The Keplerian processor

A Keplerian processor is a set of high-level routines for the computation of series expansions commonly used in celestial mechanics and astrodynamics. From a conceptual point of view, it stands at a higher level than the manipulation routines we have seen in the previous lectures.

While earlier versions of piranha include Keplerian processors, the present version does **not** at the present time. In this lecture we will develop a few classical series expansions to be used later.

In [1]:
```
from pyranhapp0x import *
from fractions import Fraction as Frac
```

Bessel functions

The Maclaurin expansion for the Bessel function of the first kind reads

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+n)!} \left(\frac{x}{2}\right)^{2m+n},$$

where $n \in \mathbb{N}$. The pyranha implementation is straightforward enough:

In [2]:
```
def besselJ(n,x,order):
    retval = 0
    for m in range(0,int((order-n)/2) + 1):
        retval = retval + Frac((-1)**m,math.factorial(m)*math.factorial(m+n)) * (x/2)**(2*m+n)
    return retval
```

In [3]:
```
pt = polynomial.get_type('rational')
x = pt('x')
besselJ(0,x,4)
```

Out[3]: $1 + \frac{1}{64}x^4 - \frac{1}{4}x^2$

In [4]:
```
besselJ(1,x,10)
```

Out[4]: $\frac{1}{2}x + \frac{1}{1474560}x^9 - \frac{1}{16}x^3 + \frac{1}{384}x^5 - \frac{1}{18432}x^7$

Binomial theorem

The generalised binomial theorem is useful to calculate negative and rational powers of series. The general statement reads:

$$\left(x+y\right)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^{r-k} y^k,$$

where r is any complex number (although we will limit ourselves to the rational and integer cases) and x and y real numbers with $|x| > |y|$. When r is a non-negative integer, the series has a finite number of terms, otherwise the series has an infinite number of terms.

The pyranha implementation:

In [5]:
```
def binomial_exp(x,y,r,order):
    retval = 0
    for k in range(0,order + 1):
        retval = retval + math.binomial(r,k) * x**(r-k) * y**k
    return retval
```

$\sqrt{1+x}$:

In [6]:
```
binomial_exp(1,x,Frac(1,2),5)
```

Out[6]: $1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5$

$1/(1-x)^2$:

In [7]:
```
binomial_exp(1,-x,-2,5)
```

Out[7]: $1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5$

$1/\left(1+e\cos f\right)$:

In [8]:
```
pst = poisson_series.get_type('polynomial_rational')
e,f = pst('e'),pst('f')
binomial_exp(1,e*math.cos(f),-1,5)
```

Out[8]: $$\left(1+\frac{3}{8}e^4+\frac{1}{2}e^2\right)+\left(-e-\frac{5}{8}e^5-\frac{3}{4}e^3\right)\cos\left(f\right)+\left(\frac{1}{2}e^2+\frac{1}{2}e^4\right)\cos\left(2f\right)+\left(-\frac{1}{4}e^3-\frac{5}{16}e^5\right)\cos\left(3f\right)+\frac{1}{8}e^4\cos\left(4f\right)-\frac{1}{16}e^5\cos$$

Elliptic expansions

Elliptic expansions are Fourier-Taylor expansions that allow to transform cartesian and polar coordinates into expressions depending on the orbital elements. In Hamiltonian perturbation theories, they often are a stepping stone towards expressing the Hamiltonian in action-angle variables.

A few examples (from Murray and Dermott):

$$\frac{r}{a}=1+\frac{1}{2}e^2-2e\sum_{s=1}^{\infty}\frac{1}{s^2}\frac{dJ_s(se)}{de}\cos(sM)$$

Here a couple of difficulties arise in deciding how to stop the expansion: we have a partial differentiation which lowers the degree of e and an external multiplication by e which offsets it. It is best to rewrite

$$\frac{dJ_s(se)}{de}=s\frac{dJ_s(se)}{d(se)}$$

and use the recurrence relation

$$\frac{dJ_n(x)}{dx}=\frac{1}{2}\left[J_{n-1}(x)-J_{n+1}(x)\right],$$

so that

$$\frac{r}{a}=1+\frac{1}{2}e^2-e\sum_{s=1}^{\infty}\frac{1}{s}\left[J_{s-1}(se)-J_{s+1}(se)\right]\cos(sM).$$

In [9]:
```
def r_a(e,M,order):
    retval = 1 + Frac(1,2) * e**2
    for s in range(1,order + 1):
        retval = retval - e * Frac(1,s) * besselJ(s-1,s*e,order - 1) * math.cos(s*M)
    for s in range(1,order - 1):
        retval = retval + e * Frac(1,s) * besselJ(s+1,s*e,order - 1) * math.cos(s*M)
    return retval
```

In [10]:
```
e,M = pst('e'),pst('M')
r_a(e,M,4)
```

Out[10]: $$\left(\frac{1}{2}e^2+1\right)+\left(-e+\frac{3}{8}e^3\right)\cos\left(M\right)+\left(-\frac{1}{2}e^2+\frac{1}{3}e^4\right)\cos\left(2M\right)-\frac{3}{8}e^3\cos\left(3M\right)-\frac{1}{3}e^4\cos\left(4M\right)$$

In [11]:
```
r_a(e,M,6)
```

Out[11]: $$\left(\frac{1}{2}e^2+1\right)+\left(-e-\frac{5}{192}e^5+\frac{3}{8}e^3\right)\cos\left(M\right)+\left(\frac{1}{3}e^4-\frac{1}{2}e^2-\frac{1}{16}e^6\right)\cos\left(2M\right)+\left(-\frac{3}{8}e^3+\frac{45}{128}e^5\right)\cos\left(3M\right)+\left(-\frac{1}{3}e^4+\frac{2}{5}e^6\right)$$

