# Chapter 4

# Global Behavior: simple examples

Different local behaviors have been analyzed in the previous chapter. Unfortunately, such analysis is insufficient if one wants to understand the *global* behavior of a Dynamical System. To make precise what we mean by global behavior we need some definitions.

**Definition 4.0.1** *Given a Dynamical System* $(X, \phi_t)$*,* $t \in \mathbb{N}$ *or* $\mathbb{R}_+$*, a set* $A \subset X$ *is called* invariant *if, for all* $t$*,* $\emptyset \neq \phi_t^{-1}(A) \subset A$*.*

Essentially, the global understanding of a system entails a detailed knowledge of its invariant set and of the dynamics in a neighborhood of such sets. This is in general very hard to achieve, essentially the rest of this book devoted to the study of some special cases.

**Remark 4.0.2** *We start with some simple considerations in the case of continuous Dynamical Systems (this is part of a general theory called Topological Dynamical Systems*[1]*) and then we will address more subtle phenomena that depend on the smoothness of the systems.*

## 4.1 Long time behavior and invariant sets

First of all let us note that if we are interested in the long time behavior of a system and we look at it locally (i.e. in the neighborhood of a point) then three cases are possible: either the motion leaves the neighborhood and never returns, or leaves the neighborhood but eventually it comes back or never leaves. Clearly, in the first case the neighborhood in question has little interest in the study of the long time behavior. This is made precise by the following.

[1] Recall that a Topological Dynamical Systems is a couple $(X, \phi_t)$ where $X$ is a topological space and $\phi_t$ is a continuous action of $\mathbb{R}$ (or $\mathbb{R}_+, \mathbb{N}, \mathbb{Z}$) on $X$.

**Definition 4.1.1** *Given a Dynamical System $(X, \phi_t)$, a point $x \in X$ is called* wandering *if there exists a neighborhood $U$ of $x$ and a $t_0 \geq 1$ such that, for all $t \geq t_0$, $\phi_t(U) \cap U = \emptyset$. A point that is not wandering is called* non-wandering. *The set of non-wandering points is called $NW(\{\phi_t\})$ or simply $NW$ if no confusion arises.*

**Problem 4.1** *If $\phi_t \in \mathcal{C}^0$, then the set $NW$ is closed and forward invariant (i.e. $\phi_t(NW) \subset NW$ for each $t \geq 0$). If the $\phi_t$ are open maps, then $NW$ is also invariant.*

**Problem 4.2** *Construct an example of a topological dynamical systems in which the non-wandering set is not invariant.*

**Problem 4.3** *Show that if $A$ is invariant, then the sets $\Lambda = \cap_{t=0}^{\infty} \phi_t^{-1} \overline{A}$ and $\Omega = \overline{\cup_{t=0}^{\infty} \phi_t(A)}$ are non-empty, invariant and, more, $\phi_t^{-1}(\Lambda) = \Lambda$ and $\phi_t^{-1}(\Omega) = \Omega$*

The relevance for the long time behavior is emphasized by the following lemma.

**Lemma 4.1.2** *If $K \subset X$ is compact and $K \cap NW = \emptyset$, then for all $x \in K$ there exists $T$ such that $\phi_t(x) \notin K$ for all $t \geq T$. In addition, if $K$ is invariant, then $T$ can be chosen independent of $x$.*

PROOF. If all the points in $K$ are wandering, then for each $x \in K$ there exists a neighborhood $U(x)$ and a time $t(x)$ such that $\phi_t U(x) \cap U(x) = \emptyset$ for all $t \geq t(x)$. Clearly $\{U(x)\}_{x \in K}$ is an open covering of $K$, hence we can extract a finite subcover. Let $\{U_i\}_{i=1}^N$ be such a subcover, let $\{t_i\}$ be the corresponding associated times. If $x \in K$ then $x \in U_i$ for some $i \in \{1, \ldots, N\}$, and $\phi_t(x) \notin U_i$ for $t \geq t_i$. If $\phi_t(x) \notin K$ for all $t \geq t_i$, then we are done. If there exists $t \geq t_i$ such that $\phi_t(x) \in K$, then $\phi_t(x)$ must belong to another $U_j$, that will leave forever for $t \geq t_j$. It is then clear that $\phi_t(x)$ cannot remain in $K$ for a time longer than $\sum_i t_i$, nor can the trajectory return for more than $N$ times.

