Chapter 5

Global behavior: more stuff is out there

Every *Dynamical System* studied so far exhibited fairly simple motions, allowing for a detailed understanding of its behavior. Yet, we have not addressed yet the problem of *long time predictions* in systems with more than two dimensions.

Although this is not the proper occasion for an historical excursus, it is worthwhile to stress that the first Dynamical Systems widely investigated have been the planetary motions. Not surprisingly the main emphasis in such investigations was accurate prediction of future positions. Nevertheless, exactly from the effort of predicting accurately future motions stemmed the consciousness of the existence of very serious obstructions to such a program. Specifically, in the work of Poincaré [Poi87] appeared for the first time the phenomena of instability with respect to initial conditions, a central concept in the understanding of modern Dynamical Systems. In fact, we will see briefly that such instability phenomena can be already observed in very simple systems–such as a periodically forced pendulum–that exhibit a so called "homoclinic tangle" [Mos01, PT93].

The realization that many relevant systems are very sensitive with respect to the initial conditions dealt a strong blow to the idea that it is always possible to predict the future behavior of a system,[1] yet the work of many physicist (and we must mention at least Boltzmann) and mathematicians (in particular, the so called *Russian School* with people like Kolmogorov, Anosov, Sinai, but also some western mathematicians, like Birkhoff, Smale, Ruelle and Bowen, gave important contributions) led to the understanding that, although precise predictions where not possible, it was possible and, at times, even

[1]Without going to the extreme of some authors of the eighteen century arguing that, given the present state of the universe, a sufficiently powerful mind (maybe God) could predict all the future. Think, more modestly, of an isolated system and imagine to use some numerical scheme to try to solve the equations of motion for an arbitrarily long time with an arbitrary precision.

easy to make statistical predictions. The concept of statistical properties of a Dynamical System will be addressed in the following chapters This chapter is dedicated to making precise, in a simple example, the nature of the above mentioned instability.

5.1 A pendulum–The model and a question

We will study a seemingly trivial example: a forced pendulum. To be more concrete, let us imagine a pendulum of length $l = 1$ meter, mass $m = 1$ kilogram and remember that the gravitational constant (on the earth surface) is approximately $g = 9.8$ meters per second squared. The Hamiltonian of the system reads [Gal83]

$$H = \frac{1}{2l^2m}p^2 - mgl\cos\theta, \tag{5.1.1}$$

where θ is the angle, counted counterclockwise, formed by the pendulum with the vertical direction ($\theta = 0$ corresponds to the configuration in which the pendulum assumes the lowest possible position) and $p = l^2m\dot{\theta}$ is the associated momentum. Thus (θ, p) are the coordinates of the pendulum. The phase space $\mathcal{M}$ where the motion takes place consists of $\mathbb{T}^1 \times \mathbb{R}$.

The equations of motion associated to the Hamiltonian (5.1.1) represent the motion of an ideal pendulum in the vacuum feeling only the force of gravity. Clearly, this is an highly idealized situation with no counterpart in realty. Every system interacts with the rest of the universe. Thus the only hope for the idea of *isolated systems* to be fruitful is that the interaction with the exterior does not affect significantly the behavior of the system. Let us try to see what this can mean in reality.

The first issue is clearly friction. Let us imagine that we have set up the pendulum in a reasonable vacuum and reduced the friction at the suspension point so that the loss of energy is negligible on the time scale of few minutes. Does such a system behaves as an isolated pendulum within such a time frame? One problem is that the suspension point is still in contact with the rest of the world. If the pendulum is in a lab not so distant from an street (a rather common situation), then the traffic will induce some vibrations. It is then natural to ask: what happens if the suspension point of the pendulum vibrates?

In fact, nothing much happens for small pendulum oscillations (this is a consequence of Komogorv-Arnold-Moser theory, an highly non trivial fact), but if we start close to the vertical configuration it is conceivable that a motion that would be oscillatory for the unperturbed pendulum could gather enough energy from the external force as to change its nature and become rotatory, this would create a substantial difference between the unperturbed (ideal) and the perturbed (more realistic) case.

This is exactly the question we want to address:

Question: *Can we really predict the motion for a reasonable time if the initial condition is close to the vertical ?*

We will assume that the frequency of vibration ω is of the order of one hertz[2] and the amplitude of the oscillations is very very small. Hence, as good mathematicians, we will call such an amplitude ε. In other words, the suspension point moves vertically according to the law $\varepsilon \cos \omega t$.

The Hamiltonian of the vibrating pendulum is then given by (see Problem 5.1)

$$H_\varepsilon(\theta, p, t) = \frac{1}{2l^2 m} p^2 - mgl\cos\theta - \varepsilon m\omega^2 l \cos\omega t \cos\theta. \tag{5.1.2}$$

Accordingly the equation of motion are (see Problem 5.1)[3]

$$\begin{aligned} \dot\theta &= \frac{\partial H_\varepsilon}{\partial p} = \frac{p}{l^2 m} \\ \dot p &= -\frac{\partial H_\varepsilon}{\partial \theta} = -mgl\sin\theta - \varepsilon m\omega^2 l\cos\omega t \sin\theta. \end{aligned} \tag{5.1.3}$$

It is well known that the function H is an integral of motion for the solutions of (5.1.3) for $\varepsilon = 0$, that is: H computed along the solutions of the associated equations of motion is constant.[4] The physical meaning of H is the energy of the system. Clearly, the energy H_ε is not constant in general since the vibration can add or subtract energy to the pendulum.

5.2 Instability–unperturbed case

Let us first recall few basic facts about the unperturbed pendulum. The equation of motions are given by the (5.1.3) setting $\varepsilon = 0$. It is obvious that there exists two fixed points: $(0, 0)$ which corresponds to the pendulum at rest and is clearly stable , and $(\pi, 0)$ which corresponds to the pendulum in the vertical position and is certainly unstable. Our interest here is to analyze the motions that start close to the unstable equilibrium and to make more precise what it is meant by *instability*.

[2]One hertz corresponds to one oscillation every second, and it can be the order of magnitude for the frequency of a vibration transmitted through the ground (R waves) at a reasonable distance. Thus we are assuming $\omega = 2\pi$.

[3]Here we write the Hamilton equations associated to the Hamiltonian, see [Arn99, Gal83] for the general theory.

[4]See [Arn99, Gal83] for this general fact or do Problem 5.4 for the simple case at hand.

5.2.1 Unstable equilibrium

If we want to have an idea of how the motion looks like near a fixed point the natural first step is to study the linearization of the equation of motion near such a point. In our case, using the coordinates $(\theta_0, p) = (\theta - \pi, p)$, they look like

$$\begin{aligned} \dot{\theta}_0 &= \frac{p}{l^2 m} \\ \dot{p} &= mgl\theta_0. \end{aligned} \tag{5.2.4}$$

Let $\omega_p = \sqrt{\frac{g}{l}}$, the general solution of (5.2.4) is

$$(\theta_0(t), p(t)) = (\alpha e^{\omega_p t} + \beta e^{-\omega_p t}, ml^2\omega_p\{\alpha e^{\omega_p t} - \beta e^{-\omega_p t}\}),$$

where α and β are determined by the initial conditions. Note that if the initial condition has the form $\alpha(1,\, ml\sqrt{gl})$ it will evolve as $\alpha e^{\omega_p t}(1,\, ml\sqrt{gl})$. While if the initial condition is of the form $\beta(1,\, -ml\sqrt{gl})$ it will evolve as $\beta e^{-\omega_p t}(1,\, -ml\sqrt{gl})$. In other words the directions $(1,\, ml\sqrt{gl})$ and $(1,\, -ml\sqrt{gl})$ are invariant for the linear dynamics. The first direction is expanded (and because of this is called *unstable direction*) while the second is contracted (*stable direction*).

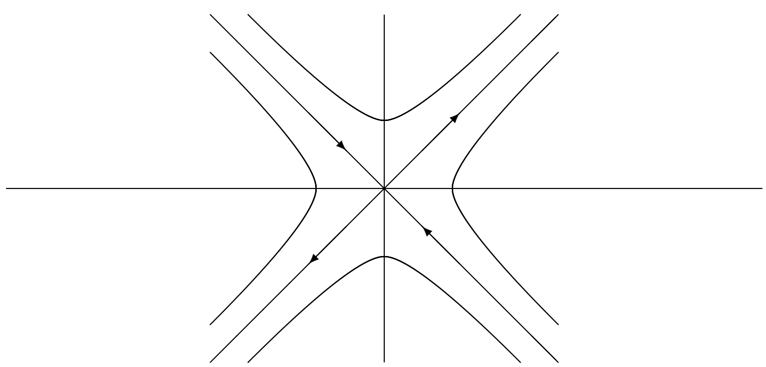

Figure 5.1: Unstable fixed point (phase portrait)

Let us imagine to start the motion from an initial condition of the type $(\pi + \theta_0, 0)$, $\theta_0 \in [-\delta, \delta]$, where $\delta \leq 10^{-4}$ represents the precision with which we are able to set the initial condition (one tenth of a millimeter); what will happen under the linear dynamics?

