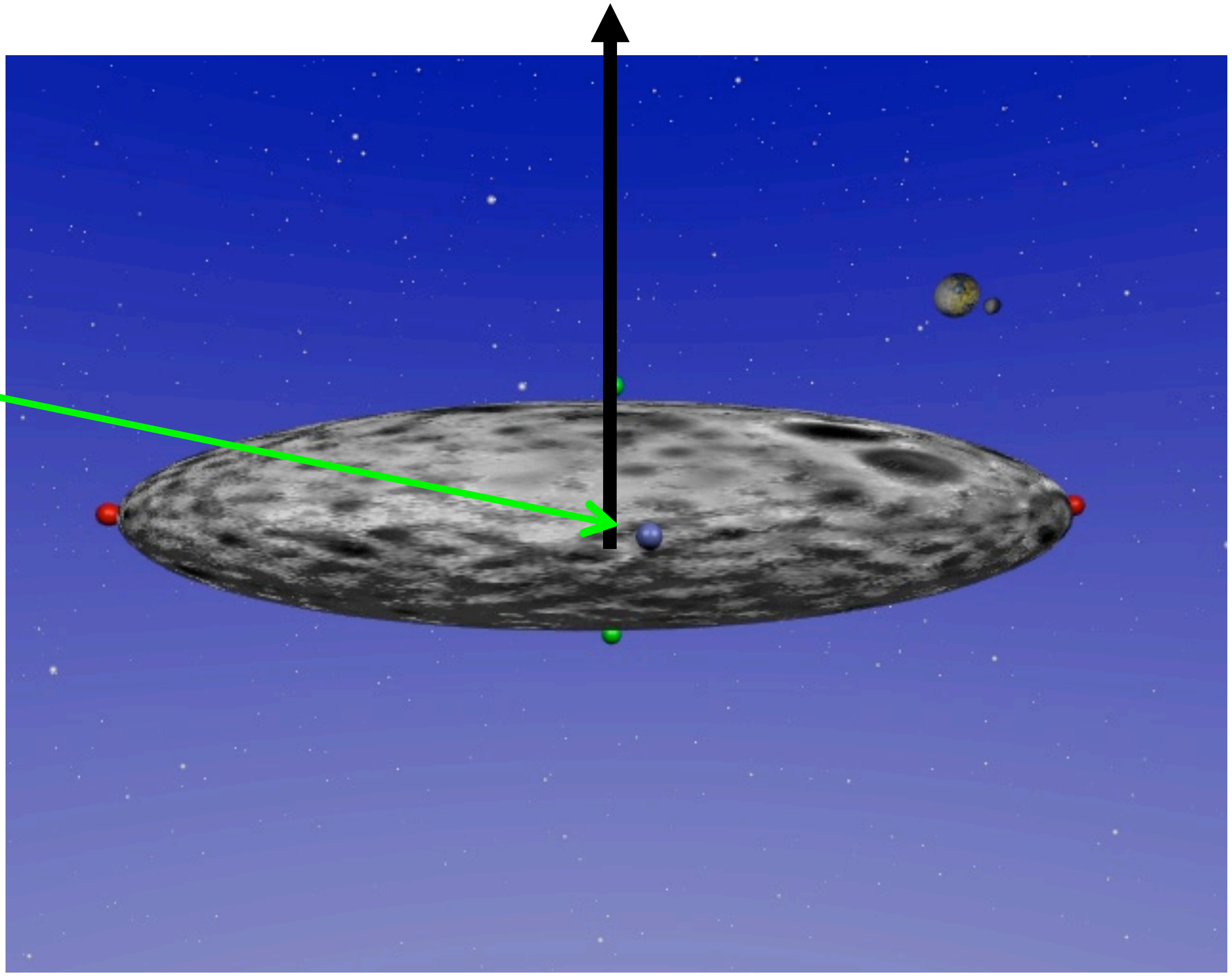
Intermediate rotation



Migration of Surface Material

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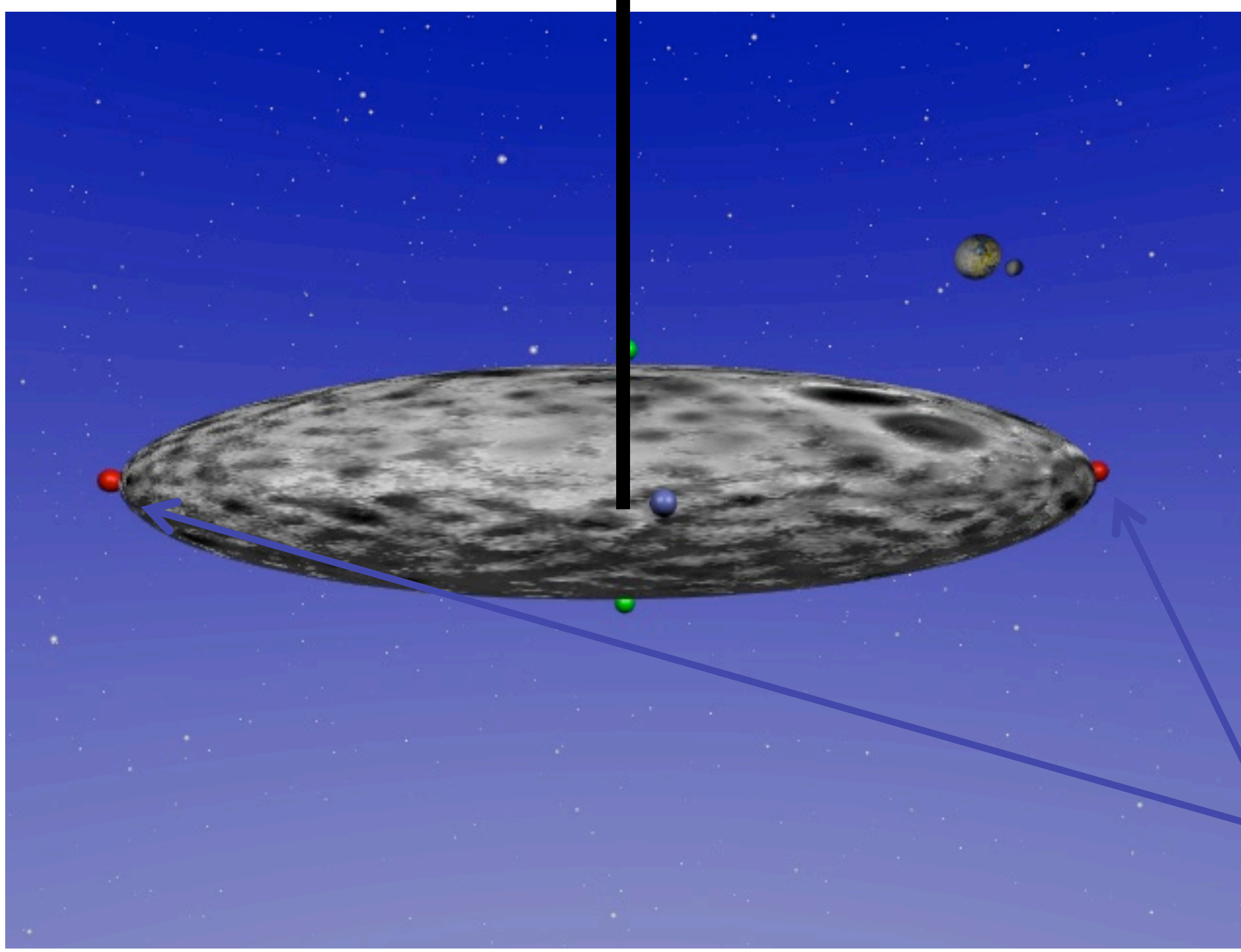

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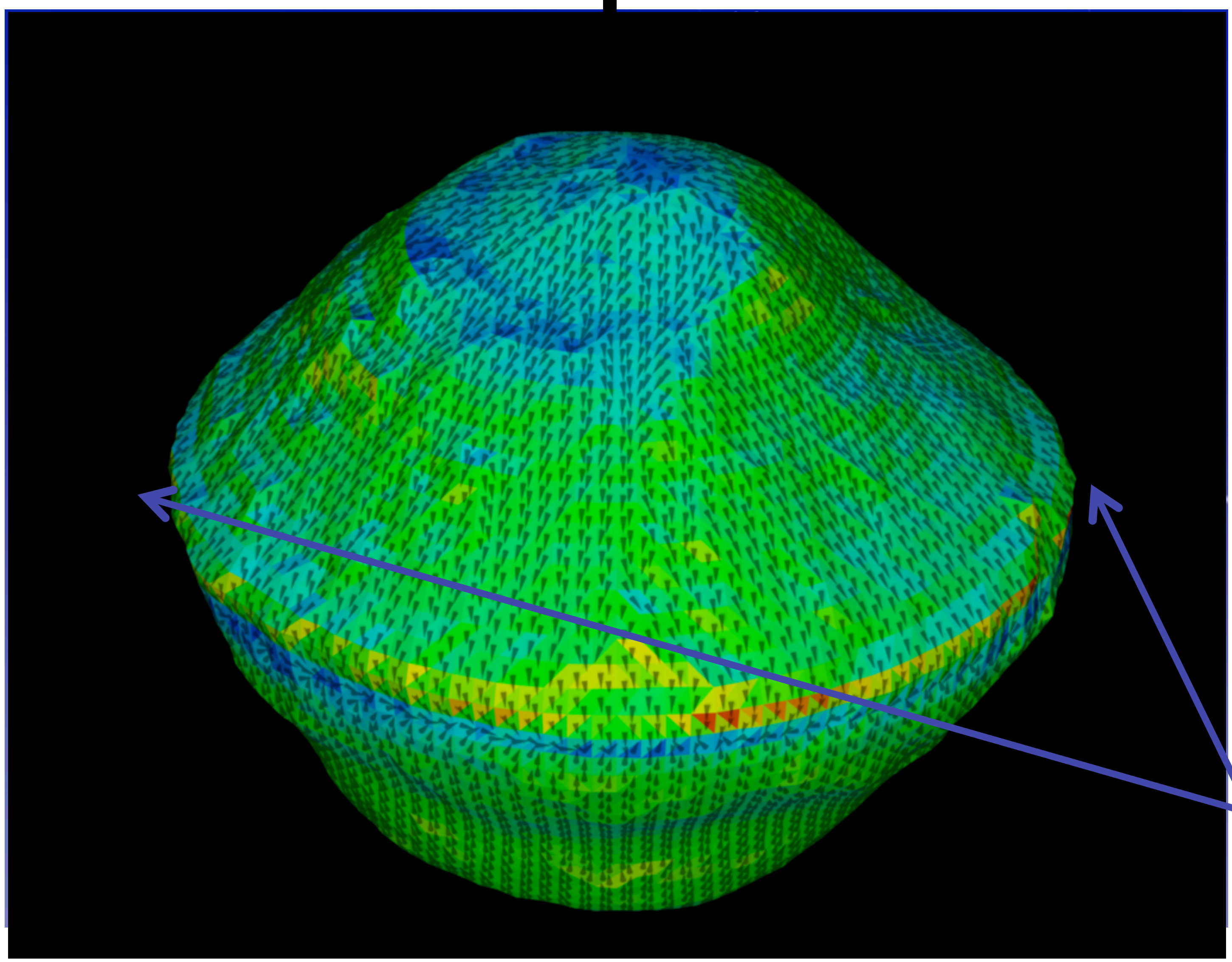
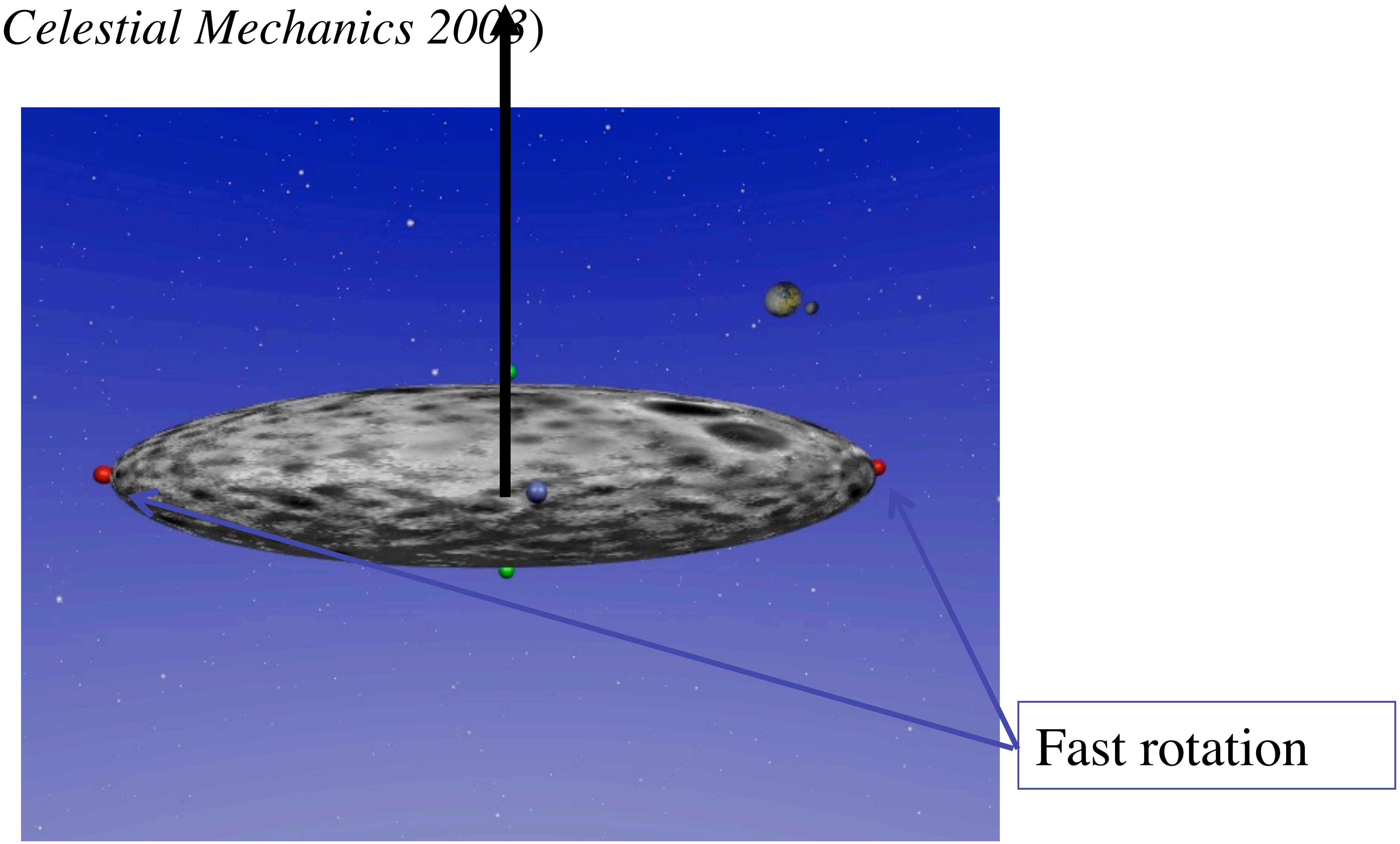