If $K$ is invariant then it follows that if $x \notin K$ then $\phi_t x \notin K$ for all $t \geq 0$. Thus once a point exits $K$ it can never come back. The above argument then show that each point must exit forever in a time at most $\sum_i t_i$. □

**Corollary 4.1.3** *If $K \subset X$ is compact and invariant, then either there exists $n \in \mathbb{N}$ such that $T^{-n} K = \emptyset$ or $NW \cap K \neq \emptyset$.*

PROOF. If $NW \cap K = \emptyset$, then Lemma 4.1.2 imply that there exists $n \in \mathbb{N}$ such that $T^n K \cap K = \emptyset$, hence $T^{-n} K = \emptyset$. □

To see the connection to long time behavior and invariant sets we need an extra definition

**Definition 4.1.4** *Given a topological Dynamical System $(X, \phi_t)$, $t \in I \in \{\mathbb{R}, \mathbb{Z}, \mathbb{R}_+, \mathbb{N}\}$, and $x \in X$ we call $\omega(x)$ (*the $\omega$-limit set of $x$*) the accumulation points of the set $\cup_{t \geq 0}\{\phi_t(x)\}$. If $t$ belongs to $\mathbb{R}$ or $\mathbb{Z}$, then the $\alpha$-limit set is defined analogously with $t \leq 0$.*

**Problem 4.4** *Show that the $\omega$-limit sets are closed sets such that $\phi_t(\omega) = \omega$ (hence if $\phi_t$ is invertible then the omega limits are invariant).*

**Theorem 4.1.5** *For each $x \in X$ we have $\omega(x) \subset NW$. In addition, if $X$ is a proper metric space,[2] then for each $z \in X$ either holds $\lim_{t\to\infty} d(\phi_t(x), z) = \infty$, or $\lim_{t\to\infty} d(\phi_t(x), NW) = 0$.*

PROOF. Let $x \in X$. If $z \in \omega(x)$, then for each neighborhood $U$ of $z$ we have $\{t_n\} \subset \mathbb{R}_+$ such that $\phi_{t_n}(x) \in U$. Thus $\phi_{t_{n+1}-t_n} U \cap U \supset \{\phi_{t_{n+1}}(x)\} \neq \emptyset$. Hence $z \in NW$.

Let us come to the second part of the Theorem. If the two alternatives do not hold, then there exists a compact set (a closed ball) that contains infinitely many points of the orbit of $x$ all at a finite distance from $NW$. This implies that the orbit has an accumulation point (hence an element of $\omega(x)$) not in $NW$ contradicting the first part of the Theorem. $\square$

In particular the above Theorem shows that all the interesting long time dynamical behavior happens in a neighborhood of the non-wandering set.

**Problem 4.5** *Given a discrete topological dynamical system $(X, T)$, let $A = NW(T)$. Since $A$ is forward invariant, one can consider the restriction $S$ of $T$ to $A$. Find an example in which $NW(S)$ is strictly smaller than $A$.*

**Definition 4.1.6** *Given a Dynamical System $(X, \phi_t)$, a point $x \in X$ is called* recurrent *if $x \in \omega(x)$. The set of recurrent points is called $R(\{\phi_t\})$, or simply $R$ if no confusions arises.*

**Problem 4.6** *Consider a linear system $\dot{x} = Ax$. Show that if $A$ is hyperbolic, then $NW = \{0\}$.*

**Problem 4.7** *Consider a saddle-node bifurcation in one dimension. Show that in a small neighborhood of the bifurcation point, when two fixed points $x_1, x_2$ are present, $NW = \{x_1, x_2\}$. Show that this may not be the case in higher dimensions.*

**Problem 4.8** *Consider the ODE $\dot{x} = \begin{pmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{pmatrix}$, $\omega_0 > 0$, $2\pi\omega_0 \notin \mathbb{Q}$. Show that $NW = \mathbb{R}^2$, while for each $x \in \mathbb{R}^2$ holds $\omega(x) = \{z \in \mathbb{R}^2 \;:\; \|z\| = \|x\|\}$.*

[2] That is, a distance $d$ is defined and the base for the topology is made of the sets $B_r(x) = \{y \in X \;:\; d(x, y) < r\}$ (this is called a *metric space*). A *proper* metric space is one in which all the closed balls $\{y \in X \;:\; d(x, y) \leq r\}$ are compact.

**Problem 4.9** *In the case of the Hopf bifurcation in two dimensions when the fixed point $O$ is repelling, and hence the periodic orbit $\gamma$ is attracting, show that (in a neighborhood of $O$ for the bifurcation parameter small enough) $NW = \{O\} \cap \gamma$.*

**Remark 4.1.7** *We have thus seen examples in which the $\omega$-limit sets can be a point or a periodic orbit,* **do other possibilities exists?**

This question is going to lead us to a long journey.

## 4.2 Poincaré-Bendixon

See [HS74].