Our initial condition correspond to choosing, at time zero, $\alpha = \beta \leq \frac{\delta}{2}$. As time goes on the coefficient of β becomes exponentially small while the coefficient of α increases exponentially, thus a good approximation of the position of the pendulum after some time is given by

$$\theta_0(t) \approx \alpha e^{\omega_p t}. \tag{5.2.5}$$

Since $\omega_p \approx 3.13$ seconds^{-1}, it follows that after about 2.5 seconds the position of the pendulum can be anywhere up to a distance of about 10 centimeters from the unstable position.

This means that the unstable position is really unstable and if we tray, as best as we can, to put the pendulum in the unstable equilibrium (always imagining that the friction has been properly reduced) it will typically fall after few seconds and it will fall in a direction that we are not able to predict (since it depend on the sign of δ, our unknown mistake). Nevertheless, after the ideal pendulum starts falling in one direction the subsequent motion is completely predictable, as we will see shortly.

An obvious objection to the above analysis is that I did not show that the linearized equation describes a motion really close to the one of the original equations. The answer to this question is particularly simple in this setting and is addressed in the next subsection.

5.2.2 The unstable trajectories (separatrices)

Given the already noted fact that, for $\varepsilon = 0$, H is a constant of motion, the phase space $\mathcal{M}$ is naturally foliated in the level curves of H, on which the motion must take place. This allows us to obtain a fairly accurate picture of the motions of the unperturbed pendulum. In fact, the level curves are given by the equations

$$\frac{p^2}{2l^2m} - mgl\cos\theta = E$$

where E is the energy of the motion. It is easy to see that $E = -mgl$ corresponds to the stable fixed point $(\theta, p) = (0, 0)$; $-mgl < E < mgl$ corresponds to oscillations of amplitude $\arccos\left[\frac{E}{mgl}\right]$; $E > mgl$ corresponds to rotatory motions of the pendulum. The last case $E = mgl$ is of particular interest to us: obviously it corresponds to the unstable fixed point $(\pi, 0)$, yet there are other two solution that travel on the two curves

$$p = \pm ml\sqrt{2lg(1 + \cos\theta)}.$$

This two curves are the ones that separate the oscillatory motions from the rotatory ones and, for this reasons, are called *separatrices*. It is very important to understand the motion along such trajectories, luckily the two differential equations

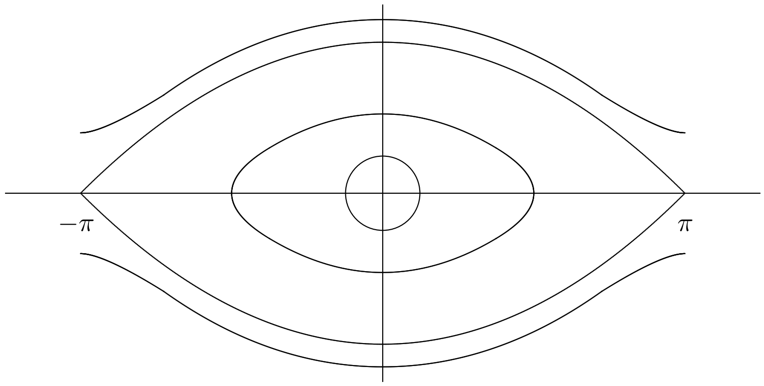

Figure 5.2: Unperturbed pendulum (phase portrait)

$$\dot{\theta} = \pm\sqrt{2\frac{g}{l}(1+\cos\theta)}. \tag{5.2.6}$$

can be integrated explicitly (see Problem 5.5) yielding, for $\theta(0) = 0$,

$$\theta(t) = 4 \arctan e^{\pm\omega_p t} - \pi. \tag{5.2.7}$$

This orbits are asymptotic to the unstable fixed point both at $t \to +\infty$ and at $-\infty$ and, for $|t|$ large, agree with the linear behaviour of section 5.2.1. This situation is somewhat atypical as we will see briefly.

5.3 The perturbed case

5.3.1 Reduction to a map

The motion of the above system takes place on the cylinder $\mathcal{M} = S^1 \times \mathbb{R}$. By the theorem of existence and uniqueness for the solutions of differential equations follows immediately the possibility to define the maps $\phi_\varepsilon^t : \mathcal{M} \to \mathcal{M}$ associating to the point (θ, p) the point reached by the solution of (5.1.3) at time t, when starting at time 0 from the initial condition (θ, p). In such a way we define the *flow* ϕ_ε^t associated to the (5.1.3).

Clearly $\phi_\varepsilon^0(\theta, p) = (\theta, p)$, that is the map corresponding at time zero is the identity. Moreover, if $\varepsilon = 0$ the system is autonomous (the vector field does not depend on the time) hence the flow defines a group: for each $t, s \in \mathbb{R}$

$$\phi_0^{t+s}(\theta, p) = \phi_0^t(\phi_0^s(\theta, p)).$$

This corresponds to the obvious fact that the motion for a time $t + s$ can be obtained first as the motion from time 0 to time s, and then pretending that the time s is the initial time and following the motion for time t.

Of course, the above fact does not hold anymore when $\varepsilon \neq 0$. In this case, the maps ϕ_ε^t depend from our choice of the initial time (if we define them by starting from time 1 instead then time 0, in general we obtain different maps). Nevertheless, due to the fact that the external force is periodic something can be saved of the above nice property.

Let us define the map $T_\varepsilon : \mathcal{M} \to \mathcal{M}$ by

$$T_\varepsilon = \phi_\varepsilon^{\frac{2\pi}{\omega}},$$

then (see Problem 5.3), for each $n \in \mathbb{Z}$,

$$T_\varepsilon^n = \phi_\varepsilon^{\frac{2n\pi}{\omega}}. \tag{5.3.8}$$

The interest of (5.3.8) is that, for many purposes, we can study the map T_ε instead than the more complex object ϕ_ε^t. Morally, it means that if we look at the system *stroboscopically*, that is only at the times $\frac{2\pi}{\omega}n$ with $n \in \mathbb{Z}$, then it behaves like an autonomous (time independent) system.[5] Another interesting fact is that the flow ϕ_ε^t (and hence also the map T_ε) is area preserving (see Problem 5.7).[6]

5.3.2 Perturbed pendulum, $\varepsilon \neq 0$

The situation for the case $\varepsilon \neq 0$ is more complex and no easy way exists to study these motions.

As a general strategy, to study the behavior of a system (in our case the map T_ε) it is a good idea to start by investigating simple cases and then move on from there. In our systems the simplest motion consists of the equilibrium solutions. These are the time independent solutions.[7] Because of the special type of perturbation chosen the fixed points of the system for the case $\varepsilon = 0$ remain unchanged when $\varepsilon \neq 0$ (see Problem 5.8 for a brief discussion of a more general case).

Next, we can study the infinitesimal nature of the fixed points. It is natural to expect that the nature of the two fixed points does not change if ε is small,

[5]This is a very simple case of a very fruitful an general strategy: to look at the system only when some special event happens–in our case at each time in which the suspension point has its maximum height. See 6.2 if you want to know more.

[6]This also is a special instance of a more general fact: the Hamiltonian nature of the system, see [Arn99, Gal83] if you want to know more.

[7]That is, equilibrium solutions for the map T_ε. These are *periodic* solutions for the flows of period $\frac{2\pi}{\omega}$. In fact, $T_\varepsilon x = x$ means $\phi^{\frac{2\pi}{\omega}} x = x$.

yet to verify this requires some checking. We will discuss explicitly only the fixed point $(\pi, 0)$.