Fast rotation

Migration of Surface Material

- If an asteroid's rotation rate changes the minimum energy resting points of particles will change (Guibout & Scheeres, *Celestial Mechanics 2003*)



Migration of Surface Material

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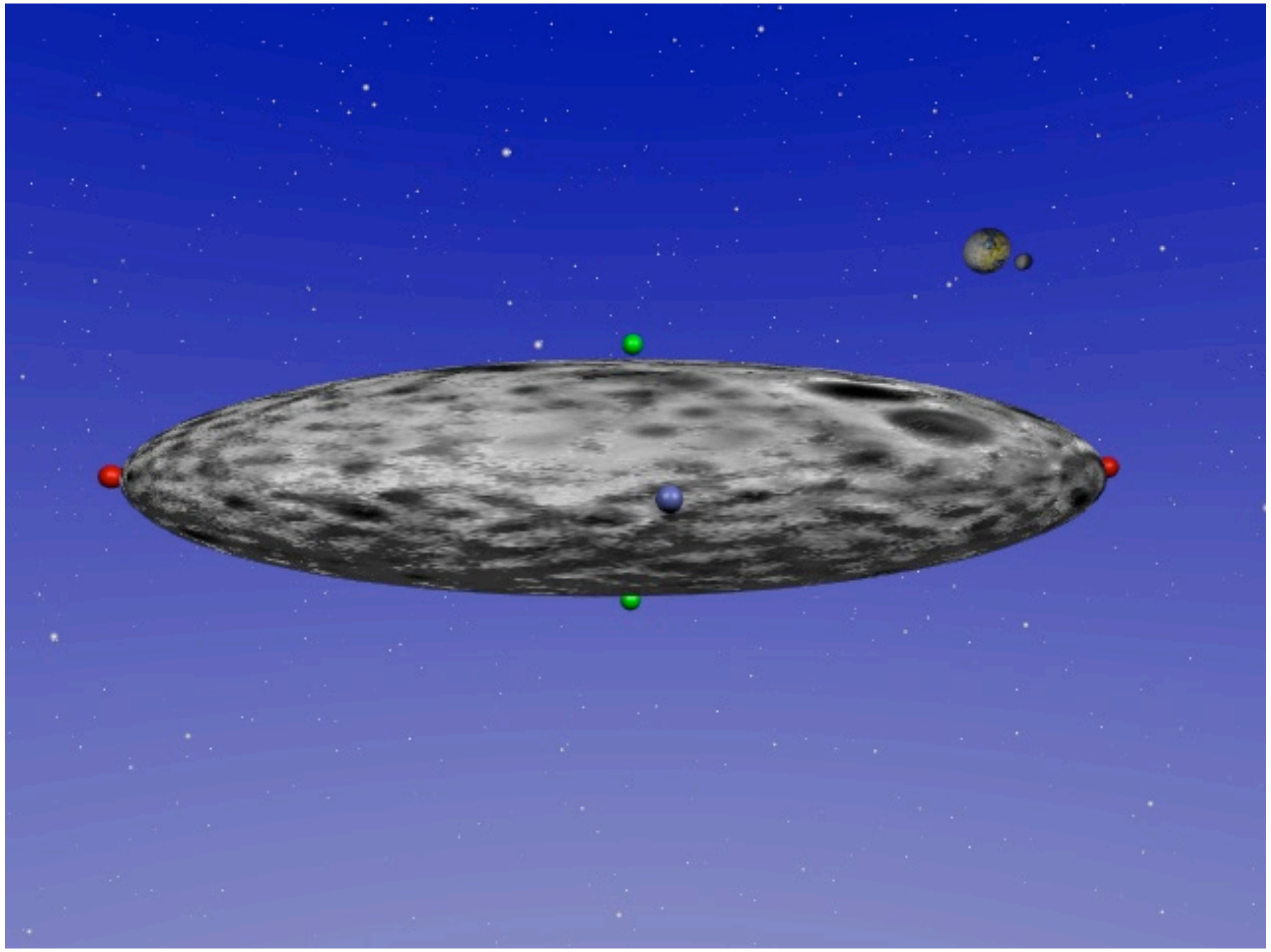