## 4.3 Equations on the Torus

As we have seen a generic family of vector fields in $\mathbb{R}^2$ can have a very limited choice of bounded invariant sets: either a fixed point and the associated stable and unstable manifolds, or (by Poincaré-Bendixon) a periodic orbit. Yet one can have a differential equation on different manifolds, notably the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

**Problem 4.10** *Consider the vector fields $V(x) = \omega \in \mathbb{R}^2$ on $\mathbb{T}^2$ and show that the orbit of the associated flow can be everywhere dense.*

The above problem shows that on $\mathbb{T}^2$ it is possible to have a new $\omega$-limit set: $\mathbb{T}^2$ itself! Can such a situation take place for an open set of vector fields? To understand the situation it is useful to generalize the setting of Problem 4.10.

**Definition 4.3.1** *A closed non self-intersecting curve $\gamma \in \mathcal{C}^r(S^1, \mathbb{T}^2)$, $r \geq 1$, is called a* global (cross) section *for the flow associated to $V$ if*

*a) $\gamma'$ is always transversal to $V$.*[3]

*b) for each $x \in \mathbb{T}^2$ there exists $t \in \mathbb{R}_+$ such that $\phi_t(x) \in \gamma$.*

Given a cross section $\gamma$ we can define the return time $\tau : \gamma \to \mathbb{R}_+$ as the first $t > 0$ such that $\phi_t(x) \in \gamma$ and the Poincaré map $f : \gamma \to \gamma$ as $f(x) = \phi_{\tau(x)}(x)$.

**Problem 4.11** *Show that if $\gamma \in \mathcal{C}^r(S^1, \mathbb{T}^2)$ is a global cross section and $f$ is the associate Poincaré map, then $f \in \mathcal{C}^r$ and $(\gamma, f)$ is a Dynamical Systems that describe the dynamics when it returns to $\gamma$.*

[3] That is, the vectors $\{\gamma'(t), V(\gamma(t))\}$ span $\mathbb{R}^2$ for all $t \in S^1$.

**Lemma 4.3.2 (Siegel)** *Let $V \in \mathcal{C}^r(\mathbb{T}^2, \mathbb{R}^2)$ be a nowhere zero vector field. If the associated flow has no periodic orbits, then there exists a global section $\gamma$. In addition, if $f : \gamma \to \gamma$ is the Poincaré map associated to the flow, then $f \in \mathcal{C}^r(\gamma, \gamma)$.*

Proof. The (nice) idea is to construct a section close to an orbit. Let $\phi_t$ be the flow associated to the vector field $V$. Let $x \in NW$ and consider an open segment, of length less than $1/2$, $\Sigma$, $x \in \Sigma$, transversal to the vector field (similar to the construction in the Flow Box Theorem 2.1.1). Since $x$ is non-wandering and due to Theorem 2.1.1, there exists $z \in \Sigma$, $z \neq x$, and $t \in \mathbb{R}$ such that $\phi_t(z) \in \Sigma$, this being the first return to $\Sigma$. Since there are no periodic orbits $z \neq \phi_t(z)$.

We will construct a global section close to $\{\phi_s(z)\}_{s=0}^t \cup \Sigma$. Note that the closed curve that one obtains joining $z$ to $\phi_t(z)$ along $\Sigma$ cannot be omotopic to a point. Otherwise the curve would have an interior homeomorphic to a disk in $\mathbb{R}^2$ from which the orbits cannot escape either in the future or the past. By Poincarè–Bendixon this would imply the existence of a periodic orbit contrary to the hypothesis. To properly explain the construction it is convenient to introduce a flow box type system of coordinates near such an orbit.

For $s \in [-1/2, 1/2]$ let $\varphi(s) = z + s(x - z)\|x - z\|^{-1}$. Clearly $\varphi(0) = z$, $\varphi(\|x - z\|) = x$, and holds $\varphi([-1/2, 1/2]) \supset \Sigma$. Next, for each $y \in \Sigma$ let $s \in [-1/2, 1/2]$ be the unique number such that $y = \varphi(s)$ and $\tau(s) = \inf\{t > 0 : \phi_t(y) \in \Sigma\}$ be the first return time to the section. By Theorem 2.1.1 and Corollary 1.1.13 there exists $\delta \in (0, 2\|x - z\|)$ such that $\tau \in \mathcal{C}^r([-\delta, \delta], \mathbb{R}_+)$. For $A := \{(s, t) \in \mathbb{R}^2 \ : \ s \in [-\delta, \delta], t \in [0, \tau(s))\}$ let us define the map $\Xi : A \to \mathbb{T}^2$ by $\Xi(s, t) = \phi_t(\varphi(s))$. Note that this map is $\mathcal{C}^r$ and invertible (provided $\delta$ is chosen small enough), hence it can be used as a change of coordinates. Note that this are essentially the coordinates used in the flow box theorem, only now they are used in a long neighborhood of an orbit.