The first step is to make precise the sense in which the case $\varepsilon \neq 0$ is a perturbation of the case $\varepsilon = 0$. This can be achieved by obtaining an explicit estimate on the size of

$$R_\varepsilon = \varepsilon^{-1}(T_0 - T_\varepsilon).$$

Let $z(t) = (z_1(t), z_2(t)) = \phi_0^t(x) - \phi_\varepsilon^t(x)$, then substituting in (5.1.3) and subtracting the general case to the case $\varepsilon = 0$ it yields

$$\begin{aligned} |\dot{z}_1| &\leq \frac{|z_2|}{ml^2} \\ |\dot{z}_2| &\leq mgl|z_1| + \varepsilon m\omega^2 l. \end{aligned}$$

In order to get better estimates it is convenient to define the new variables $\zeta_1 = z_1$ and $ml^2\omega_p\zeta_2 = z_2$. In these new variables the preceding equations read

$$\begin{aligned} |\dot{\zeta}_1| &\leq \omega_p|\zeta_2| \\ |\dot{\zeta}_2| &\leq \omega_p|\zeta_1| + \varepsilon\frac{\omega^2}{\omega_p l}. \end{aligned} \tag{5.3.9}$$

Which implies $\|\dot{\zeta}\| \leq \omega_p\|\zeta\| + \varepsilon m\omega^2 l$. Taking into account that, in our situation, $ml^2\omega_p > 1$, it follows (see Problem 5.9)

$$\|R\|_{\mathcal{C}^0} \leq \frac{m\omega^2}{l\omega_p}(e^{2\pi\frac{\omega_p}{\omega}} - 1) \leq 69.$$

Unfortunately, the above norm does not suffice for our future needs. We will see quite soon that it is necessary to estimate also the first derivatives of R, that is the $\mathcal{C}^1$ norm.

To do so the easiest way is to use the differentiability with respect to the initial conditions of the solutions of our differential equation. Fixing any point $x \in \mathcal{M}$ and calling $\xi^\varepsilon(t) = d_x\phi_\varepsilon^t\xi(0)$ we readily obtain:[8]

$$\begin{aligned} \dot{\xi}_1^\varepsilon &= \frac{\xi_2^\varepsilon}{l^2 m} \\ \dot{\xi}_2^\varepsilon &= -mgl\cos\theta\,\xi_1^\varepsilon - \varepsilon m\omega^2 l\cos\omega t\cos\theta\,\xi_1^\varepsilon \end{aligned} \tag{5.3.10}$$

[8]The vector $\xi_\varepsilon(t)$ is nothing else than the derivative $\frac{d\phi_\varepsilon^t(x+s\xi(0))}{ds}|_{s=0}$, the following equation is then obtained by exchanging the derivative with respect to t with the derivative with respect to s.

One can then estimate the $\mathcal{C}^1$ norm of R by estimating $\|\xi^\varepsilon(\frac{2\pi}{\omega}) - \xi^0(\frac{2\pi}{\omega})\|$, since $\xi^\varepsilon(\frac{2\pi}{\omega}) = D_{(\theta,p)}T_\varepsilon\xi^\varepsilon(0)$. Doing so one obtains[9]

$$\|R\|_{\mathcal{C}^1} \leq \frac{2m\omega^2}{l\omega_p}e^{3\pi\frac{\omega_p}{\omega}} := d_1 \leq 690. \tag{5.3.11}$$

5.4 Infinitesimal behavior (linearization)

As a first application of the above considerations let us study the linearization of T_ε at $x_f = (\pi, 0)$. From (5.3.10) follows (see Problem 5.12)

$$\begin{aligned} D_{x_f}T_0 &= \begin{pmatrix} \cosh\frac{2\pi\omega_p}{\omega} & \frac{\sinh\frac{2\pi\omega_p}{\omega}}{ml^2\omega_p} \\ ml^2\omega_p\sinh\frac{2\pi\omega_p}{\omega} & \cosh\frac{2\pi\omega_p}{\omega} \end{pmatrix} \\ D_{x_f}T_\varepsilon &= D_{x_f}T_0 + \mathcal{O}(d_1\varepsilon) \end{aligned} \tag{5.4.12}$$

The eigenvalues of $D_{x_f}T_\varepsilon$ are then $\lambda_\varepsilon = e^{\frac{2\pi\omega_p}{\omega}} + \mathcal{O}(d_2\varepsilon)$,[10] λ_ε^{-1}, where $d_2 = 2d_1\omega_p ml^2 \simeq 4400$. In addition, calling v_ε, $\langle v_\varepsilon, v_0\rangle = 1$, the eigenvector associate to λ_ε, holds true $\|v_0 - v_\varepsilon\| \leq d_3\varepsilon$, $d_3 = 4\lambda_0^{-1}\omega_p^2\omega^2l^4d_1 \simeq 1200$.[11]

Clearly, if ε is sufficiently small, then $\lambda_\varepsilon > 1$. This means that the hyperbolic nature of the unstable fixed point remains unchanged under small perturbations (see Problem 5.13 for a case when the perturbation is not so small).[12]

If one does a similar analysis at the fixed point $(0,0)$ one finds that the eigenvalues have modulus one: that is the infinitesimal motion is a rotation around the fixed point, exactly as in the $\varepsilon = 0$ case.

Hence the comments made at the end of subsection 5.2.1 for the unperturbed pendulum hold for the perturbed pendulum as well. Only now the is no longer an integral of motion (the energy) that controls globally the behavior of the system.

Imagining that the map is linear (which is clearly false but, as we will see, qualitatively not so wrong) this would mean that the distance between two trajectories can be expanded by almost a factor 23 in a second. Initial

[9] The following bounds are not sharp, working more one can obtain better estimates but this would not make much of a difference in the sequel.

[10] In this chapter we will adopt the strict convention that $\mathcal{O}(x)$ means a quantity bounded, in absolute value, by x.

[11] This follows by the fact that the eigenvalues of $D_{x_f}T_0$ are $e^{\pm\frac{2\pi\omega_p}{\omega}} \simeq (23)^{\pm 1}$, a simple perturbation theory of matrices (see Problems 5.10, 5.11) and the already mentioned fact that the map T_ε is area preserving, thus the determinant of its derivative must be one.

[12] As we will see later in detail, hyperbolicity means that there is a direction in which the maps expands (the eigenvector v_ε^u associated to the eigenvalue λ_ε) and a direction in which the map contracts (the eigenvector v_ε^s associated to the eigenvalue λ_ε^{-1})

conditions that are δ close at time zero will be about 23δ far apart after 1 second. If such a state of affair could persist (and we will see it may) after one minute the two configurations would differ roughly by a factor $10^{80}\delta$, which means that not even knowing the initial condition plus or minus a quark could we predict the final one. This is certainly a rather worrisome perspective but much more work it is needed to decide if this may be indeed the case.

5.5 Local behavior (Hadamard-Perron Theorem)

The next step is to try to go from the above infinitesimal analysis to a local picture in a small neighborhood of the fixed points.

It is natural to expects that the two fixed points are still stable and unstable respectively, yet this is a far from trivial fact.

The stability of the point $(0, 0)$ can be proven by invoking the so called KAM Theorem (this exceeds the scope of the present book and we will not discuss such matters, see [Gal83] for such a discussion).[13]

The study of the local behavior around the point x_f is instead a bit easier and can be performed by applying the Hadamard-Perron Theorem 2.4.2 to conclude that, in a neighborhood of $(\pi, 0)$, there exists two curves $x^u_\varepsilon(s) = (\theta^u_\varepsilon(s), p^u_\varepsilon(s))$, $x^s_\varepsilon(s)$ that are invariant with respect to the map T_ε. Namely, there exists $\delta_\varepsilon > 0$ such that $T_\varepsilon x^s_\varepsilon([-\delta_\varepsilon, \delta_\varepsilon]) \subset x^s_\varepsilon([-\delta_\varepsilon, \delta_\varepsilon])$ and $T_\varepsilon^{-1} x^u_\varepsilon([-\delta_\varepsilon, \delta_\varepsilon]) \subset x^u_\varepsilon([-\delta_\varepsilon, \delta_\varepsilon])$; this are called the local stable and unstable manifold of zero, respectively. Essentially δ_ε is determined by the requirement that the non-linear part of T_ε be smaller than the linear part.

Clearly, for $\varepsilon = 0$ $x^s_0 = x^u_0 = x_0$ and it coincides with the homoclinic orbit of the unperturbed pendulum. In addition, by Hadarmd-Perron and the estimates of the previous section, we can choose δ_ε such that

$$\|x^u_\varepsilon - x_0\| \leq 2d_3\varepsilon\|x_0\|. \tag{5.5.13}$$

and the analogous for the stable manifold. We have so obtained a local picture of the behavior of the map T_ε, yet this does not suffice to answer to our original question. To do so we need to follow the motion for at least a full oscillation: this requires really a global information.

[13] In some sense this implies that we can indeed predict the motion for an extremely long time if we consider only oscillations close to the configuration $(0, 0)$, so in that case the assumption that the pendulum is isolated is legitimate. Yet, this depends on the precision we are interested in and tends to degenerate if the amplitude of the oscillations is rather large. A complete analysis would be a very complicated matter but we will have an idea of the type of problems that can arise by considering extremely large oscillations, close to a full rotation of the pendulum.