For an ellipsoid, transition rotation rates are related to the Jacobi Sequence

From Miyamoto et al.,
Science, 2007

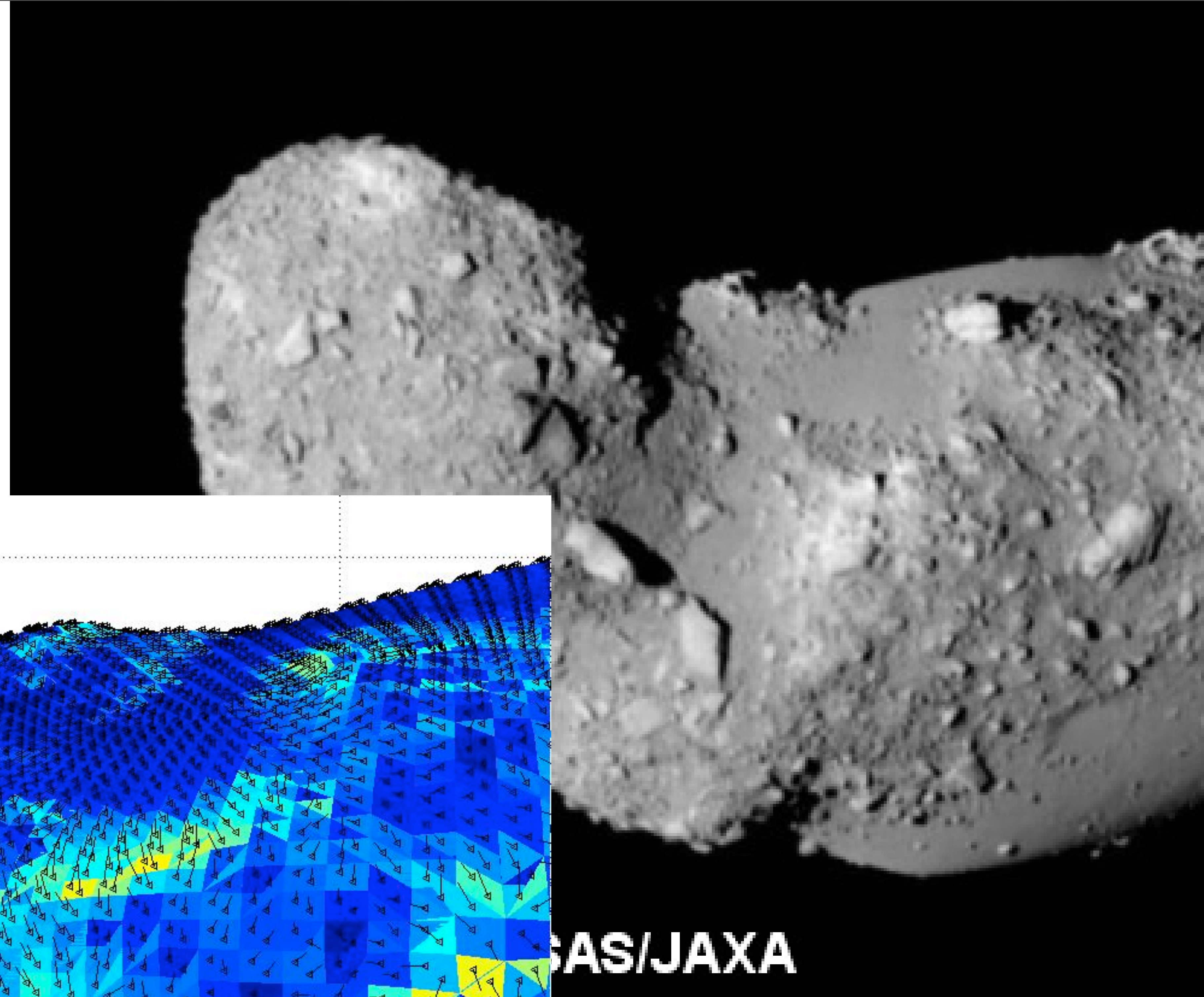
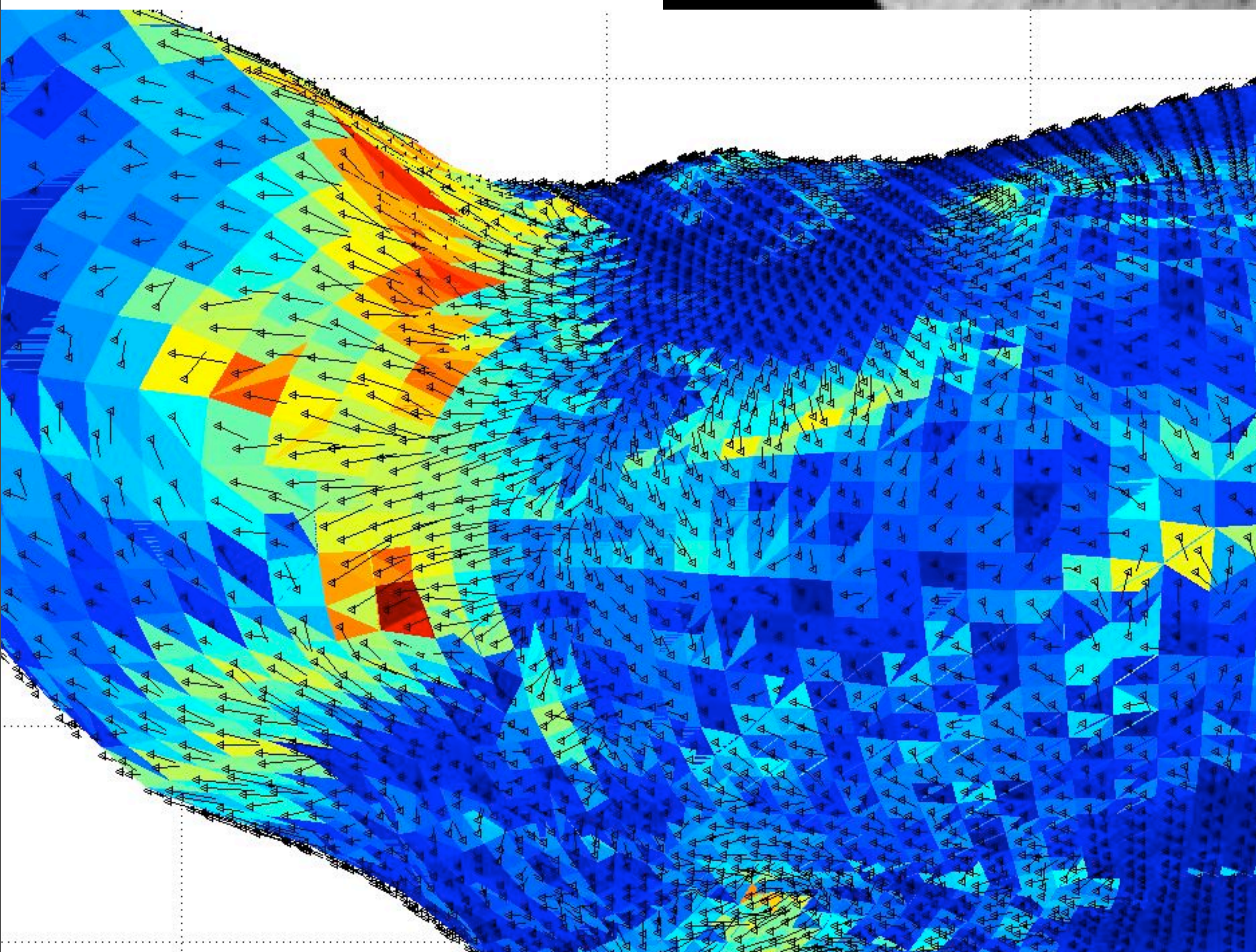
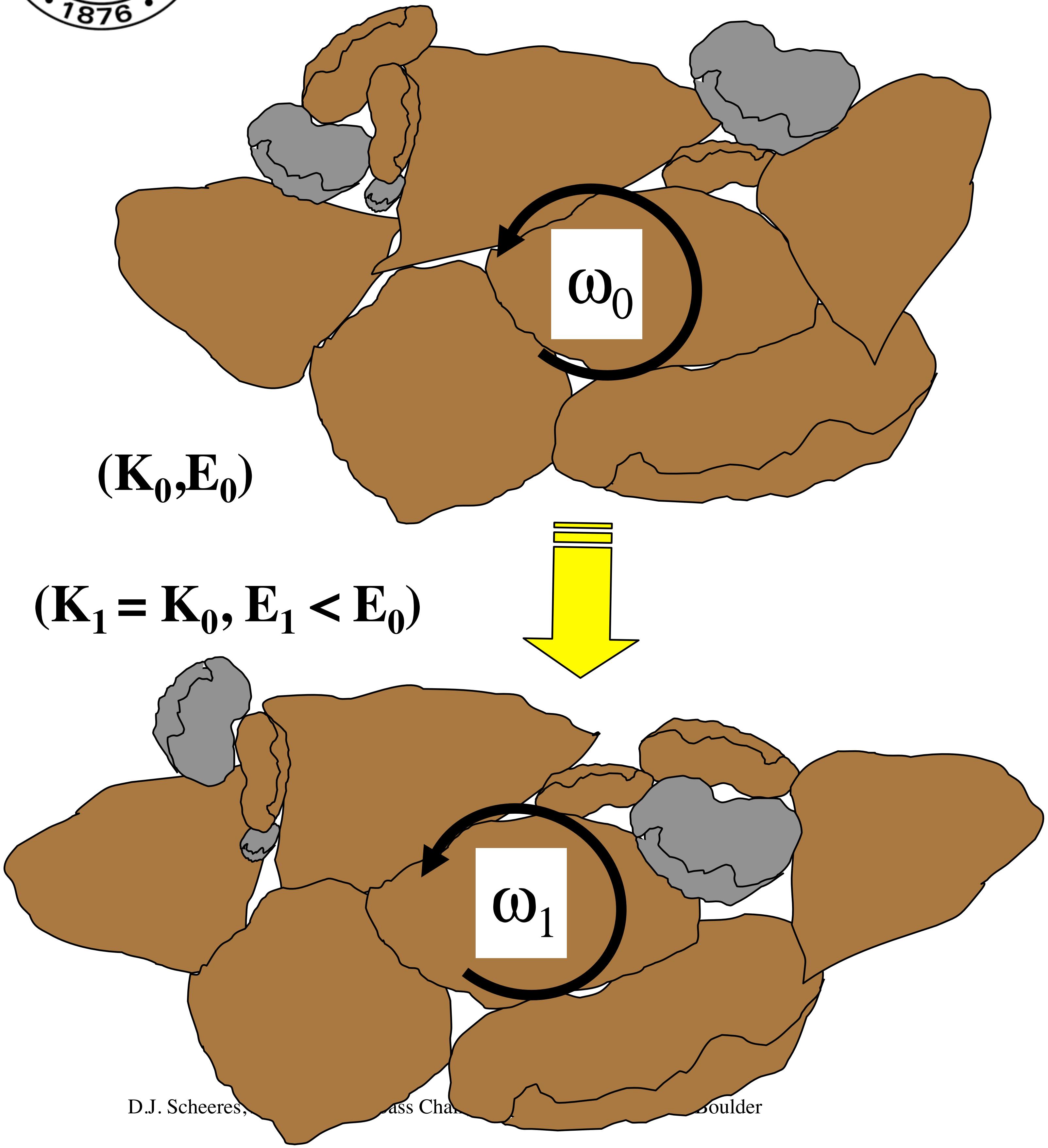


図4 イトカワの +270 度面



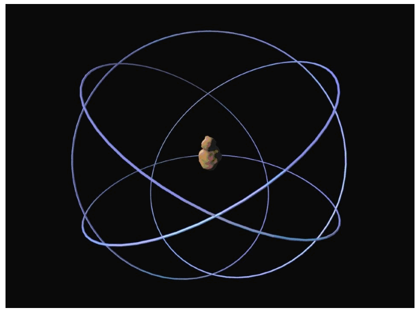
Minimum Energy Configurations



As spin rate increases or decreases, an aggregate can be placed into a non minimum energy state.

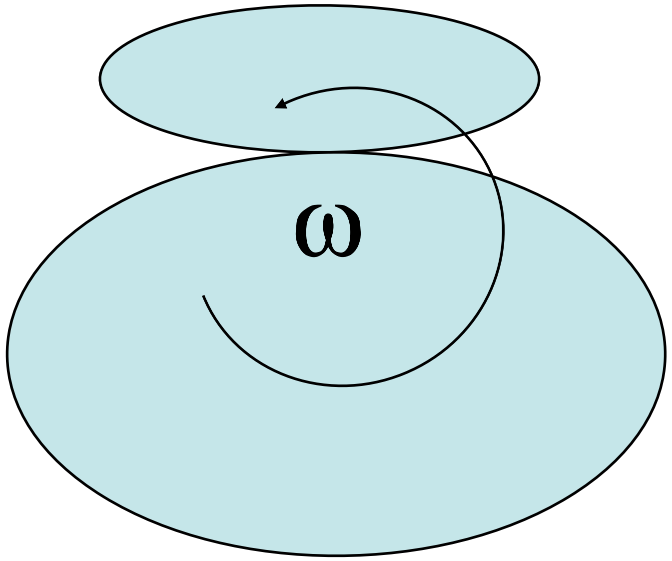
A perturbation can trigger a shape change, conserving AM, decreasing energy, and dissipating excess energy via friction and seismic waves.

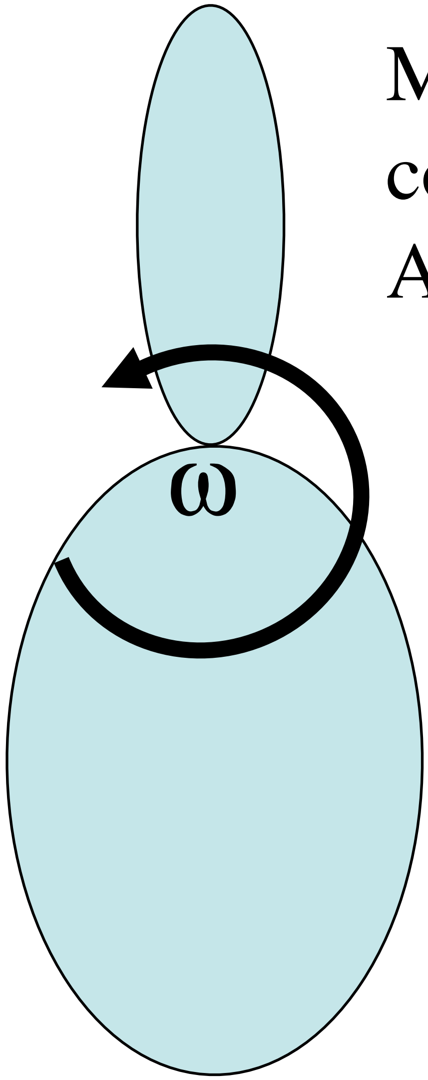
Generalization to Non-Spherical Bodies



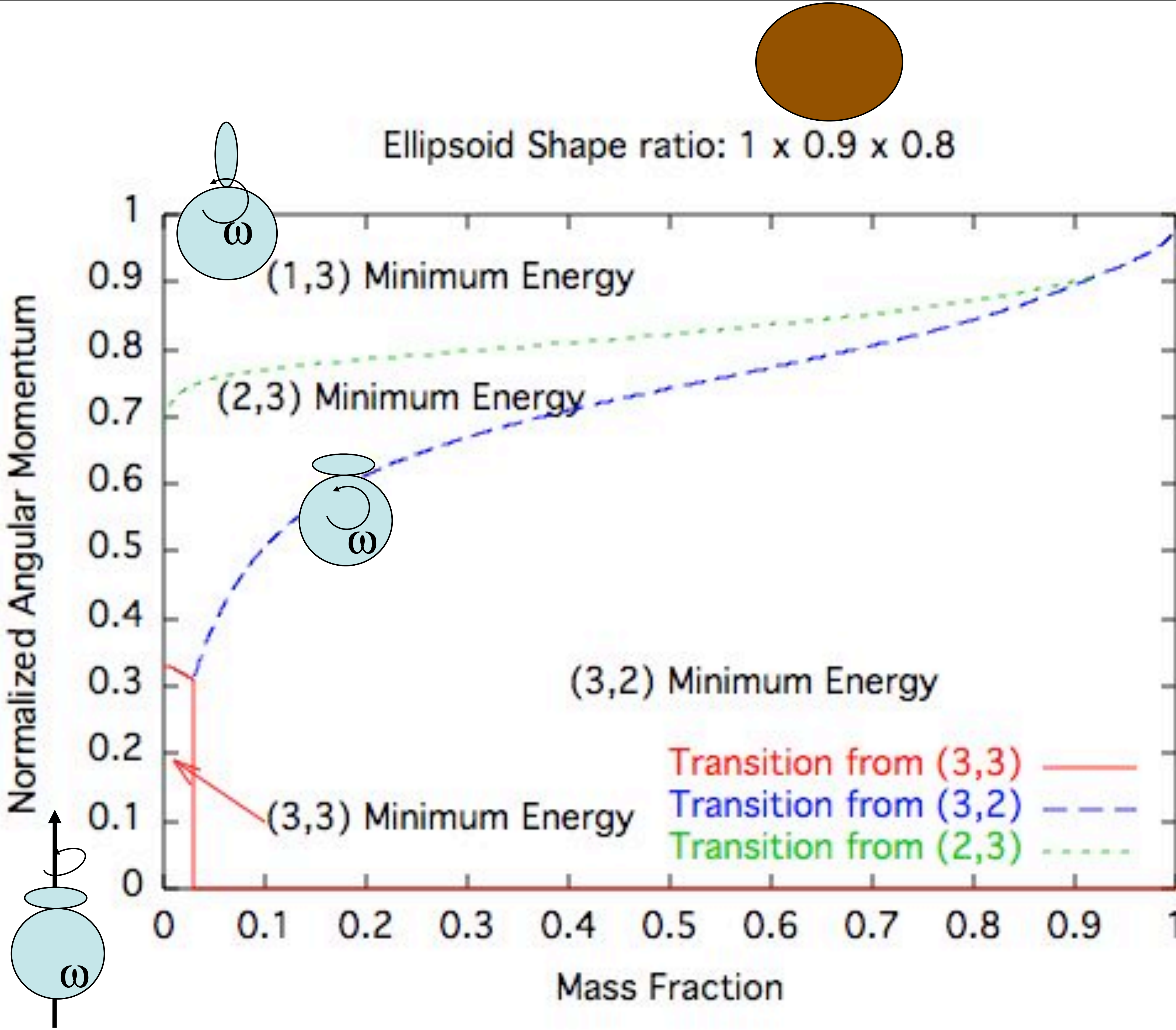
The theory of minimum energy configurations can be extended to arbitrary finite density shapes, e.g. an equal density ellipsoid/ellipsoid system