The next step is to understand how the orbit comes back. Indeed, if we use standard flow box coordiantes $(s', t')$ in a neighborhood of $\Sigma$, then $(s, t) = (s', t')$ for $t \geq 0$ but for $t$ close to $\tau(s)$ we are again in the neighborhood of $\Sigma$ corresponding to $t' < 0$. The change of coordinates can then be described by the function $\theta$ such that $\phi_{\tau(s)}\varphi(s) = \varphi(\theta(s))$. Then $(s, t)$ corresponds to $(\theta(s), t - \tau(s))$.

**Problem 4.12** *Let $\tau_0 = \tau(0)$, then $\langle x - z, \frac{d}{ds}\phi_{\tau_0}(\varphi(s))|_{s=0}\rangle > 0$.*[4]

The above problem means simply that $\theta' > 0$.

To conclude we must analyze two possibilities: either $\phi_\tau z$ is closer to $x$ than $z$ or vice versa. The two cases are treated exactly in the same way so

[4] This is really a consequence of the fact that the torus is orientable, yet it can be proven directly in several ways.

we discuss only the first, that is $\theta(0) > 0$. We can then chose $\varepsilon \in (0, \delta)$ such that $\theta(-\varepsilon) > 0$. Consider a line $(\varepsilon - 2\varepsilon\tau_0^{-1}t, t)$, $t \in [0, \tau_0]$, obviously it is always transversal to the flow. If we look at it in the standard flow box coordinates in a neighborhood of $\Sigma$ we see that it start as a decreasing curve and, since $\theta' > 0$, it reappears (for $t' < 0$) as a still decreasing curve. It is then easy to see that it can be smoothly deformed, in a neighborhood of $\Sigma$, into a closed curve that is always transversal to the flow. We have thus constructed a smooth transversal section it remains to show that it is global.

**Problem 4.13** *Consider a piecewise smooth closed curve $\Gamma$ in $\mathbb{T}^2$. Show that $\mathbb{T}^2 \setminus \Gamma$ is either disconnected (and one connected component is isomorphic to an open set in $\mathbb{R}^2$) or it is isomorphic to a cylinder.*

If the above section would not be global, then there would be trajectories that stay forever in a set (either a piece of $\mathbb{R}^2$ or a cylinder) to which Poincaré-Bendixon applies. But this would imply the presence of a periodic orbit, contrary to the asumption. $\square$

**Problem 4.14** *Show that, in the setting of the above theorem, the sign of $f'$ cannot change and that the condition $f' \neq 0$ is generic.*

It is important to notice that, given a topological Dynamical System $(M, f)$ and a function $\tau \in \mathcal{C}^0(M, \mathbb{R}_+ \setminus \{0\})$ (called *roof function*) one can always see them as a Poincaré section and a return time of a flow. The resulting object is called a *suspension* or *standard flow* and is constructed as follows.

Consider the set $\tilde{\Omega} = \{(x, s) \in M \times \mathbb{R}_+ \;:\; s \in [0, \tau(x)]\}$ with the topology induced by $M \times \mathbb{R}_+$ equipped with the product topology.

**Problem 4.15** *Consider the relation $(x, s) \sim (y, t)$ iff $x = y$ and $s = t$ or $s = \tau(x)$, $t = 0$ and $y = f(x)$ or $t = \tau(y)$, $s = 0$ and $x = f(y)$. Prove that it is an equivalence relation.*

One can then consider the space of the equivalence classes $\Omega = \tilde{\Omega}/\sim$ with the induced topology, this is the space on which the flow is defined: let $t \leq \inf \tau$, define

$$\phi_t(x, s) = \begin{cases} (x, s + t) & \text{if } t < \tau(x) - s \\ (f(x), t + s - \tau(x)) & \text{if } t \geq \tau(x) - s \end{cases}$$

and extend $\phi_t$ by the group property.

**Theorem 4.3.3** *Let $V \in \mathcal{C}^2(\mathbb{T}^2, \mathbb{R}^2)$ be a nowhere zero generic vector field with no periodic orbits. Then for each point $y \in \mathbb{T}^2$, $\omega(y) = \mathbb{T}^2$.*

PROOF. By Lemma 4.3.2 we have a smooth global section $\gamma$ with a Poincaré map $g$. Let $h : S^1 \to \gamma$ be a parametrization of $\gamma$. If we set $f = h^{-1} \circ g \circ h$, we can consider the return map as $\mathcal{C}^2$ map on the unit circle such that $f' \neq 0$ at each point. Note that a periodic point for the map $f$ corresponds to a periodic orbit for the flow, hence $f$ cannot have periodic orbits. The claim follows then by Lemma 4.5.2 in which it is proven that a smooth circle map with no periodic orbits has dense orbits. □

The final natural question is:

**In the hypotheses of Theorem 4.3.3, is it possible to conjugate the flow to a rigid rotation of the torus, and, if yes, to which one?**

Motivated by the above question and results we will now study orientation preserving circle maps. It turns out to be interesting and helpful to study their properties in relations to their increasing smoothness.