To gain a more global knowledge we can try to construct larger invariant set for the map T_ε. A natural way to do so is to iterate: define $W^u = \cup_{n=0}^{\infty} T_\varepsilon^n x^u([-\delta_\varepsilon, \delta_\varepsilon])$. Since $T_\varepsilon x^u([-\delta_\varepsilon, \delta_\varepsilon]) \supset x^u([-\delta_\varepsilon, \delta_\varepsilon])$, it is clear that each time we iterate we get a longer and longer curve. The set W^u is then clearly a manifold and it is called the global unstable manifold.[14]

The global manifold, as the name clearly states, it is a global object: it carries information on the dynamics for arbitrarily long times. Yet, the procedure by which it has been defined is far from constructive and the truth is that, besides the sketchy considerations above, at the moment we know very little of it. The next step is to gain some more detailed understanding of a large portion of W^u.

5.6 A more global understanding (the Melnikov method)

From the above considerations follows that the stable and unstable manifolds $(\theta_\varepsilon^s(s),\, p_\varepsilon^s(s))$, $(\theta_\varepsilon^u(s),\, p_\varepsilon^u(s))$, $|s| \leq \delta_\varepsilon$, of T_ε at 0, are ε close to the homoclinic orbit of the unperturbed pendulum, $(\theta_0(t),\, p_0(t))$, $\theta_0(0) = 0$.

Note, however, that while $x_0 = (\theta_0, p_0)$ is invariant under the unperturbed flow, the same does not apply to $(\theta_\varepsilon^{s,u}(s),\, p_\varepsilon^{s,u}(s))$ under ϕ_ε^t. Indeed the invariant object is the time-space surface $(\tau, x_\varepsilon^{s,u}(s,\tau)) := (\tau, \phi_\varepsilon^\tau(\theta_\varepsilon^s(s),\, p_\varepsilon^s(s)))$ where $(s,\tau) \in [-\delta_\varepsilon, \delta_\varepsilon] \times [0, \frac{2\pi}{\omega}]$ and and $\tau = t \mod \frac{2\pi}{\omega}$.[15]

Clearly, we can choose freely the parameterization of our curves in such a surface and some are more convenient than others. The separatrix of the unperturbed pendulum is most conveniently parametrized by time, hence $\phi^t(\theta_0(s), p_0(s)) = (\theta_0(s+t), p_0(s+t))$. We wish to parameterize the perturbed manifold in a convenient way, one simple possibility could be to impose $\theta_\varepsilon^u(-s) = \theta_0(-s)$, $\theta_\varepsilon^s(s) = \theta_0(s)$, yet this happens to be not very helpful for our goals. To find a more convenient parameterization it is necessary to do first some preliminary considerations.

To grow the above manifolds, as explained in the previous section, we can start from some remote time $-S_n := 2\pi\omega^{-1}n$, $n \in \mathbb{N}$, (S_n for the stable) and then iterate forward the unstable manifold and backward the stable. This is better done by using the flow and the equations of motion. To this end, it turns

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[14] Applying the above procedure to the unperturbed problem yields the full separatrix.

[15] A standard way to bring the present non-autonomous setting in the more familiar autonomous one is to introduce the fake variables $(\varphi, \eta) \in S^1 \times \mathbb{R}$ and the new, time independent, Hamiltonian $\bar{H}_\varepsilon(\theta, p, \varphi, \eta) := H_\varepsilon(\theta, p, \varphi) + \frac{2\pi}{\omega}\eta$. The Hamilton equations yield $\varphi(t) = \frac{2\pi}{\omega}t + \varphi(0)$ and hence the equations for θ, p reduce to (5.1.3). Since $\bar{H}_\varepsilon$ is now conserved under the motion we can restrict the system to the three dimensional manifold $\bar{H}_\varepsilon = 0$. In such a manifold we have the *weak* stable and unstable manifolds (now flow invariant) $(x_\varepsilon^{s,u}(s,\varphi), \varphi, -\frac{2\pi}{\omega}H_\varepsilon((x_\varepsilon^{s,u}(s,\varphi), \varphi))$.

out to be specially smart to first use global coordinates similar to the ones used to simplify equation (5.3.9) and then to consider local coordinates adapted to the separatrix of the unperturbed pendulum. Namely, let us introduce $p =: ml^2\omega_p\tilde{p}$, $\theta =: \tilde{\theta}$. Note that such a change of coordinate is not symplectic, hence we have to compute the resulting Hamiltonian in the new coordinates. It is easy to verify that the Hamiltonian becomes

$$\tilde{H}_\varepsilon := \frac{\omega_p}{2}\tilde{p}^2 - \omega_p \cos\tilde{\theta} - \varepsilon\frac{\omega^2}{l\omega_p}\cos\omega t\cos\tilde{\theta} =: H_0 + \varepsilon H_1 \tag{5.6.14}$$

which indeed yields the correct equations of motion:

$$\begin{aligned} \dot{\tilde{\theta}} &= \omega_p\tilde{p} \\ \dot{\tilde{p}} &= -\omega_p\sin\tilde{\theta} - \varepsilon\frac{\omega^2}{l\omega_p}\cos\omega t\sin\tilde{\theta} \end{aligned} \tag{5.6.15}$$

We will use the vector notation $x := (\tilde{\theta}, \tilde{p})$.[16] In such coordinates we consider the stable and unstable manifolds $x^s_\varepsilon(s), x^u_\varepsilon(s)$ for the perturbed pendulum and the separatrix $x_0(s)$ for the unperturbed pendulum and we define

$$x^{s,u}_\varepsilon(s,\, t) = \phi^t_\varepsilon x^{s,u}_\varepsilon(s). \tag{5.6.16}$$

If we call ϕ^t_ε the flow started at the time $-S_n$ (S_n, respectively),[17] and we consider $t = S_m$ ($t = -S_m$), $m < n$, we obtain new curves that are much longer than the original ones and still describe the unstable and stable manifolds (albeit with a different parameterization). Next, we define the vectors

$$\eta_1(s) := \frac{\dot{x}_0(s)}{\|\dot{x}_0(s)\|} = \frac{J\nabla_{x_0(s)}H_0}{\|\nabla_{x_0(s)}H_0\|} \quad \text{and} \quad \eta_2(s) := \frac{\nabla_{x_0(s)}H_0}{\|\nabla_{x_0(s)}H_0\|}.$$

This form an orthonormal basis of $\mathbb{R}^2$ (see Problem 5.14). We can then consider the map $F(a,b) := x_0(a) + b\eta_2(a)$. One can check that $\det DF_{(a,0)} \neq 0$, hence F defines a change of coordinates in a neighborhood of x_0. Note that in the new coordinates the unperturbed separatrix x_0 reads $\{(a,0)\}$.

In analogy with a standard approach to the Hadamard-Perron Theorem (see 2.4.2) it seems natural to have our curves parametrized so that, in the new coordinates, they have the same first component. This means that we

[16] Using such a notation equations (5.6.15) take the more compact form

$$\dot{x} = J\nabla_x\tilde{H}_\varepsilon\ ; \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

[17] Remember that the flow started at such times is exactly the same than the flow started at time zero, see subsection 5.3.1.

would like to have $\langle x^u_\varepsilon(s)-x_0(s), \eta_1(s)\rangle = 0$. We can obviously arrange such a property for the original curve at the tine $-S_n$, but can we keep it thruought the growth process? A simple possibility is to flow different points for different times as to maintain the wanted property. That is to look for a τ such that,[18]

$$G(s,t,\tau) := \langle x^u_\varepsilon(s,t+\tau) - x_0(s+t), \eta_1(s+t)\rangle = 0. \tag{5.6.17}$$

Since, by construction, $G(s,0,0) = 0$ we can apply the implicit function theorem, to prove the existence of the wanted function $\tau(s,t)$. The necessary condition to do so is a lower bound on $|\partial_\tau G|$. Next, setting $x^u_\varepsilon(s,t+\tau) =: x_0(s+t) + \varepsilon x^u_1(s,t,\tau)$,

$$\begin{aligned}\partial_\tau G(s,t,\tau) &= \langle J\nabla_{x^u_\varepsilon(s,t+\tau)}\tilde H_\varepsilon, \eta_1(s+t)\rangle \\ &= \|\nabla_{x^u_0(s+t)}H_0\| + \varepsilon\mathcal{O}(\|D^2\tilde H_\varepsilon\|\,\|x^u_1\| + \|\nabla_{x^u_0}H_1\|).\end{aligned} \tag{5.6.18}$$

By (5.5.13) we have $\|x^u_\varepsilon(s) - x_0(s)\|\,\|x_0(s)\|^{-1} \le 2d_3\varepsilon$, for $s \le -T_{n_0}$. In addition, from (5.2.7) and Problem 5.6 follows $\sin\tilde\theta_0(t) = 2\frac{\sinh\omega_p t}{(\cosh\omega_p t)^2} \simeq 2e^{\omega_p t}$, for $t \ll 0$. Moreover $\tilde p_0 = \sqrt{2(1+\cos\tilde\theta_0)} = 2(\cosh\omega_p t)^{-1}$. Then $\|\nabla_{x_0(t)}H_0\| \ge \frac{\omega_p}{\sqrt 2}e^{-\omega_p|t|}$.