Minimum energy configuration for small Angular Momentum





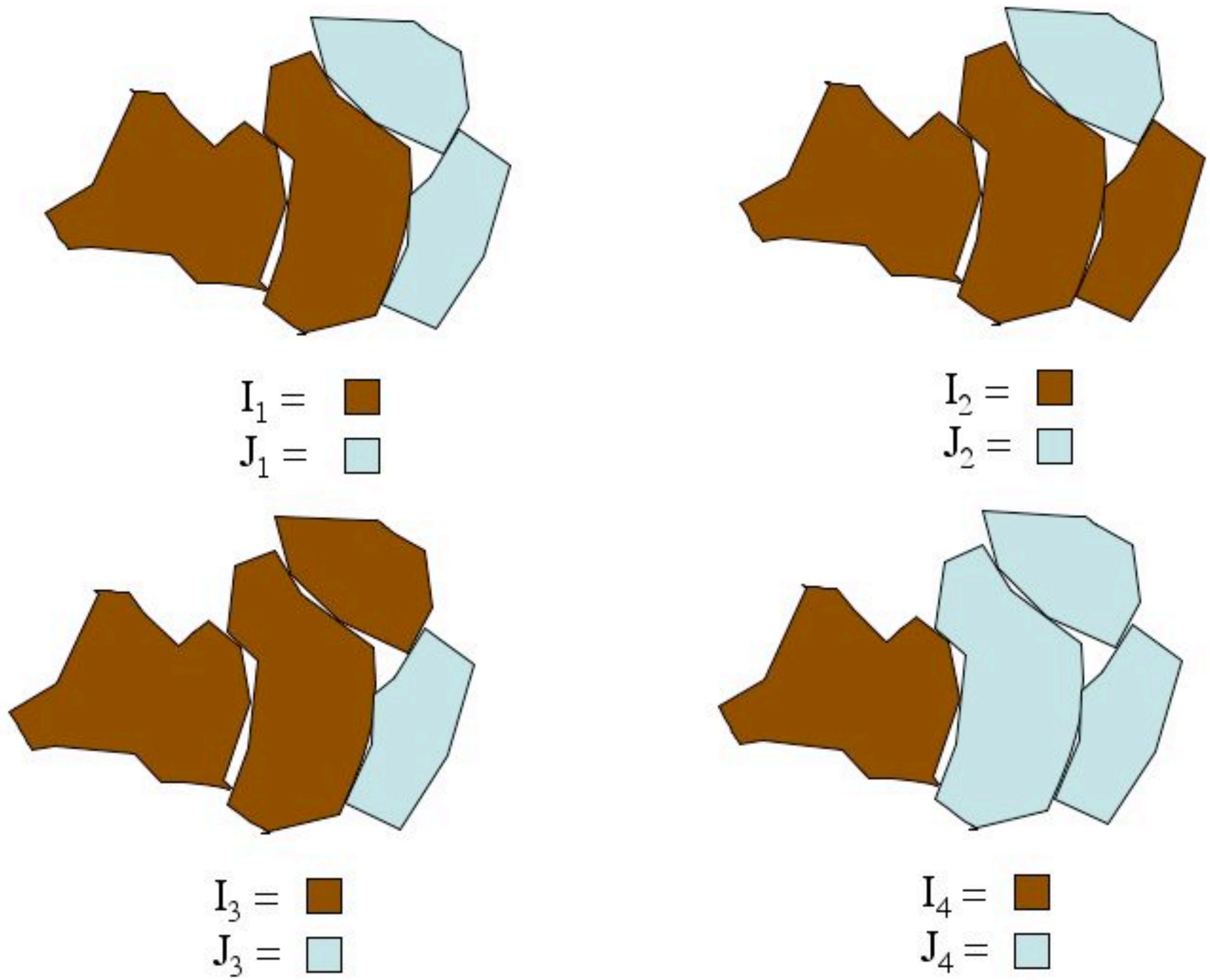
Minimum energy configuration for large Angular Momentum



(*Icarus* 189: 370-385)

Rubble pile partitions

As spin rate increases, the transition limit for each "mutual" set should be computed and compared with each other.



Fission Conditions

- For fission of an arbitrary rubble pile split into two collections I and J the general condition becomes:

$$\boldsymbol{R}_{IJ} \cdot \tilde{\boldsymbol{\omega}} \cdot \tilde{\boldsymbol{\omega}} \cdot \boldsymbol{R}_{IJ} \quad \geq \quad -\frac{M_I + M_J}{M_I M_J} \frac{\partial U_{IJ}}{\partial \boldsymbol{R}_{IJ}} \cdot \boldsymbol{R}_{IJ}$$

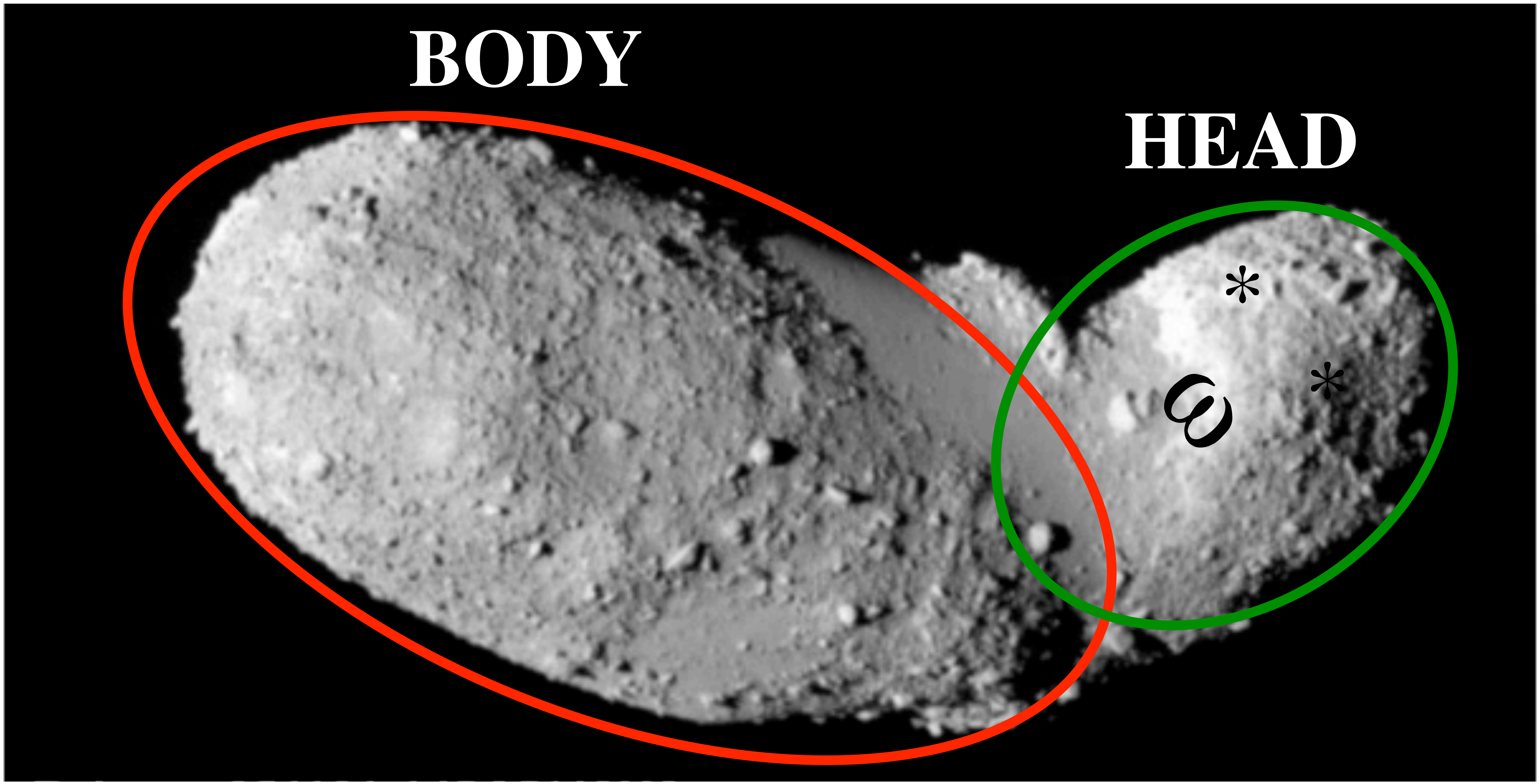
  - For mass distributions only a weak form of Euler's Theorem of Homogenous functions applies which allows us to reduce this inequality to:

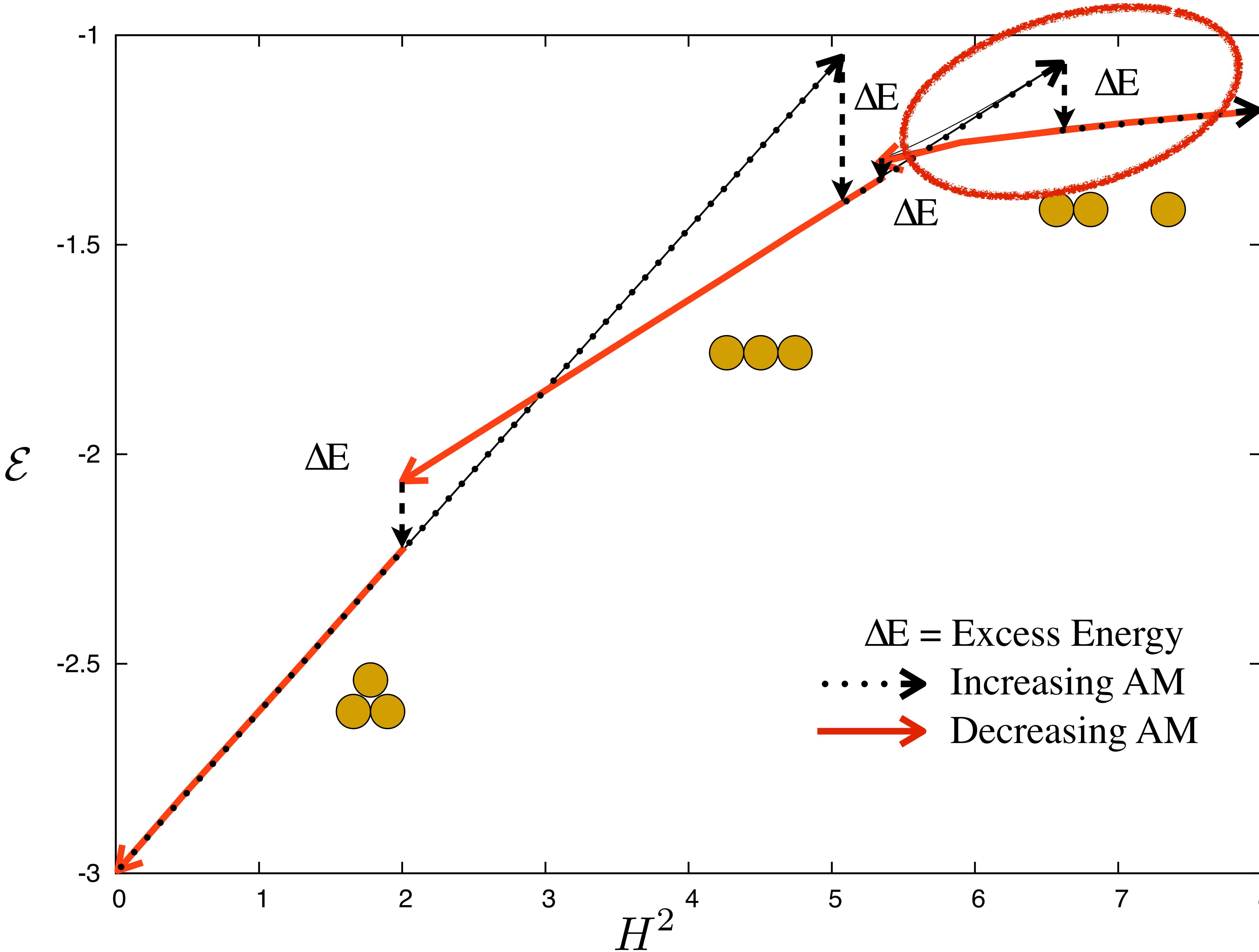
$$T_{IJ} + \alpha U_{IJ} \geq 0$$

  - where $\alpha > 1$
- This is equivalent to "the two components with the largest separation between their centers of mass will fission first at the lowest spin rate"

Asteroid Itokawa's peculiar mass distribution will "fission" when its rotation period < 6 hours – spin period can change due to the "YORP Effect", slowly changes total angular momentum...

Body = 490 x 310 x 260 *meters* **Head** = 230 x 200 x 180 *meters*

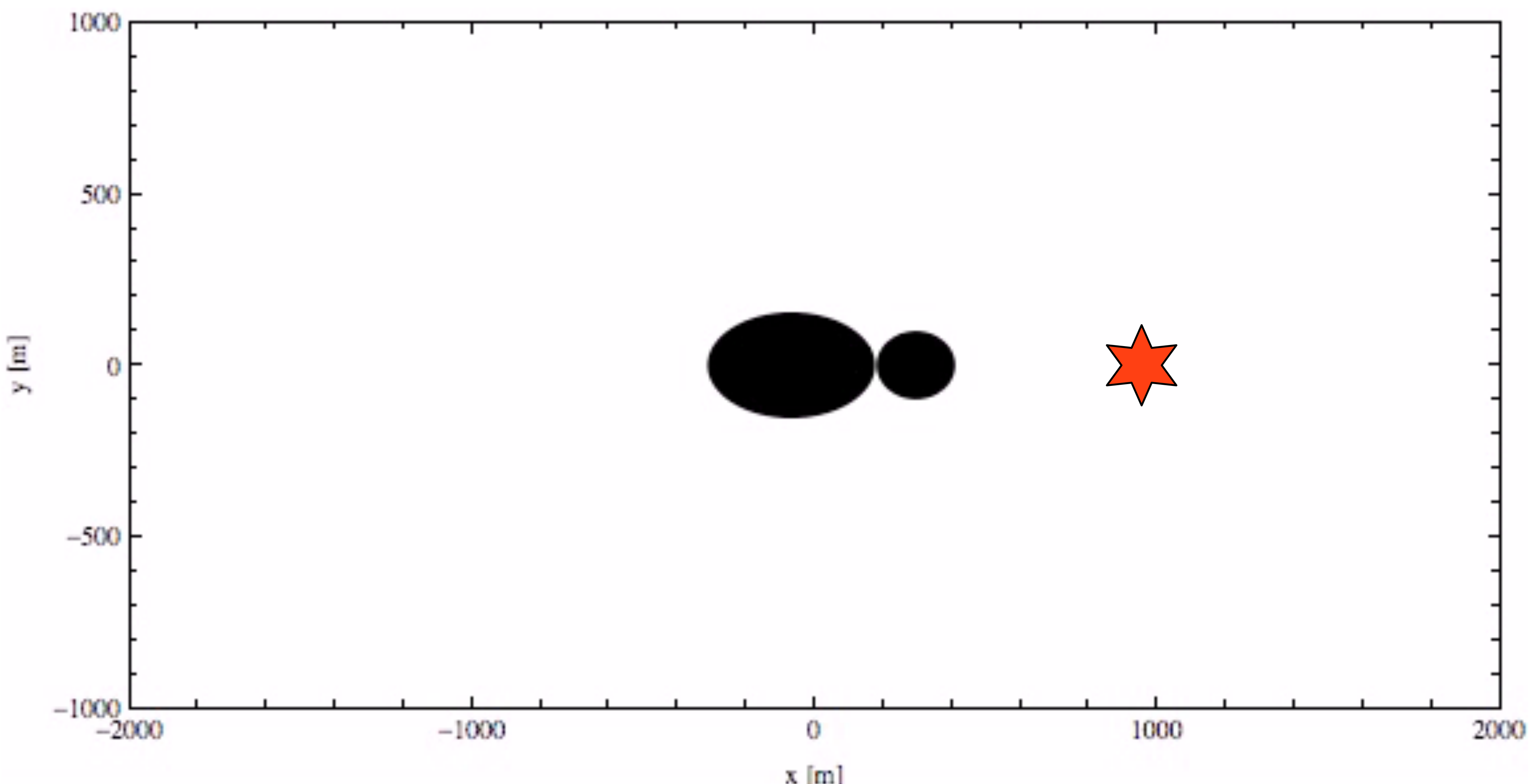




ΔE
ΔE
ΔE
ΔE
$\mathcal{E}$
-1
-1.5
-2
-2.5
-3
0
1
2
3
4
5
6
7
8
H^2
ΔE = Excess Energy
Increasing AM
Decreasing AM

Reconfigurations and Fission Events

- When a Local Minimum reaches its reconfiguration or fission state it cannot directly enter a different minimum energy state
  - Excess energy ensures a period of dynamics where dissipation may occur



Movie by S.A. Jacobson

Fission

- Fission can be a smooth transition for a rubble pile
- Energy and AM are ideally conserved, but are decomposed:
  – Kinetic Energy

$$\frac{1}{2}\omega \cdot I_0 \cdot \omega = \frac{1}{2}\omega \cdot I_1 \cdot \omega + \frac{1}{2}\omega \cdot I_2 \cdot \omega + \frac{1}{2}\frac{M_1 M_2}{M_1 + M_2}(R\omega)^2$$

  – Potential Energy

$$\mathcal{U}_{00} = \mathcal{U}_{11} + \mathcal{U}_{22} + \mathcal{U}_{12}$$

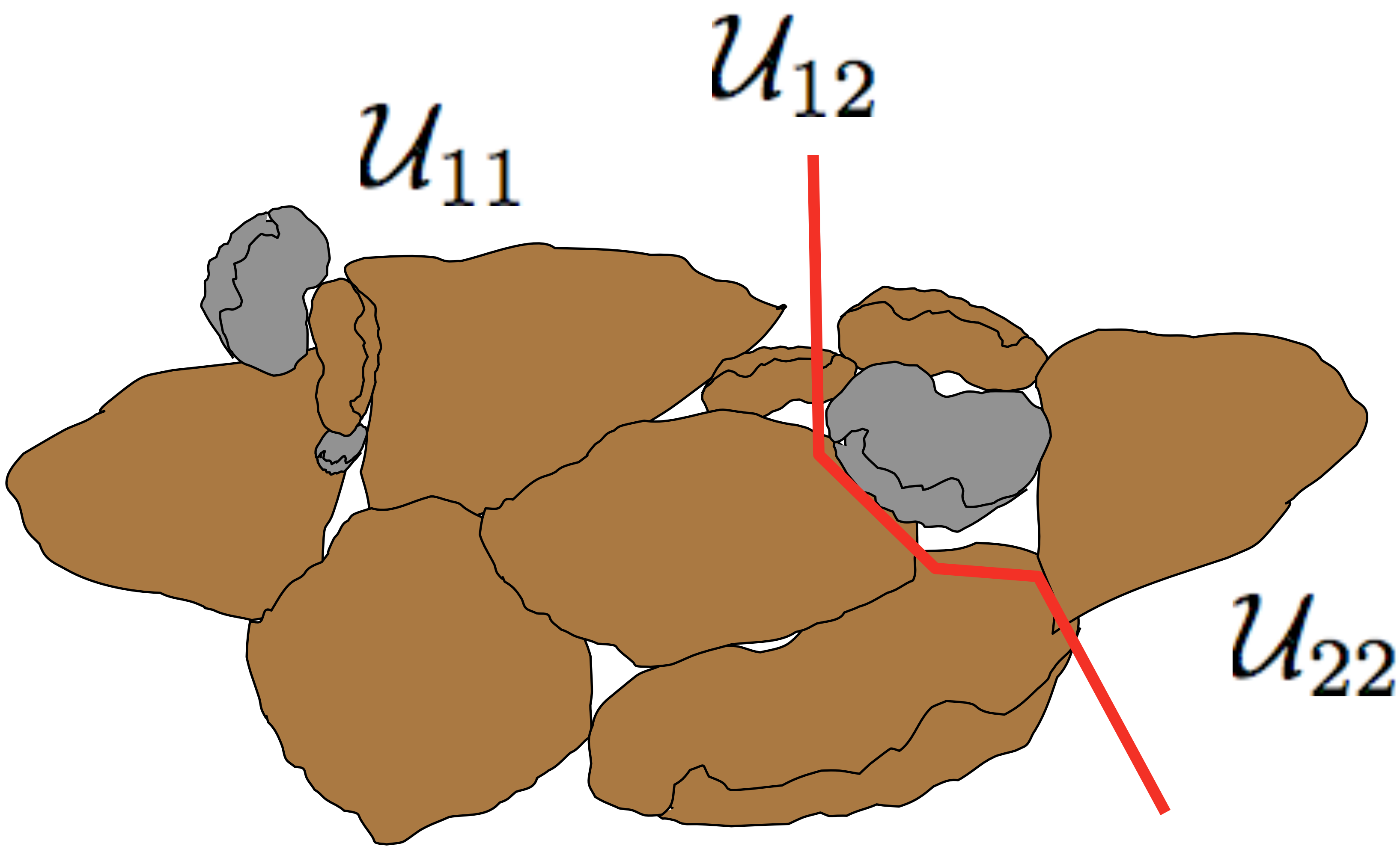
  – The mutual potential energy is "liberated" and serves as a conduit to transfer rotational and translational KE

Fission

$$\mathcal{U}_{00} = \mathcal{U}_{11} + \mathcal{U}_{22} + \mathcal{U}_{12}$$

Fission spin rate is the minimum rate for two partitions of the asteroid to enter orbit.