## 4.4 Circle maps: topology

Here , and in the following, we study a Dynamical System $(S^1, f)$ where $f$ is a homeomorphism of $S^1$ (i.e. $f$ is invertible and $f(S^1) = S^1$).

We start with some facts that follow from the simple hypothesis of continuity.

First of all note that one can lift the map $f$ to the universal cover $\mathbb{R}$ of the circle, that is defining $\pi : \mathbb{R} \to S^1$ as $\pi(x) = x \mod 1$, it is possible to find $F \in \mathcal{C}^0(\mathbb{R}, \mathbb{R})$ such that

$$f \circ \pi = \pi \circ F.$$

**Problem 4.16** *Construct explicitly such an $F$. Show that $F(x+1) = F(x)+1$.*

**Problem 4.17** *If there exists $L > 0$ such that $-L \leq a_{m+n} \leq a_n + a_m + L$ for all $n, m \in \mathbb{N}$, then the limit $\lim_{n\to\infty} \frac{a_n}{n}$ exists.*

**Lemma 4.4.1** *Let $f : S^1 \to S^1$ be an homeomorphism and $F \in \mathcal{C}^0(\mathbb{R}, \mathbb{R})$ a lift of $f$. Then the limit*

$$\tau(f) := \lim_{|n|\to\infty} \frac{F^n(x)}{n} \mod 1$$

*exists and is independent both from the point and the lift.*

PROOF. Applying Problem 4.17 to the sequence $F^n(x)$ the existence of the limit follows. The other assertions depend on the already mentioned equality $F(x+1) = F(x) + 1$. □

**Lemma 4.4.2** *Show that $\tau(f) \in \mathbb{Q}$ if and only if $f$ has a periodic orbit.*

PROOF. If $f^q(x) = x$ and $F$ is a lift then it must be $F^q(x) = x + p$ for some $p \in \mathbb{N}$. This immediately implies $F^{kq}(x) = x + kp$ and hence $\tau(f) = \frac{p}{q} \in \mathbb{Q}$. On the other hand, if $\tau(f) = \frac{p}{q} \in \mathbb{Q}$, we have $\tau(f^q) = p \mod 1 = 0$. It thus suffices to prove that $\tau(f) = 0$ implies $f$ has a fixed point. Let us do a proof by contradiction: we suppose that $f$ has no fixed points. Note that this is the same than saying that $G(\mathbb{R}) \cap \mathbb{Z} = \emptyset$ where $G(x) = F(x) - x$. Since $G$ is continuous this implies $\max G - \min G < 1$. Let $\alpha = \min G$, $\beta = \max G$. Note that, by properly choosing the lift $F$, one can insure tat $[\alpha, \beta] \subset (0, 1)$. Then

$$F^n(x) = G(F^{n-1}(x)) + F^{n-1}(x) \geq \alpha + F^{n-1}(x) \geq n\alpha$$

hence $\tau(f) \geq \alpha$, analogously $\tau(f) \leq \beta$ which contradicts $\tau(f) = 0$. □

**Problem 4.18** *Given $f \in \mathcal{C}^0(S^1, S^1)$, for any interval $I \subset S^1$, if $f(I) \subset I$, then $f$ has a fixed point in $I$.*

**Problem 4.19** *If $\tau(f) \notin \mathbb{Q}$, then for each $n \in \mathbb{N} \setminus \{0\}$ and $x, y \in S^1$, $\{f^k(y)\}_{k \in \mathbb{N}} \cap [x, f^n(x)] \neq \emptyset$.*

**Problem 4.20** *If $\tau(f) \notin \mathbb{Q}$, then for each $x \in S^1$ there exist infinitely many $n \in \mathbb{Z}$ such that $\{f^k x\}_{|k|<n} \cap [x, f^n x] = \emptyset$.*

**Lemma 4.4.3** *For any homomorpfism $f : S^1 \to S^1$ with $\tau(f) \notin \mathbb{Q}$ and any $x, y \in S^1$ holds $\omega(x) = \omega(y)$.*

PROOF. If $z \in \omega(x)$, then there exists $\{n_j\}$ such that $\lim_{j\to\infty} f^{n_j}(x) = z$. But then Problem 4.19 implies that for each $j \in \mathbb{N}$ there exists $k_j \in \mathbb{N}$ such that $f^{k_j}(y) \in [f^{n_j}(x), f^{n_{j+1}}(x)]$. Clearly $\lim_{j\to\infty} f^{k_j}(y) = z$, thus $z \in \omega(y)$. Reversing the role of $x$ and $y$ the Lemma follows. □

**Problem 4.21** *Let $f$ be a homeomorphism of $S^1$ with irrational rotation number show that for each $\varepsilon > 0$ there exists a homeomorphism $f_\varepsilon$, $\|f - f_\varepsilon\|_\infty \leq \varepsilon$, with $\tau(f_\varepsilon) \in \mathbb{Q}$.*