A second example:

$$\sin f=2\sqrt{1-e^2}\sum_{s=1}^{\infty}\frac{1}{s}\frac{dJ_s(se)}{de}\sin(sM).$$

As done previously, we can rewrite this as

$$\sin f=\sqrt{1-e^2}\sum_{s=1}^{\infty}\left[J_{s-1}(se)-J_{s+1}(se)\right]\sin(sM).$$

Now we need to take into account the presence of the square root, that will be developed using the binomial theorem into its own series. For instance $\sqrt{1-e^2}$ to the fifth order in e^2:

In [12]:
```
binomial_exp(1,-e**2,Frac(1,2),5)
```

Out[12]: $$\left(1-\frac{5}{128}e^8-\frac{1}{2}e^2-\frac{7}{256}e^{10}-\frac{1}{8}e^4-\frac{1}{16}e^6\right)$$

By multiplying the expansion for the square root by the expansions for the Bessel functions, we will generate terms of higher order than desired. We can remove the excess terms via the `filter()` method, and the implementation will look like this:

In [13]:
```
def sin_f(e,M,order):
    retval = 0
    for s in range(1,order + 2):
        retval = retval + besselJ(s-1,s*e,order) * math.sin(s*M)
    for s in range(1,order):
        retval = retval - besselJ(s+1,s*e,order) * math.sin(s*M)
    retval = retval * binomial_exp(1,-e**2,Frac(1,2),int(order/2))
    return retval.transform(lambda t: (t[0].filter(lambda u: u[1].degree(['e']) <= order),t[1]))
```

In [14]:
```
sin_f(e,M,4)
```

Out[14]:
$$\left(1 + \frac{17}{192}\,e^4 - \frac{7}{8}\,e^2\right)\sin(M) + \left(e - \frac{7}{6}\,e^3\right)\sin(2M) + \left(-\frac{207}{128}\,e^4 + \frac{9}{8}\,e^2\right)\sin(3M) + \frac{4}{3}\,e^3\sin(4M) + \frac{625}{384}\,e^4\sin(5M)$$

In [15]:
```
sin_f(e,M,5)
```

Out[15]:
$$\left(1 + \frac{17}{192}\,e^4 - \frac{7}{8}\,e^2\right)\sin(M) + \left(e + \frac{1}{3}\,e^5 - \frac{7}{6}\,e^3\right)\sin(2M) + \left(-\frac{207}{128}\,e^4 + \frac{9}{8}\,e^2\right)\sin(3M) + \left(-\frac{34}{15}\,e^5 + \frac{4}{3}\,e^3\right)\sin(4M) + \frac{625}{384}\,e^4\sin$$

In [16]:
```
sin_f(e,M,6)
```

Out[16]:
$$\left(1 + \frac{17}{192}\,e^4 - \frac{7}{8}\,e^2 - \frac{271}{9216}\,e^6\right)\sin(M) + \left(e + \frac{1}{3}\,e^5 - \frac{7}{6}\,e^3\right)\sin(2M) + \left(-\frac{207}{128}\,e^4 + \frac{9}{8}\,e^2 + \frac{3681}{5120}\,e^6\right)\sin(3M) + \left(\frac{4}{3}\,e^3 - \frac{34}{15}\,e^5\right)\mathrm{s}$$

Finally, let's take a look at $\cos f$:

$$\cos f = -e + \frac{2\left(1-e^2\right)}{e}\sum_{s=1}^{\infty} J_s(se)\cos(sM)$$

Here we have to take into account the fact that we have a negative power of e outside the summation. It is better to use the recurrence relation

$$\frac{J_n(x)}{x} = \frac{J_{n-1}(x) + J_{n+1}(x)}{2n},$$

valid for $n \neq 0$, and rewrite

$$\cos f = -e + \left(1-e^2\right)\sum_{s=1}^{\infty}\left[J_{s-1}(se) + J_{s+1}(se)\right]\cos(sM).$$

A possible implementation thus reads:

In [17]:
```
def cos_f(e,M,order):
    retval = 0
    for s in range(1,order + 2):
        retval = retval + besselJ(s-1,s*e,order) * math.cos(s*M)
    for s in range(1,order):
        retval = retval + besselJ(s+1,s*e,order) * math.cos(s*M)
    retval = retval * (1 - e**2) - e
    return retval.transform(lambda t: (t[0].filter(lambda u: u[1].degree(['e']) <= order),t[1]))
```

In [18]:
```
cos_f(e,M,4)
```

Out[18]:
$$-e + \left(1 + \frac{25}{192}\,e^4 - \frac{9}{8}\,e^2\right)\cos(M) + \left(e - \frac{4}{3}\,e^3\right)\cos(2M) + \left(-\frac{225}{128}\,e^4 + \frac{9}{8}\,e^2\right)\cos(3M) + \frac{4}{3}\,e^3\cos(4M) + \frac{625}{384}\,e^4\cos(5M)$$

In [19]:
```
cos_f(e,M,5)
```

Out[19]:
$$-e + \left(1 + \frac{25}{192}\,e^4 - \frac{9}{8}\,e^2\right)\cos(M) + \left(e + \frac{3}{8}\,e^5 - \frac{4}{3}\,e^3\right)\cos(2M) + \left(-\frac{225}{128}\,e^4 + \frac{9}{8}\,e^2\right)\cos(3M) + \left(-\frac{12}{5}\,e^5 + \frac{4}{3}\,e^3\right)\cos(4M) + \frac{62}{38}$$