Accordingly, remembering equations (5.5.13) and (5.6.18) we can apply the Implicit Function Theorem provided $\|x^u_1(s,t,\tau)\| \le 4d_3 e^{-\omega_p|s+t|}$ and $\varepsilon \le (8d_3)^{-1} \simeq 10^{-4}$. Hence the wanted function $\tau(s,t)$ is well defined and[19]

$$\frac{\partial\tau}{\partial t} = -\frac{\partial_t G}{\partial_\tau G} = \mathcal{O}(64 d_3\varepsilon). \tag{5.6.19}$$

It is then convenient to define

$$\Delta^u(s,\,t) = \varepsilon^{-1}\langle x^u_\varepsilon(s,t+\tau) - x_0(s,t),\, \nabla_{x_0(s+t)}H_0\rangle = \|x^u_1\|\,\|\nabla_{x_0}H_0\|.$$

Using (5.1.3) we can differentiate Δ^u with respect to t and since $J\nabla_{x^u_\varepsilon}H_\varepsilon = J\nabla_{x_0}H_\varepsilon + \varepsilon JD^2_{x_0}H_\varepsilon x^u_1 + \mathcal{O}(\frac{\varepsilon^2}{2}\|D^3H_0\|\,|x^u_1|^2)$, we have

$$\begin{aligned}&\frac{d\Delta^u}{dt}(s,\,t) = \varepsilon^{-1}\langle J\nabla_{x^u_\varepsilon}H_\varepsilon(1+\dot\tau), \nabla_{x_0}H\rangle + \langle x^u_1, D^2_{x_0}H_0 J\nabla_{x_0}H_0\rangle \\ &= \Big\{\langle J\nabla_{x_0}H_1, J\nabla_{x_0}H_0\rangle + \mathcal{O}(2\varepsilon\omega_p d_3 e^{\omega_p(t+s)}|\Delta^u_1|)\Big\}(1+|\dot\tau|_\infty) \\ &\quad + \mathcal{O}(|\dot\tau|_\infty|\Delta^u_1|\omega_p).\end{aligned} \tag{5.6.20}$$

---

[18]Note that, in so doing, we will construct an object different from the starting one associated to a fixed Poincarè section.

[19]Indeed, $\partial_t G = \langle J\nabla_{x^u_\varepsilon}\tilde H_\varepsilon - J\nabla_{x_0}H_0, \eta_1\rangle + \varepsilon\langle x^u_1, \dot\eta_1\rangle = \varepsilon\mathcal{O}(3\omega_p\|x^u_1\| + \|\nabla_{x_0}H_1\|)$.

We can thus integrate the Gronwald type inequality (5.6.20), (if in doubt, see Problem 5.9), and, assuming $256\omega_p d_3\varepsilon < 1$ (roughly $\varepsilon \leq 10^{-5}$),

$$|\Delta^u(s,t)| \leq \frac{8\omega^2}{l\omega_p} e^{2\omega_p(t+s)}.$$

Hence, $\|x_1^u\| \leq \frac{24\omega^2 e^{\omega_p(t+s)}}{l\omega_p^2} < 4d_3 e^{-\omega_p|t+s|}$, provided it holds true $t+s \leq (2\omega_p)^{-1}\ln\left[\frac{d_3 l\omega_p^2}{6\omega^2}\right] =: t_0 \simeq 0.6$.

To gain complete control on the stable manifold we need only to discuss the issue of the time shift. On the one hand, all is needed is to change $t+\tau(s,t)$ to zero ($\mod \frac{2\pi}{\omega}$). On the other hand if $\rho \in [0, \frac{2\pi}{\omega}]$, then $\zeta(\rho) := \phi_\varepsilon^\rho(x) - \phi_0^\rho(y)$ can be estimated, slightly refining (5.3.9), by integrating $\|\dot{\zeta}\| \leq (\omega_p + \varepsilon\frac{\omega^2}{\omega_p l})\|\zeta\| + \varepsilon\frac{\omega^2}{\omega_p l}|\theta_0(t+s+\rho)|$. This shows that we can extend the unstable manifold till a neighborhood of $x_0(-S_2)$ and still keep the an inequality of the type $\|x_\varepsilon^u - x_0\| \leq 3d_3\|x_0\|$.

Finally, substituting the above estimate in (5.6.20), yields

$$\frac{d\Delta^u}{dt}(s,\,t) = \langle J\nabla_{x_0}H_1, J\nabla_{x_0}H_0\rangle + \mathcal{O}\left(544\cdot l^{-1}\omega^2 d_3\varepsilon e^{2\omega_p(t+s)}\right).$$

Integrating from 0 to S_m, $m\in\mathbb{N}$ for $s+S_m \leq t_0$, yields

$$\begin{aligned}\Delta^u(s,S_m) &= \int_0^{S_m}\{H_1(\cdot,t_1),\,H\}_{x_0(s+t_1)}dt_1 + \Delta^u(s,0) + \mathcal{O}\left(\varepsilon d_4 e^{2\omega_p(s+S_m)}\right)\\ &= \int_{-S_m}^{0}\{H_1(\cdot,t_1),\,H\}_{x_0(s+S_n+t_1)}dt_1 + \Delta^u(s,0) + \mathcal{O}\left(\varepsilon d_4 e^{2\omega_p(s+S_m)}\right)\end{aligned} \tag{5.6.21}$$

where $d_4 := 272\cdot\frac{\omega^2}{\omega_p l}d_3 \simeq 4\cdot 10^6$ and the curly brackets stand for the so called *Poisson brackets* ($\{f,\,g\}_x = \langle J\nabla_x f,\,\nabla_x g\rangle$).

The stable manifold can be studied similarly, yet it is faster to define the transformation $\Psi(\theta,p) = (-\theta,p)$, and note that $\phi_\varepsilon^{-t}(\Psi(x)) = \Psi(\phi_\varepsilon^t(x))$. Accordingly, $x_\varepsilon^s(s,-t) = \Psi(x_\varepsilon^u(-s,t))$. Also, one easily checks that, calling $\tau^s(s,t)$ the time shift arising from the analogous of (5.6.17), $\tau^s(s,-t) = \tau(-s,t)$. In addition, $|\tau(s,S_m)| \leq 65d_3\varepsilon S_m$.

Setting $\Delta(\sigma) := \Delta^u(-s-S_m,S_m) - \Delta^s(s+S_m,-S_m)$, for all $\sigma\in[-t_0,t_0]$, we finally have

$$\begin{aligned}&\|x_\varepsilon^u(-\sigma-S_m,S_m) - x_\varepsilon^s(S_m-\sigma,S_m)\| \leq 64\frac{4\pi\omega-p}{\omega}d_3\varepsilon S_m + \Delta(\sigma)\\ &\Delta(\sigma) = \int_{-\infty}^{\infty}\{H_1,\,H\}_{x_0(t+\sigma)}dt + \mathcal{O}\left(\varepsilon 2d_4 e^{2\omega_p|\sigma|}\right),\end{aligned} \tag{5.6.22}$$

provided $m > 2$. The integral in (5.6.22) is called *Melnikov integral* and provides an expression, at first order in ε, of the distance between the stable and the unstable manifold. All we are left with is to compute the integrals in (5.6.22). This turns out to be an exercise in complex analysis and it is left to the reader (see Problem 5.15), the result is:[20]

$$\int_{-\infty}^{\infty} \{H_1(\cdot, t),\, H\}_{x_0(t+\sigma)} dt = 8\pi ml \frac{\omega^4 e^{-\frac{\pi\omega}{2\omega_p}}}{\omega_p^2 (e^{\frac{\pi\omega}{\omega_p}} - 1)} \sin \omega\sigma.$$

We have thus gained a very sharp control on the shape of the above manifolds.[21] In particular, $\Delta(\pm 1/4) \simeq \pm 76 + \mathcal{O}(4 \cdot 10^7 \varepsilon) \neq 0$ provided $\varepsilon \leq 1.5 \cdot 10^{-6}$, that is the two manifolds intersect. To understand a bit better such an intersection (we would like to know that in the region $\sigma \in [-1/4, 1/4]$ there is only one *transversal* intersection) its suffices to notice that (5.6.20) provides a control on the angle between x^u_ε and x_0.