**Problem 4.22** *Note that $\tau$ is a map from circle homomorphisms to $[0, 1]$. Show that it is a continuous map.*

**Problem 4.23** *Let $f_\lambda$ be a one parameter family of homeomorphisms such that $\tau(f_0) < \tau(f_1)$. Suppose that $\tau(f_\lambda)$ is increasing, what can you say on the possible intervals in which it is not strictly increasing?*

## 4.5 Circle maps: differentiable theory

In this section we assume $f \in \mathcal{C}^2(S^1, S^1)$ and $\ln f' \in \mathcal{C}^1(S^1, \mathbb{R})$.[5]

**Lemma 4.5.1** *If $\tau(f) \notin \mathbb{Q}$ and $x_0 \notin \omega(x_0)$, then*

$$\sum_{n=0}^{\infty} (f^n)'(x_0) < \infty.$$

Proof. Let $U(x_0) \ni x_0$ be the largest open interval not intersecting $\omega(x_0)$, call $K(x_0)$ its closure. First of all we see that the invariance of the $\omega$-limit set implies $\{f^n(\partial K(x_0))\}_{n=1}^{\infty} \subset \omega(x_0)$. This implies that either $f^n K(x_0) \cap K(x_0) = \emptyset$ or $f^n K(x_0) \supset K(x_0)$ but the latter would imply the existence of a fixed point for $f^n$, which is impossible, hence all the sets $\{f^n K(x_0)\}_{n\in\mathbb{Z}}$ must be disjoint. We can now conclude thanks to a typical distortion estimate: let $K_n(x_0) := f^n(K(x_0))$, then, setting $D := \left|\frac{f''}{f'}\right|_\infty$,

$$\begin{aligned}
1 > \sum_{n\in\mathbb{N}} |K_n(x_0)| &= \sum_{n\in\mathbb{N}} \int_{K(x_0)} (f^n)'(x)dx = \sum_{n\in\mathbb{N}} (f^n)'(x_0) \int_{K(x_0)} \frac{(f^n)'(x)}{(f^n)'(x_0)} dx \\
&\geq \sum_{n\in\mathbb{N}} (f^n)'(x_0) \int_{K(x_0)} e^{-\sum_{k=0}^{n-1} |\ln f'(f^k(x)) - \ln f'(f^k(x_0))|} dx \\
&\geq \sum_{n\in\mathbb{N}} (f^n)'(x_0) \int_{K(x_0)} e^{-\sum_{k=0}^{n-1} D|K_k(x_0)|} dx \geq |K(x_0)| e^{-D} \sum_{n\in\mathbb{N}} (f^n)'(x_0).
\end{aligned}$$

□

**Lemma 4.5.2** *If $\tau(f) \notin \mathbb{Q}$, then, for all $x \in S^1$, $\omega(x) = S^1$.*

Proof. We use the same notation as in Lemma 4.5.1. If the Lemma is false then there exists $x \in S^1$ such that $\omega(x) \neq S^1$. But by Lemma 4.4.3 all the omega limit sets are equal, hence there exists $x_0 \in S^1$ such that $x_0 \notin \omega(x_0)$. Note that if there exists $n \in \mathbb{N}$, $n \neq 0$, such that $f^n(x_0) \in K(x_0)$ then, by the invariance of $\omega(x_0)$, it must be $f^n(x_0) \neq \partial K(x_0) \subset \omega(x_0)$ and then Problem 4.19 implies that there are infinitely many $k$ such that $f^k(x_0) \in [x_0, f^n(x_0)] \subset K(x_0)$, but this is impossible since such an interval does not contain accumulation points of the forward trajectory. Thus, for each $n \in \mathbb{Z}$, $n \neq 0$, $f^n(x_0) \notin K(x_0)$, accordingly there exist $\delta > 0$ such that each interval $[x_0, f^n(x_0)]$ has length at least $\delta$.

Next, choose $L > 0$, by Lemma 4.5.1 there exists $m \in \mathbb{N}$ such that $(f^n)'(x_0) < L^{-1}$, for all $n > m$. We can then apply Problem 4.20 to find an

[5] These hypotheses can be slightly weakened, see [HK95].