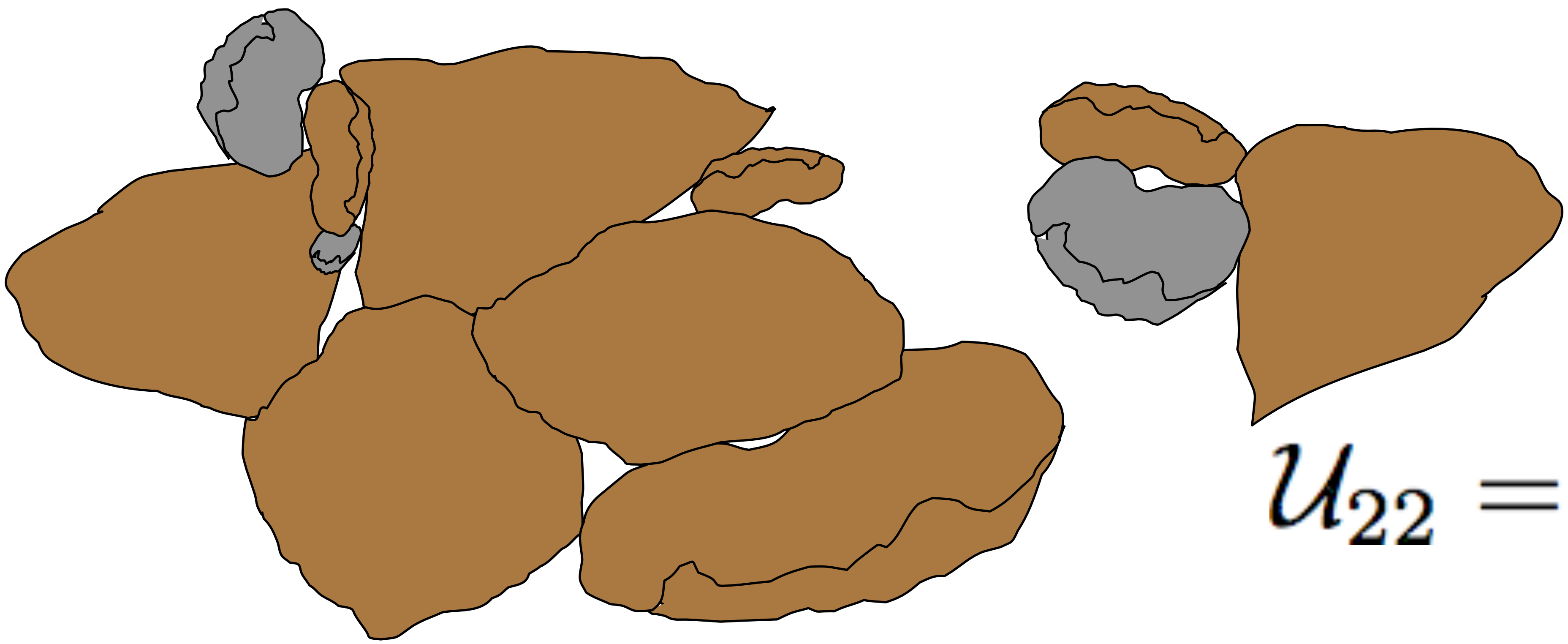
Corresponds to the maximum center of mass separation across all possible partitions



$$\mathcal{U}_{12} = -G \int_{B_1} \int_{B_2} \frac{dm_1 dm_2}{|\rho_{12}|}$$

Orbital Evolution

$$\Delta T_{\text{rot}} + \Delta T_{\text{trans}} + \Delta \mathcal{U}_{12} = 0$$

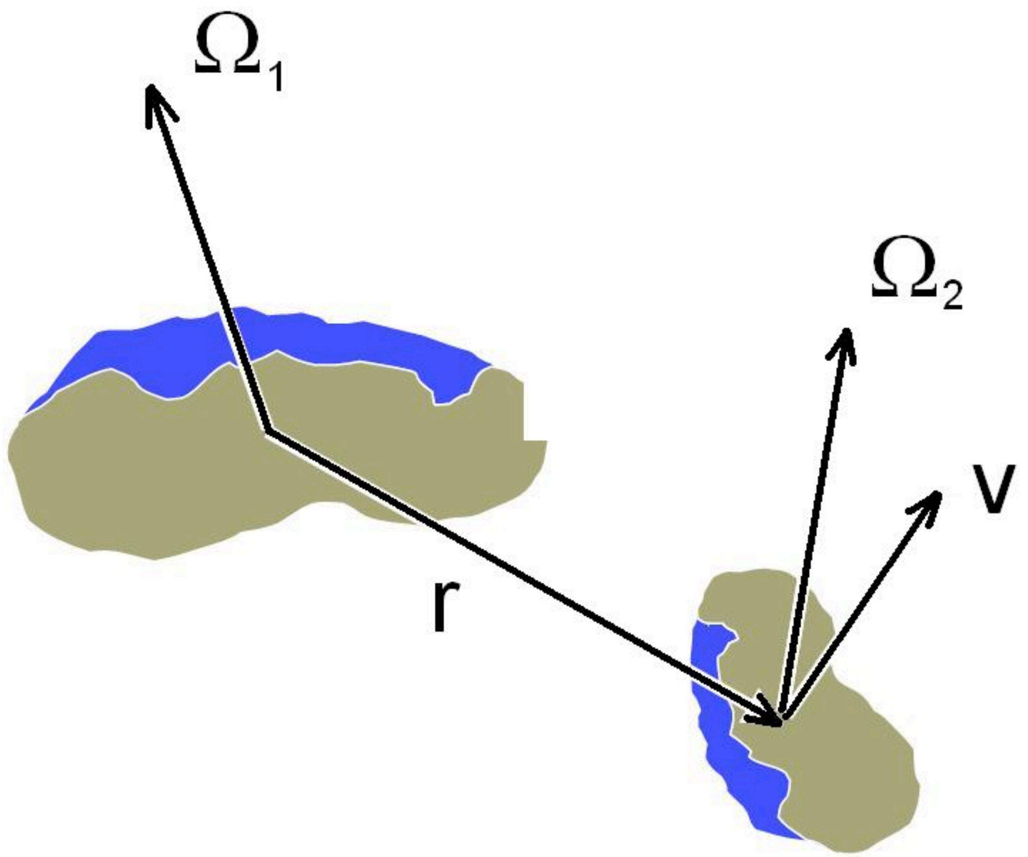


$$\mathcal{U}_{22} = \text{ Constant}$$

$$\mathcal{U}_{11} = \text{ Constant}$$

Post-Fission Dynamics

- The problem changes to the dynamical motion of two arbitrary mass distributions orbiting about each other
  - Strong coupling between translational and rotational motion
  - Generalized version of a classical celestial mechanics problem



Orbit Mechanics after Fission

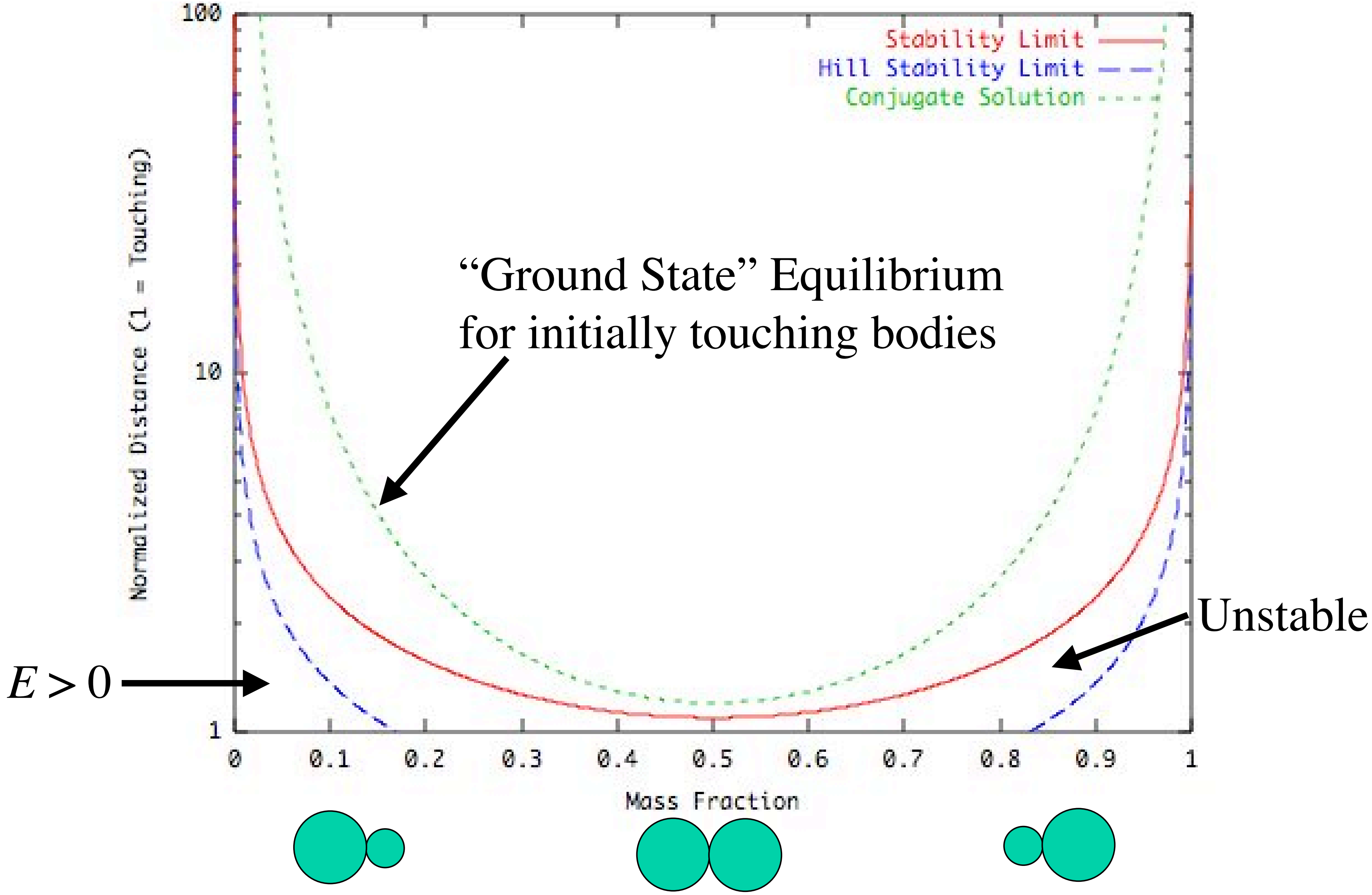
- The relevant energy for orbital motion is the "free energy," which is conserved under dynamical evolution:

$$E_{\text{Free}} = E - \mathcal{U}_{11} - \mathcal{U}_{22}$$

$$E_{\text{Free}} = \frac{1}{2}\left[\omega_1 \cdot I_1 \cdot \omega_1 + \omega_2 \cdot I_2 \cdot \omega_2 + \frac{M_1 M_2}{M_1 + M_2} V \cdot V\right] + \mathcal{U}_{12}$$

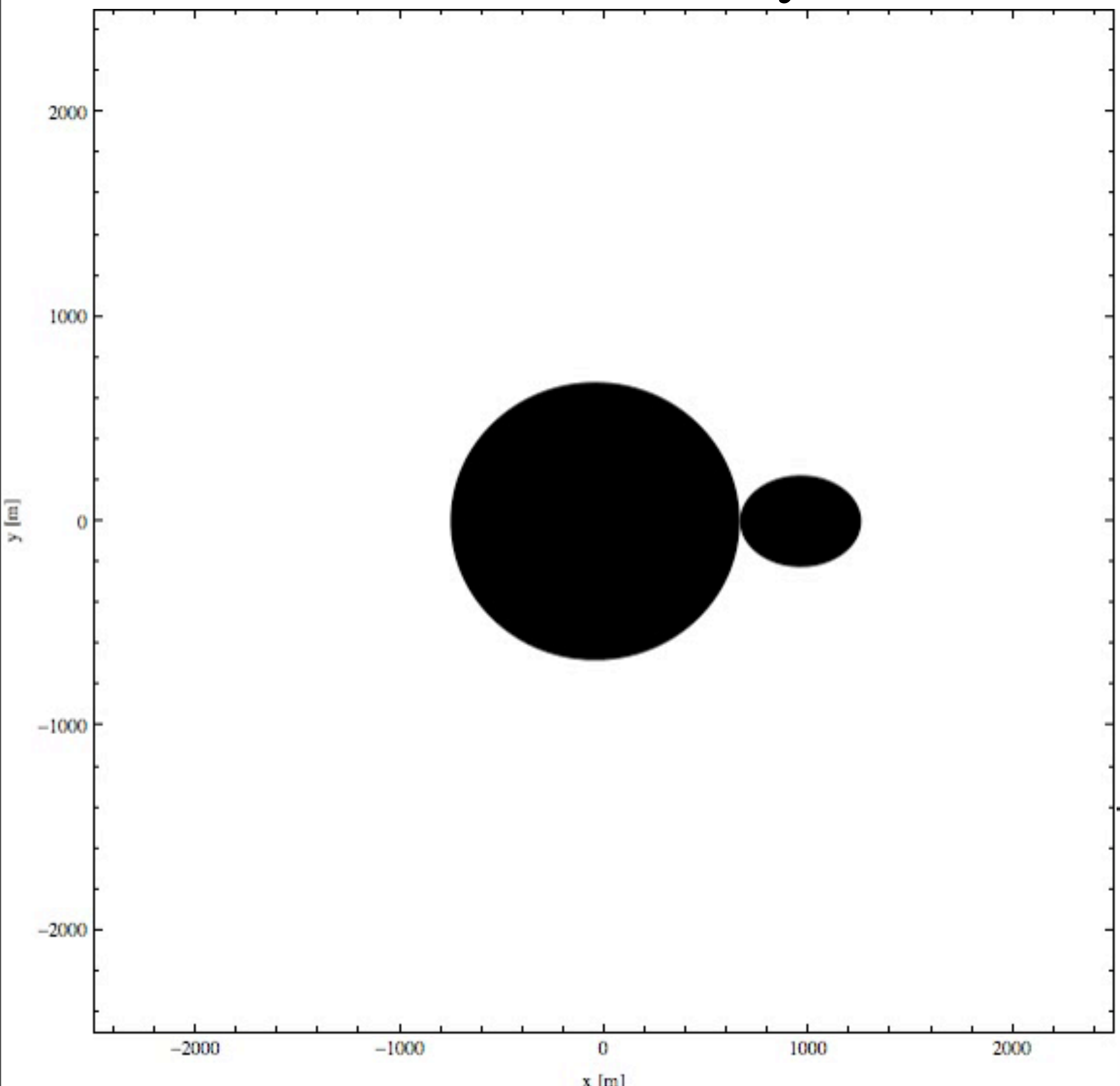
- Energy transfer between orbit and rotation happen rapidly
  - If $E_{\text{Free}} > 0$, system can "catastrophically disrupt"
  - If $E_{\text{Free}} < 0$, system cannot "catastrophically disrupt"
- Orbits with $E_{\text{Free}} > 0$ are highly unstable and usually will send the components away on hyperbolic orbits