This intersections are called *homoclinic* intersection and their very existence is responsible for extremely interesting phenomena as can be readily seen by trying to draw the stable and unstable manifolds (see Figure 5.3 for an approximate first idea); we will discuss this issue in detail shortly.[22]

We have gained much more global information on the map T_ε, yet it does not suffice to answer to our question. The next section is devoted to obtaining a really global picture. Up to now we have used mainly analytic tools. Next, geometry will play a much more significant rôle.[23]

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[20] A simple computation yields:

$$\{H_1,\, H\}_{x_0(t+s)} = -\frac{\omega^2}{l} p(t+s) \cos \omega t \sin \theta(t+s).$$

Then, by using (5.2.7) and looking at Problem 5.6, one readily obtains:

$$\{H_1,\, H\}_{x_0(t)} = 4\frac{\omega^2}{l} \frac{\cos \omega(t-s) \sinh \omega_p t}{(\cosh \omega_p t)^3}.$$

Finally, use Problem 5.15.

[21] Note that ε must be exponentially small with respect to ω. In many concrete problems (notably the so called *Arnold diffusion* [**?**]) it happens that this it is not the case. One can try to solve such an obstacle by computing the next terms of the ε expansion of Δ. In fact, it turns out that it is possible to express Δ as a power series in ε with all the terms exponentially small in ω [**?**]. Yet this is a quite complex task far beyond our scopes.

[22] Note that the intersection corresponds to an homoclinic orbit for the map T_ε (that is, an orbit which approaches the fixed point x_f both in the future and in the past). This is what it is left of the homoclinic orbit of the unperturbed pendulum.

[23] What comes next is the first example in this book of what is loosely called *a dynamical argument*.

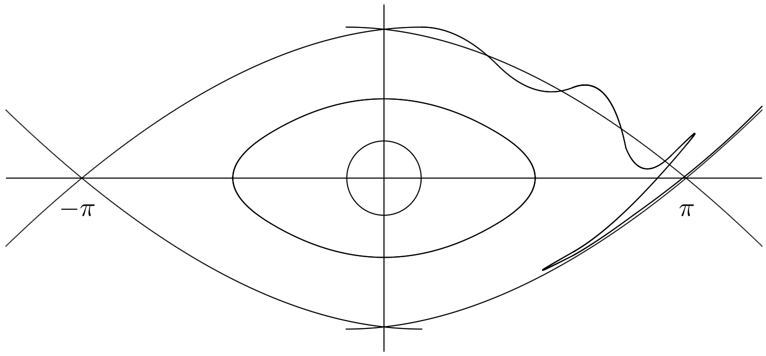

Figure 5.3: Perturbed pendulum

5.7 Global behavior (an horseshoe)

We want to explicitly construct trajectories with special properties. A standard way to do so is to start by studying the evolution of appropriate regions and to use judiciously the knowledge so gained. Let us see what this does mean in practice.

The starting point is to note that we understand the shape of the invariant manifold but not very well the dynamics on them, this is our next task. Since points on the unstable manifolds are pulled apart by the dynamics, the estimate must be done with a bit of care. In fact, we will use a way of arguing which it typical when instabilities are present, we will see many other instances of this type of strategy in the sequel.

For each x in the unstable manifold (zero included) let us call $D^u_x T_\varepsilon := D_x T_\varepsilon v^u(x)$, where $v^u(0) = v^u$ and if $x = x^u_\varepsilon(t)$ then $v^u(x) = \|\dot{x}^u_\varepsilon(t)\|^{-1}\dot{x}^u_\varepsilon(t)$, that is the derivative of the map computed along the unstable manifold. A useful idea in the following is the concept of *fundamental domain*. Define $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ by $x^u_\varepsilon(t) = x^u_\varepsilon(\alpha(t))$. Then $[t, \alpha(t)]$ is a fundamental domain and has the property that, setting $t_i := \alpha^i(t)$, the sets $\alpha^i[t_0, t_1]$ intersect only at the boundary.

Lemma 5.7.1 (Distortion) *For each x, y in the same fundamental domain of the unstable manifold, $\delta_0 > 0$, and $n \in \mathbb{N}$ such that $\|T^n_\varepsilon x\| \le \delta_0$, holds*[24]

$$e^{-\delta_0 C_2} \le \left|\frac{D^u_x T^n_\varepsilon}{D^u_y T^n_\varepsilon}\right| \le e^{\delta_0 C_2},$$

[24] This quantity is commonly called *Distortion* because it measures how much the map differs from a linear one (notice that if T is linear then $\frac{D_x T}{D_y T} = 1$). Although apparently an innocent quantity, it is hard to overstate its importance in the study of hyperbolic dynamics.

where $C_2 = \sup_{t\leq 0}\left|\frac{\ddot\alpha(t)}{\dot\alpha(t)}\right|$.

Proof. The proof is a direct application of the chain rule:

$$\left|\frac{D^u_x T^n_\varepsilon}{D^u_y T^n_\varepsilon}\right| = \prod_{i=1}^n \left|\frac{D^u_{T^i x} T_\varepsilon}{D^u_{T^i y} T_\varepsilon}\right| \leq \operatorname{Exp}\left[\sum_{i=1}^n |\log(|D^u_{T^i x}T_\varepsilon) - \log(|D^u_{T^i y}T_\varepsilon|)|\right]$$
$$\leq \operatorname{Exp}\left[\sum_{i=1}^n C_2\|T^i x - T^i y\|\right] = \operatorname{Exp}\left[\sum_{i=1}^n C_2\|x^u_\varepsilon(t_i) - x^u_\varepsilon(t_{i-1})\|\right] \leq e^{C_2\delta_0}.$$

The other inequality is obtained by exchanging the rôle of x and y. □

Next we would like to consider the evolution of a small box constructed around the fix point.

Consider the following small parallelogram: $Q_\delta := \{\xi \in \mathbb{R}^2 \mid \xi = av^u + bv^s \text{ for some } a, b \in [-\frac{\delta}{2}, \frac{\delta}{2}]\}$, $\delta \ll \delta_0$. Next consider the first $n \in \mathbb{N}$ such that $T^n_\varepsilon Q_\delta \cap \{\theta = 0\} \neq \emptyset$. Our first task is to understand the shape of $T^n_\varepsilon Q_\delta$ near $\{\theta = 0\}$. Since a fundamental domain in the latter region is of order one, while at the boundary of Q_ε is of order δ, Lemma 5.7.1 implies that the expansion is proportional to $C\delta^{-1}$. By the area preserving of the map it follows that $T^n_\varepsilon Q_\delta$ must be contained din a $C\delta^2$ neighborhood of the unstable manifold, see Figure 5.4.

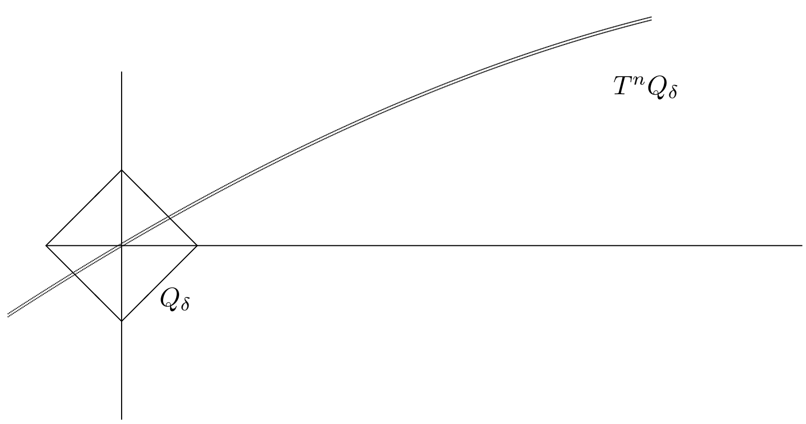

Figure 5.4: The evolution of the small box Q_δ

By the previous section considerations on the shape of the invariant manifolds $T^n Q_\delta \cap T^n Q_\delta \neq \emptyset$, moreover they intersect *transversally*.[25]