$|n| > m$ such that $\{f^k x\}_{|k|<n} \cap [x_0, f^n(x_0)] = \emptyset$. Suppose $n < 0$ and let $J_- = [x_0, f^n(x_0)]$, then for each $k \in \{1, \dots, -n-1\}$, $f^k J_- = [f^k x_0, f^{n+k} x_0]$, since the extreme of such an interval do not belong to $J$ it follows that $f^k J_- \cap J_- = \emptyset$ (otherwise the first would be contained in the second and there would be a fixed point). Thus, setting $J = [x_0, f^{|n|}(x_0)]$, for all $k \in \{1, \dots, -n-1\}$, holds $f^k J \cap J = \emptyset$. The same result follows, setting $J_- = [x_0, f^{-n}(x_0)]$ , for $n > 0$. Finally we conclude with another distortion argument

$$\begin{aligned} |f^{-|n|} J| &= \int_J (f^{-|n|})'(x) dx = \frac{1}{(f^{|n|})'(x_0)} \int_J \frac{(f^{|n|})'(f^{-|n|}(f^{|n|}(x_0))}{(f^{|n|})'(f^{-|n|} x)} dx \\ &\geq \frac{1}{(f^{|n|})'(x_0)} \int_J e^{-\sum_{k=0}^{|n|-1} D|f^k J|} dx \geq L e^{-D} \delta. \end{aligned}$$

Then choosing $L > e^D \delta^{-1}$ leads a length of $|f^{-|n|} J|$ larger than one, which contradicts the fact that $f$ is an homeomorphism. □

The above fact can be used to prove the following result (due to Poincaré).

**Theorem 4.5.3** *If $\tau(f) = \omega \notin \mathbb{Q}$, then $f$ is $\mathcal{C}^0$-conjugate to $R_\omega(x) = x + \omega$* mod 1.

PROOF. See [HK95] Theorem 11.2.7. □

## 4.6 Circle maps: smooth theory

We have seen that the qualitative behavior of smooth circle maps with irrational rotation number is similar to the behavior of the rigid rotation in Problem 4.10. What it is not clear is if the two dynamics can be smoothly conjugated (i.e. in the spirit of the flow box theorem, but globally). This latter problem turns out to be extremely subtle and to require much finer number theoretical consideration than distinguishing between rational and irrationals.

Since we have seen that more smoothness allows to obtain stronger results, it is natural to start by considering analytic functions.

To make the following easier, we will limit ourselves to the case of a maps close to the identity. That is maps with a covering $F : \mathbb{R} \to \mathbb{R}$ of the form $F(x) = x + \omega + f(x)$, where $f(x+1) = f(x)$ is "small".

### 4.6.1 Analytic KAM theory

To define the sense in which $f$ is small we assume first that $f$ is an analytic function. That is $f$ is a restriction to the real axes of a function, that abusing notation we will still call $f$, holomorphic in a strip. Let $D_\alpha = \{z \in \mathbb{C} \;\;:$

$|\Im(z)| \leq \frac{\alpha}{2\pi}\}$ and consider the function space $\mathbb{B}_\alpha = \{g \in \mathcal{C}^0(D_\alpha, \mathbb{C}) \ : \ g(z+1) = g(z) \ \forall z \in D_\alpha \, , \ g$ holomorphic in $\mathring{D}_\alpha\}$. This is a Banach space when equipped with the norm $\|g\|_\alpha = \sup_{z\in D_\alpha} |g(z)|$.

**Theorem 4.6.1** *If $\tau(F) = \omega$ and there exist $\alpha_0 \in (0,1)$, $C_0 > 0$ such that if $\|f - \int_{S^1} f\|_{\alpha_0} \leq C_0\alpha_0^3 10^{-10}$ and $\omega > 0$ satisfies*

$$\left|\omega - \frac{p}{q}\right| \geq \frac{C_0}{q^2}$$

*for each $p, q \in \mathbb{N}$, then there exists $h \in \mathbb{B}_{\alpha_0/2}$ such that, setting $H(x) = x + h(x)$, $\|h\|_{\alpha_0/2} \leq 3C_0^{-\frac{1}{3}}\|f\|_{\alpha_0}^{\frac{1}{3}}$ and, for all $x \in \mathbb{R}$,*

$$H^{-1} \circ F \circ H(x) = x + \omega. \tag{4.6.1}$$

A natural question is: do irrational numbers with the above properties exists? The answer is yes (for example all the quadratic irrational satisfy such inequalities), but a bit of theory is needed to see it. For a quick introduction to these problems solve the Problems 4.28, 4.29, 4.30, 4.31, 4.32, 4.33, 4.34.