Stability of a fissioned system



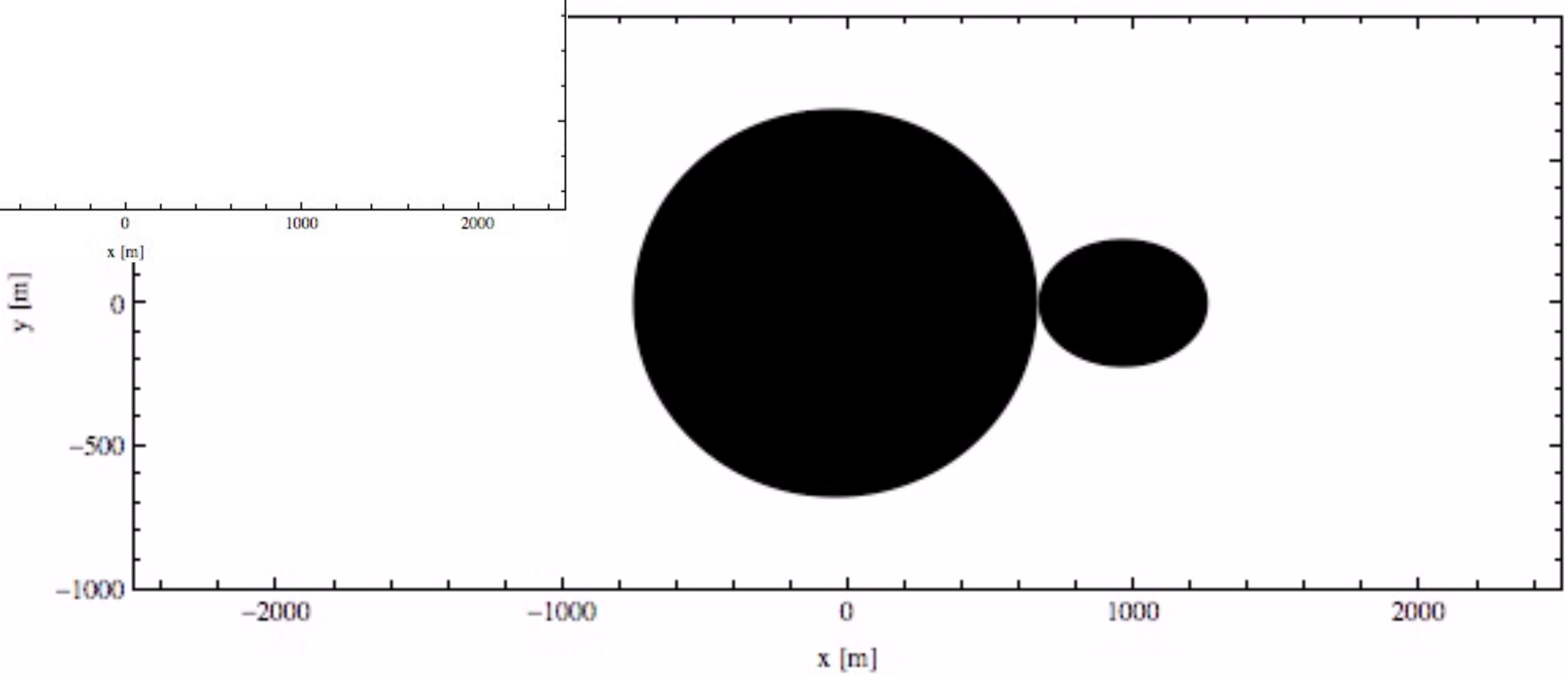
– Fissioned binaries with a mass ratio < ~0.2 have sufficient energy to escape in a timescale of days to months (Scheeres, *Celestial Mechanics & Dynamical Astronomy 2009*).

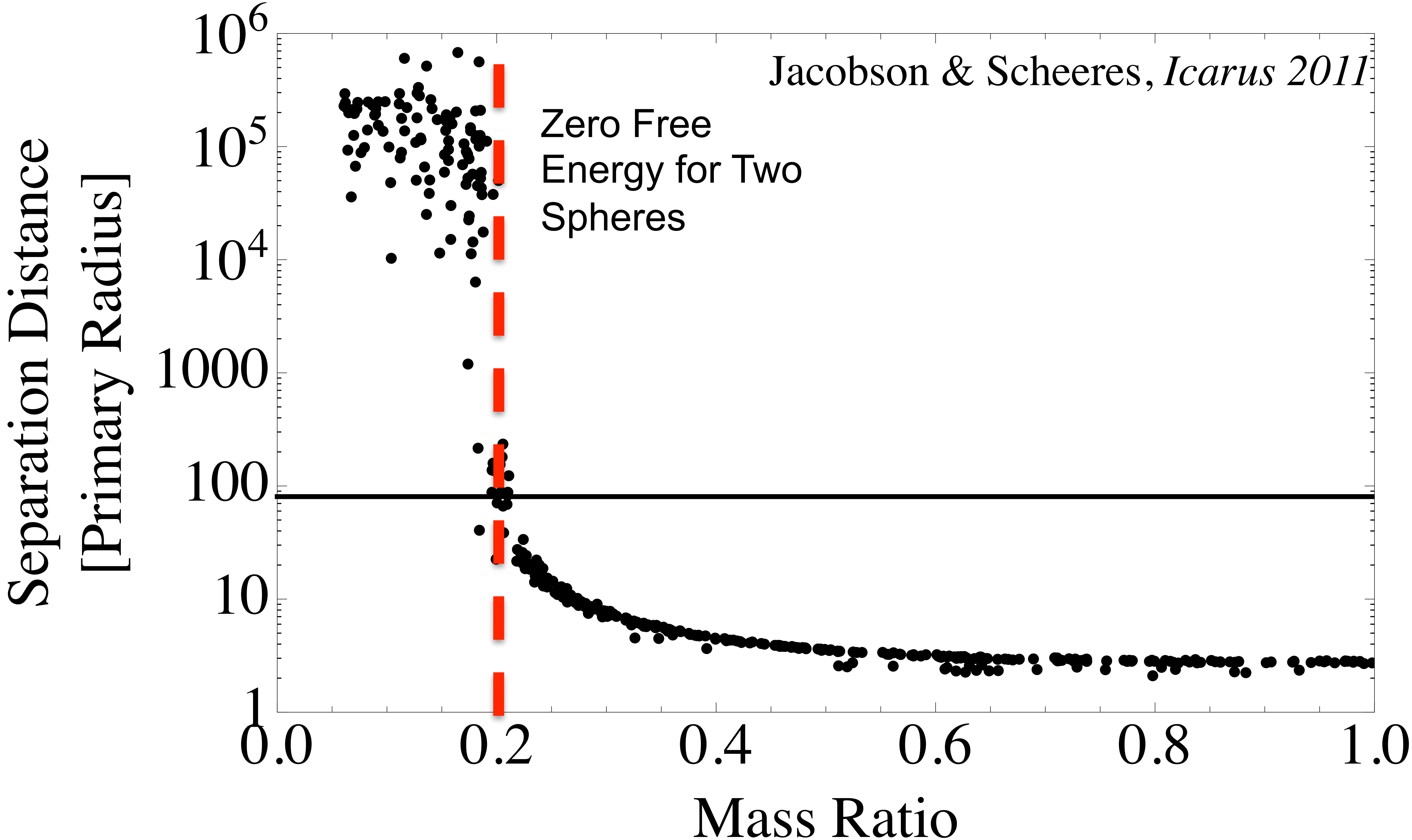
Fission Dynamics of a proto 1999 KW4 System



Movie by S. Jacobson

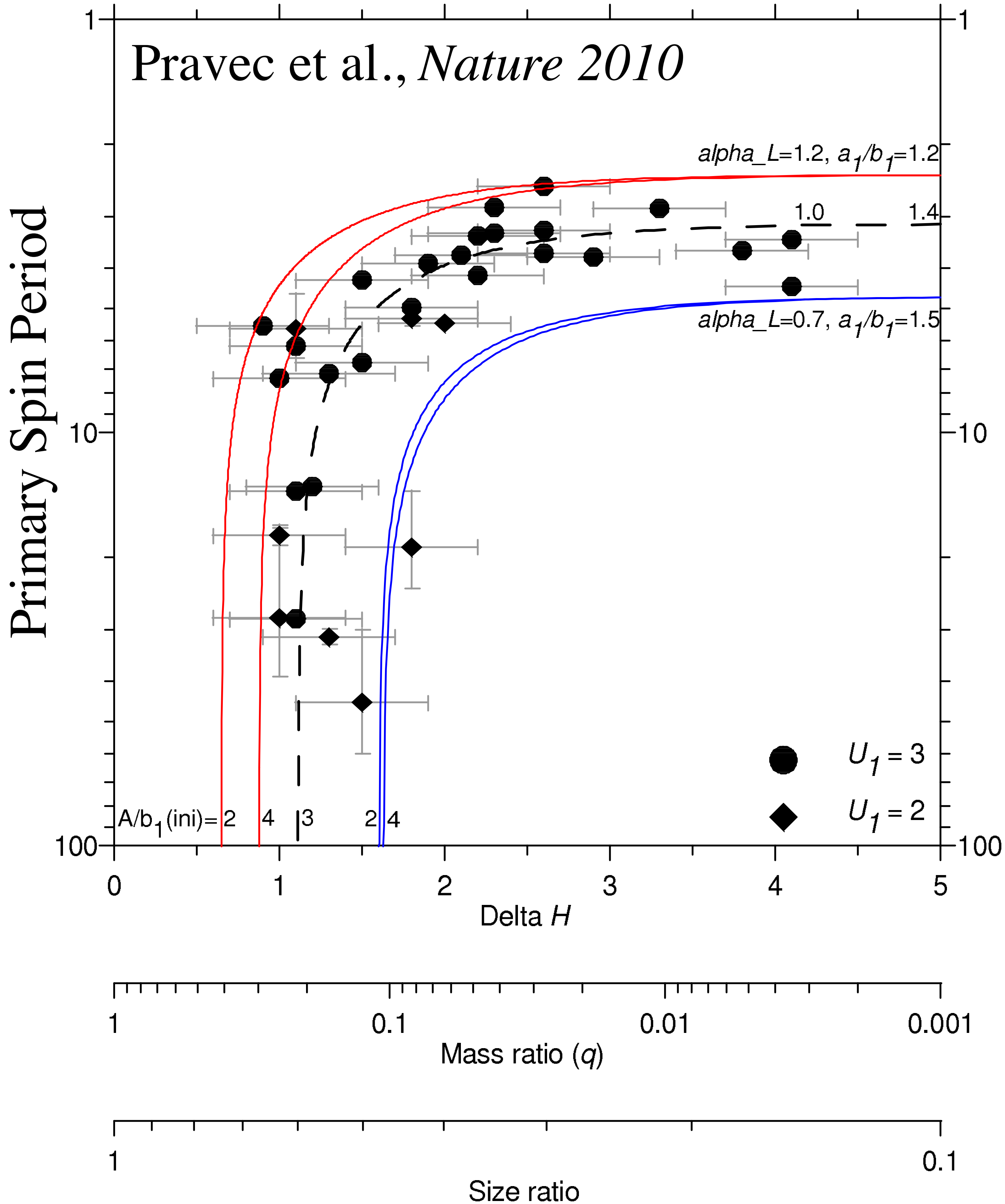
Initial conditions are chosen at the precise “fission limit” when the two components enter orbit about each other. Total movie duration is several days.