[25]The meaning of *transversally* is the following: the square Q_δ has two sides parallel to v^u (the unstable direction), which we will call unstable sides, and two sides parallel to v^s

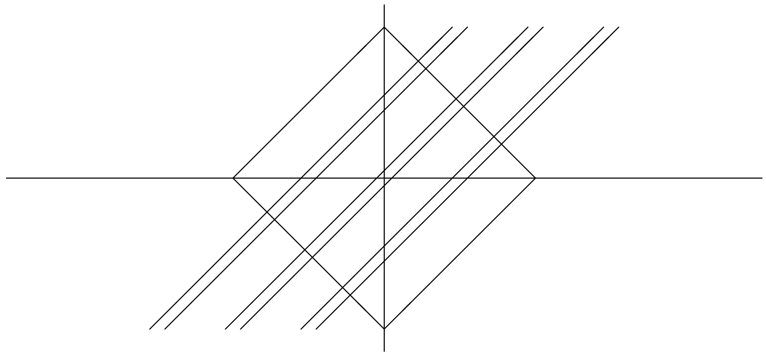

Figure 5.5: Horseshoe construction

This is all is needed to construct an horseshoe (see section ???). In particular, in our case it means that $T^{2n_0}Q_\delta \cap Q_\delta \neq \emptyset$, in fact the intersection are transversal and consist of three strips almost parallel to the unstable sides. One contains zero, and it is the lest interesting for us, the other two cross above and below the unstable manifold respectively. The with of such strip is about δ^{-3}. We will discuss in the next chapters all the implications of this situation, here it suffices to notice that if we have two initial conditions in $T^{-2n_0}Q_\delta \cap Q_\delta$ at a distance h, after $2n_0$ iterations the two points will be in Q_δ again but at a distance $h\varepsilon^{-1}$. Since to decide if after that there will be a rotation or an oscillation we need to know the final position with a precision of order δ, we need to know the initial position with a precision $\mathcal{O}(\delta\varepsilon) = \mathcal{O}(\delta^3)$.

Note that in the above construction we have lost almost all the points, only the ones that come back to Q_δ at time $2n_0$ are under control. Nevertheless, we can consider the set $\Lambda := \cup_{k\in\mathbb{Z}} \bigcap T_\varepsilon^{2kn_0} Q_\delta$. This is clearly a measure zero set, yet it is far from empty (it contains uncountably many points) and it is made of points that at times multiple of $2n_0$ are always in Q_δ. When they arrive in Q_δ they will rotate if they are above the separatrices and oscillate otherwise. Let us call this two subset of Q_δ R and O. Given a point $\xi \in Q_\delta$ we can associate to it the doubly infinite sequence $\sigma \in \{0,1\}^{\mathbb{Z}}$ by the rule $\sigma_i = 1$ iff $T^{2n_0 i}\xi \in R$. The reader can check that the correspondence is onto.

(the stable direction), which we will call stable sides. Then the intersection is transversal if it consists of a region with again four sides: two made of the image of the unstable sides and two made of images of stable sides of Q_δ.

5.8 Conclusion–an answer

If $\varepsilon = 10^{-6}$ and δ is a millimeter then we need to know the initial condition with a precision of 10^{-9} meters if we want to decide if the point will come back or rotate when it will get almost vertical again (this will happen in about 6 seconds). By the same token if we want to answer the same question, but for the second time the pendulum get close to the unstable position, we need to know the initial condition with a precision of the order 10^{-15} meters, and this just to predict the motion for about 12 seconds.[26]

We can finally answer to our original question:

Answer: *NO!*

Nevertheless, as we mentioned at the beginning, the above answer it is not the end of the story. In fact, there exists many other very relevant questions that can be answered.[27] The rest of the book deals with a particular type of question: can we meaningfully talk about the *statistical behavior* of a system?

Problems

5.1. Derive the Lagrangian, Hamiltonian and equations of motions for a pendulum attached to a point vibrating with frequency ω and amplitude ε. (Hint: see [LL76, Gal83] on how to do such things. Remember that two Lagrangian that differ by a total time derivative give rise to the same equation of motions and are thus equivalent.)

5.2. Consider the systems of differential equations $\dot{x} = f(x)$, $x \in \mathbb{R}^n$ and f smooth and bounded. Prove that the associated flow form a group. (Hint: use the uniqueness of the solutions of the ordinary differential equation)

5.3. Consider the systems of differential equations $\dot{x} = f(x, t)$, $x \in \mathbb{R}^n$ and f smooth, bounded and periodic in t of period τ. Let ϕ^t be the associated flow. Define $T = \phi^\tau$, prove that $T^n = \phi^{n\tau}$.

[26] Remark that it is not just a matter of precision on the initial condition, it is also a matter of how one actually does the prediction. If the method is to integrate numerically the equation of motion, then one has to insure that the precision of the algorithm is of the order of 10^{-15}. This maybe achieved by working in double precision but if one wants to make predictions of the order of one minute it is quite clear that the numerical problem becomes very quickly intractable.

[27] For example: which type of motions are possible? This is a *qualitative* question. Such type of questions give rise to the qualitative theory of Dynamical Systems [PT93, HK95], an extremely important part of the theory of dynamical systems, although not the focus here.

5.4. Show that the Hamiltonian is a constant of motion for the pendulum. (Hint: Compute the time derivative)

5.5. Prove (5.2.7). (Hint: Write (5.2.6) in the integral form

$$t = \int_0^t \frac{\dot\theta(s)}{\sqrt{\frac{2g}{l}(1+\cos\theta(s))}} ds.$$

Using some trigonometry and changing variable obtain

$$t = \int_0^{\theta(t)} \frac{1}{2\omega_p \cos\frac{\theta}{2}} d\theta.$$

and compute it.)

5.6. If $\theta(t)$ is the motion obtained in the previous problem, show that

$$\sin\theta(t) = 2\frac{\sinh\omega_p t}{(\cosh\omega_p t)^2}; \quad \cos\theta(t) = \frac{2}{(\cosh\omega_p t)^2} - 1;$$
$$\cos^2\frac{\theta(t)+\pi}{4} = \frac{1}{1+e^{2\omega_p t}}.$$

5.7. Consider the systems of differential equations $\dot x = f(x,t)$, $x \in \mathbb{R}^n$ and f smooth. Suppose further that $\mathrm{d}ivf = 0$ (that is $\sum_{i=1}^n \frac{\partial f_i}{\partial x_i} = 0$). Show that the associated flow preserves the volume. (Hint: note that this is equivalent to saying that $|\det d\phi^t| = 1$, moreover by the group property and the chain rule for differentiating it suffices to check the property for small t. See that $d\phi^t = \mathbb{1} + Dft + \mathcal{O}(t^2) = e^{Dft+\mathcal{O}(t^2)}$. Finally, remember the formula $\det e^A = e^{\mathrm{Tr}A}$.)

5.8. Let $T, T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth maps such that $T0 = 0$ and $\det(\mathbb{1} - D_0 T) \neq 0$. Consider the map $T_\varepsilon = T + \varepsilon T_1$ and show that, for ε small enough, there exists points $x_\varepsilon \in \mathbb{R}^2$ such that $T_\varepsilon x_\varepsilon = x_\varepsilon$. (Hint: Consider the function $F(x,\varepsilon) = x - T_\varepsilon x$ and apply the Implicit Function Theorem to $F = 0$.)

5.9. Let $x(t) \in \mathbb{R}^n$ be a smooth curve satisfying $\|\dot x(t)\| \le a(t)\|x(t)\| + b(t)$, $x(0) = x_0$, $a, b \in \mathcal{C}^0(\mathbb{R}, \mathbb{R}_+)$, prove that

$$\|x(t) - x_0\| \le \int_0^t e^{\int_s^t a(\tau)d\tau} \left[a(s)\|x_0\| + b(s)\right] ds.$$

(Hint: Note that $\|x(t) - x_0\| \le \int_0^t \|\dot x(s)\| ds$. Transform then the differential inequality into an integral inequality. Show that if $z(t) \le 0$ and $z(t) \le \int_0^t z(s)ds$, then $z(t) \le 0$ for each t. Use the last fact to compare a function satisfying the obtained integral inequality with the solution of the associated integral equation.)