Fissioned Asteroids with mass ratios < 0.2 eventually escape, the closer to the cutoff, the more energy is drawn from the primary spin.

Asteroid Pair Primary Spin vs Mass Ratio



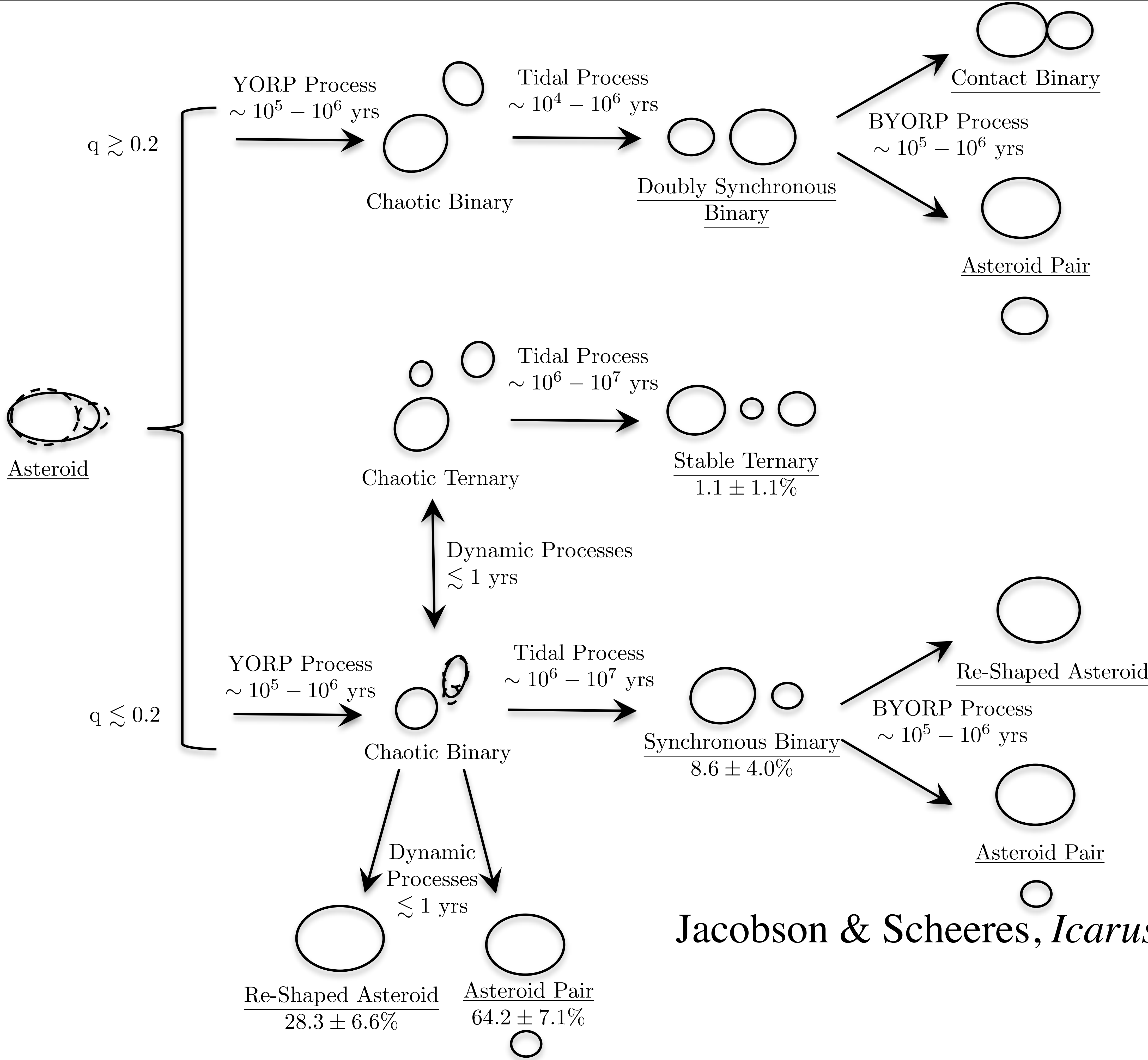
Prediction:
The mass ratio between asteroid pairs should be < 0.2
The primary spin period should grow long near the cut-off

Observation:
The mass ratios and primary spin periods of Main Belt asteroid pairs match with our Asteroid Fission Theory

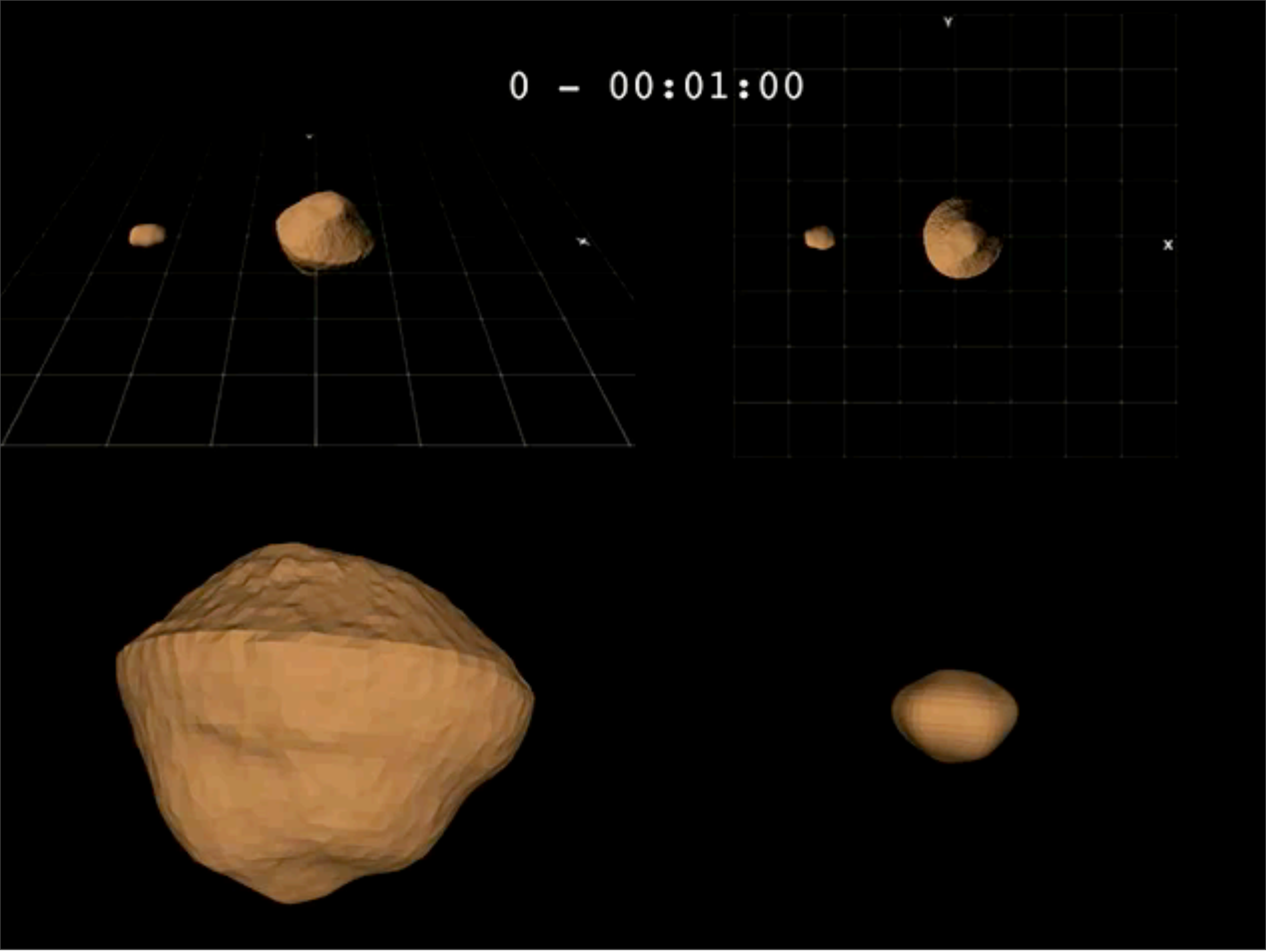
Comment:
The theory matches two independent outcomes, mass ratio cut-off and primary spin period lengthening

Energetics of a fissioned system

- When a body fissions, it's components enter a chaotic dynamical phase:
  - Fissioned binaries with a mass ratio < ~0.2 have sufficient energy to evolve to escape in a timescale of days to months (Scheeres, Celestial Mechanics & Dynamical Astronomy 2002).
  - The theory makes specific predictions that are consistent with observations of Asteroid Pairs
  - Prior to escape, a sizable fraction of the secondaries are spun fast enough to undergo fission again
    - Inner satellite usually impacts the primary, should cause reshaping
    - Outer satellite is stabilized
  - This sequence can repeat and, along with gravity, friction, and sunshine, can create the observed class of NEA asteroid systems (Jacobson & Scheeres, *Icarus 2011*).



Jacobson & Scheeres, *Icarus 2011*


0 - 00:01:00
Y
X
Y
X

Precise Modeling of the Mechanics of an Asteroid System

$$
\begin{aligned}
\frac{M_1 M_2}{M_1 + M_2}\ddot{\mathbf{r}} &= \frac{\partial U_{12}}{\partial \mathbf{r}} \\
\mathbf{r} &= \mathbf{r}_2 - \mathbf{r}_1 \\
\dot{\mathbf{H}}_1 &= \mathbf{H}_1 \times \mathbf{\Omega}_1 + \mathbf{M}_1 \\
\dot{\mathbf{H}}_2 &= \mathbf{H}_2 \times \mathbf{\Omega}_1 + \mathbf{M}_2 \\
\mathbf{\Omega}_1 &= \mathbf{I}_1^{-1} \cdot \mathbf{H}_1 \\
\mathbf{\Omega}_2 &= \mathbf{I}_2^{-1} \cdot \mathbf{A} \cdot \mathbf{H}_2 \\
\dot{\mathbf{A}} &= \mathbf{A} \times \mathbf{\Omega}_1 - \mathbf{\Omega}_2 \times \mathbf{A} \\
U_{12}(\mathbf{r}, \mathbf{A}) &= \mathcal{G} \int_{\beta_1} \int_{\beta_2} \frac{dm_1 dm_2}{|\mathbf{r} + \rho_1 - \mathbf{A} \cdot \rho_2|} \\
\dot{\mathbf{A}}_1 &= \mathbf{A}_1 \times \mathbf{\Omega}_1
\end{aligned}
$$

High dimensional system with coupled rotational and translational motions.

Interest in both short-time dynamics and evolutionary dynamics.

Summary

- Study of asteroids leads directly to study of minimum energy configurations of self-gravitating grains
  - Only possible for bodies with finite density
- For finite density bodies, minimum energy and stable configurations are defined as a function of angular momentum by studying the minimum energy function:

$$\mathcal{E}_m = \frac{H^2}{2I_H} + U$$

  - only a function of the system configuration
  - Globally minimum energy configurations are denumerable
- Simple few body systems can be fully explored
  - Need theories for polydisperse grains and $N >> 1$