5.10. Given two by two matrices A, B such that A has eigenvalues $\lambda \neq \mu$, show that the matrix $A_\varepsilon = A + \varepsilon B$, for ε small enough, has eigenvalues $\lambda_\varepsilon, \mu_\varepsilon$ analytic as functions of ε. Show that the same holds for the eigenvectors. (Hint:[28] consider z in the resolvent of A, that is $(z - A)^{-1}$ exists. Then $(z - A_\varepsilon) = (z - A)(\mathbb{1} - \varepsilon(z - A)^{-1}B)$. Accordingly, if ε is small enough, $(z - A_\varepsilon)^{-1} = \left\{\sum_{n=0}^\infty \varepsilon^n \left[(z - A)^{-1}B\right]^n\right\}(z - A)^{-1}$. Finally, if γ, γ' are curves on the complex plane containing λ and μ, respectively, verify that

$$\Pi_\varepsilon := \frac{1}{2\pi i}\int_\gamma (z - A_\varepsilon)^{-1} dz \qquad \Pi'_\varepsilon := \frac{1}{2\pi i}\int_{\gamma'} (z - A_\varepsilon)^{-1} dz$$

are commuting projectors and $A_\varepsilon = \lambda_\varepsilon \Pi_\varepsilon + \mu_\varepsilon \Pi'_\varepsilon$. Finally verify that

$$\lambda_\varepsilon \Pi_\varepsilon := \frac{1}{2\pi i}\int_\gamma z(z - A_\varepsilon)^{-1} dz \qquad \mu_\varepsilon \Pi'_\varepsilon := \frac{1}{2\pi i}\int_{\gamma'} z(z - A_\varepsilon)^{-1} dz.$$

The statement follows then from the fact that the right hand side of the above equalities is written as a power series in ε.[29])

5.11. Given two by two matrices A, B such that A has eigenvalues $\lambda \neq \mu$, show that the matrix $A_\varepsilon = A + \varepsilon B$ has eigenvalues $\lambda_\varepsilon, \mu_\varepsilon$ such that $|\lambda_\varepsilon - \lambda| \leq C\varepsilon\|B\|$ and $|\mu_\varepsilon - \mu| \leq C\varepsilon\|B\|$. Compute C. (Hint: By Problem 5.10 we know that $\lambda_\varepsilon, \mu_\varepsilon$ are differentiable function of ε and the same holds for the corresponding eigenvector $v_\varepsilon, \tilde{v}_\varepsilon$. Let us discuss λ_ε since the other eigenvalues can be treated in the same way. One possibility is to use the above formula for $\lambda_\varepsilon \Pi_\varepsilon$ to obtain the wanted estimates.

In alternative, let v, w, $\langle w, v\rangle = 1$ and $\|v\| = 1$, be the eigenvectors of A, with eigenvalue λ and of A^*, with eigenvalue $\bar{\lambda}$, respectively. Hence $\Pi_0 = v \otimes w$ and $\|\Pi_0\| = \|w\|$. Normalize v_ε such that $\langle v_\varepsilon, w\rangle = 1$. Differentiate then the above constraint and the defining equation $(A + \varepsilon B)v_\varepsilon = \lambda_\varepsilon v_\varepsilon$ obtaining (the prime refers to the derivative with respect to ε)

$$\begin{aligned} &Av'_\varepsilon + Bv_\varepsilon + \varepsilon B v'_\varepsilon = \lambda'_\varepsilon v_\varepsilon + \lambda_\varepsilon v'_\varepsilon \\ &\langle v'_\varepsilon, w\rangle = 0. \end{aligned}$$

Multiplying the first for w yields $\lambda'_\varepsilon = \langle w, Bv_\varepsilon\rangle + \varepsilon\langle w, Bv'_\varepsilon\rangle$. Setting $\tilde{A} := A - \lambda\Pi_0$ we have

$$v'_\varepsilon = (\lambda - \tilde{A})^{-1}\left[Bv_\varepsilon + \varepsilon B v'_\varepsilon - \lambda'_\varepsilon v_\varepsilon - (\lambda - \lambda_\varepsilon)v'_\varepsilon\right].$$

[28] Of course for matrices one could argue more directly by looking at the characteristic polynomial. Yet the strategy below has the advantage to work even in infinitely many dimensions (that is, for operators over Banach spaces).

[29] This is a very simple case of the very general problem of perturbation of point spectrum, see [Kat66] if you want to know more.

Next, consider ε_0 such that, for $\varepsilon < \varepsilon_0$ holds

$$\|v'_\varepsilon\| \leq 4\|(\lambda - \tilde{A})^{-1}\| \, \|B\| \, \|w\| = 4\|(\lambda - \tilde{A})^{-1}\| \, \|B\| \, \|\Pi_0\| =: C_0, \tag{5.8.23}$$

then $\|v_\varepsilon - v\| \leq \varepsilon C_0$ and $|\lambda'_\varepsilon| \leq \|B\| \, \|w\|(1 + 2\varepsilon C_0)$. If $4\varepsilon_0 C_0 < 1$, then, indeed, (5.8.23) holds true.)

5.12. Compute $D_0 T$. (Hint: solve (5.3.10) for $\varepsilon = 0$, $\theta = \pi$, $p = 0$ and $t = \frac{2\pi}{\omega}$.)

5.13. Compute $D_0 T_\varepsilon$ and see that, if ω is sufficiently large, the eigenvalues have modulus one (the unstable point becomes stable!). (Hint: setting $\xi := \xi_1$ equation (5.3.10) yields $\ddot{\xi} = \omega_p^2 \xi + \varepsilon \frac{\omega^2}{l} \cos \omega t \xi$. It is then convenient to write $\xi := \bar{\xi} + \varepsilon\eta + \varepsilon^2 \zeta$ where $\ddot{\bar{\xi}} = \omega_p^2 \bar{\xi}$ and $\ddot{\eta} = \omega_p^2 \eta + \frac{\omega^2}{l} \cos \omega t \, \bar{\xi}$. One can look for a solution of the latter equation of the form

$$\bar{\eta} = A e^{\omega_p t} \cos \omega t + B e^{\omega_p t} \sin \omega t + C e^{-\omega_p t} \cos \omega t + D e^{-\omega_p t} \sin \omega t.$$

This allows to compute $D_0 T_\varepsilon(\alpha, \beta) = (\xi_1(\frac{2\pi}{\omega}), \xi_2(\frac{2\pi}{\omega})) + \mathcal{O}(\varepsilon^2)$, where $(\xi_1(0), \xi_2(0)) = (\alpha, \beta)$. Finally one can verify that, for ε small and ω large enough the eigenvalues of $D_0 T_\varepsilon$ are imaginary, hence the equilibrium is linearly stable.)

5.14. Given an Hamiltonian $H : \mathbb{R}^2 \to \mathbb{R}$, for each solution $x(t)$ of the associated equations of motion show that $\langle \nabla_{x(t)} H, \, \dot{x}(t) \rangle = 0$.

5.15. Compute the following integrals (5.6.22):

$$\int_{\mathbb{R}} e^{iat} (\cosh t)^{-n} \sinh t \, dt,$$

$a \in \mathbb{R}$ and $n \in \mathbb{N}$, $n > 1$.[30] (Hint: By a change of variable one can consider only the case $a > 0$. Consider the integral on the complex plane, show that the integral on the half circle $Re^{i\phi}$, $\phi \in [0, \pi]$, goes

[30] The result, for $a > 0$, is:

$$\int_{\mathbb{R}} e^{iat} (\cosh t)^{-n} \sinh t = 2\pi i \sum_{k=0}^{\infty} \frac{\phi_{n,k}^{(n-1)}(i \frac{2k+1}{2} \pi)}{(n-1)!},$$

where

$$\phi_{n,k}(z) = e^{iza} \sinh z \left(\frac{z - i \frac{2k+1}{2} \pi}{\cosh z} \right)^n.$$

For $n = 3$ the above formula yields

$$\int_{\mathbb{R}} e^{iat} (\cosh t)^{-3} \sinh t = \pi a^2 e^{-\frac{\pi}{2} a} (1 - e^{-\pi a})^{-1}.$$

to zero as $R \to \infty$, then check that the poles of the integrand, on the complex plane, lie on the imaginary axis, finally use the residue theorem to compute the integrals.)

5.16. Do the same analysis carried out for the pendulum with a vibrating suspension point in the case of a pendulum subject to an external force $\varepsilon \cos \omega t$ and in presence of a small friction $-\varepsilon^2 \gamma \dot{\theta}$.

Notes

As already mentioned in the text, the first to realize that the motions arising from differential equations can be very complex was probably Poincaré [Poi87]. At the time the main problem in celestial mechanics (the famous n-body problem) was to find all the integral of motion. Dirichlet and Weierstrass worked on this problem, but Poincaré was the first to rise serious doubt on the existence of such integrals (which would have implied regular motions). For more historical remarks see [Mos01]. In fact, all the content of this chapter is inspired by the more sophisticated, but more qualitative, analysis in [Mos01].