**Remark 4.6.2** *Note that we can always reduce to the case $\int f = 0$ by subtracting the average to $f$ and adding it to $\omega$. As an exercise you can show that given the map $F(x) = x + \omega + \xi + f(x)$, with $f$ zero average and norm small as in Theorem 4.6.1, there exists a $\xi$ for which the map is conjugated to $x + \omega$.*

**Remark 4.6.3** *The unaware reader can be horrified by the $10^{-10}$ in the statement of the above theorem. Such a ridiculous number is in part due to the fact that I have privileged readability over optimality, but in part it comes with the method. Indeed, it is well known among specialists that to obtain optimal estimates for KAM-type theorems is a very hard problem. Indeed, it is a field of research currently active.*

Proof of Theorem 4.6.1. First of all remark that $\hat{f}_0 = \int_{S^1} f$ and

$$\omega + \hat{f}_0 - \|f - \hat{f}_0\|_{\alpha_0} \leq \omega = \tau(F) \leq \omega + \hat{f}_0 + \|f - \hat{f}_0\|_{\alpha_0}$$

thus

$$|\hat{f}_0| \leq \|f - \hat{f}_0\|_{\alpha_0}. \tag{4.6.2}$$

Next, note that if $H$ is invertible, then equation (4.6.1) is equivalent to, for each $z \in D_{\alpha_0/2}$,

$$h(z + \omega) - h(z) = f(z + h(z)). \tag{4.6.3}$$

In fact, we are interested to solving the above equation only for real $z$. In the following to avoid confusion I will use $z$ for a complex variable and $x$ for a real one.

It is natural to introduce the linear operator $L_\omega g(x) = g(x+\omega) - g(x)$. If such an operator were invertible, then we could write

$$h = L_\omega^{-1} f \circ H, \tag{4.6.4}$$

that looks like a fixed point problem and hopefully can be studies with known techniques.

We have thus to study the operator $L_\omega$. The best is to compute it in Fourier series:

$$L_\omega g(x) = \sum_{k\in\mathbb{Z}} e^{2\pi ikx}(e^{2\pi i\omega k} - 1)\hat{g}_k$$

where $g(x) = \sum_{k\in\mathbb{Z}} e^{2\pi ikx}\hat{g}_k$. Thus, provided $\hat{g}_0 = 0$,

$$L_\omega^{-1} g(x) = \sum_{k\in\mathbb{Z}\setminus\{0\}} e^{2\pi ikx}\frac{\hat{g}_k}{e^{2\pi i\omega k} - 1}.$$

Thanks to the fact that $\omega \notin \mathbb{Q}$, the coefficients in the above formula are well defined. Yet, it remains the issue of the convergence of the series. Indeed, the coefficients can be very large since,[6]

$$\left|e^{2\pi i\omega k} - 1\right| \geq 2\inf_{p\in\mathbb{N}} |\omega k - p| \geq 2C_0|k|^{-1}.$$

This is the main difficulty of the present problem: the infamous *small divisors*. Clearly, due to the small divisors $L_\omega^{-1}$ is not a bounded operator. This makes it very hard to study directly (4.6.4). To bypass this problem we need an idea.

The idea that we will use if due to Kolomogorov and goes as follows: instead of solving (4.6.4) consider the change of variables $H_0(x) = x + h_0(x)$ where $h_0 = L_\omega^{-1}(f - \hat{f}_0)$. Of course such a change of variable it is not the right one since

$$h_0(x+\omega) - h_0(x) = f(x) - \hat{f}_0, \tag{4.6.5}$$

yet one can try to write

$$H_0^{-1} \circ F \circ H_0(x) = x + \omega + f_1(x) \tag{4.6.6}$$

and hope that $f_1$ is much smaller that $f$. If this is the case one can iterate the procedure and hope that it converges to a limiting change of variables that is the one we are looking for.

[6] Note that $|e^{ix} - 1| \geq |\sin x| \geq \frac{2x}{\pi}$, provided $x \in [0, \pi/2]$. On the other hand if $x \in [\pi/2, \pi]$, then $|e^{ix} - 1| \geq |1 - \cos x| \geq 1$. Hence we can use the simple, but not very sharp, estimate $|e^{2\pi ix} - 1| \geq \inf_{p\in\mathbb{Z}} 2|x - p|$.

To implement the above idea the first thing we need is to connect the analysis via Fourier series to the analytic properties of the functions.

Consider the norm
$$|g|_\alpha := \sum_{k\in\mathbb{Z}} e^{\alpha|k|}|\hat{g}_k|.$$
Let us call $\mathcal{B}_\alpha$ the Banach space of the periodic functions (of period one) on $\mathbb{R}$ equipped with the above norm.

Note that, for $\beta < \alpha$,[7]

$$\begin{aligned}|L_\omega^{-1}g|_\beta &\leq \sum_{k\in\mathbb{Z}} \frac{|k|}{2C_0} e^{\beta|k|}|\hat{g}_k| \leq \frac{|g|_\alpha}{2C_0}\sup_{k\in\mathbb{Z}} |k|e^{-(\alpha-\beta)|k|}\\ &\leq \frac{|g|_\alpha}{2eC_0(\alpha-\beta)}\end{aligned} \tag{4.6.7}$$

Thus $L_\omega^{-1} : \mathcal{B}_\alpha \to \mathcal{B}_\beta$ is a bounded operator for each $\alpha > \beta$.

The point is that there is a connection between the above Banach spaces, namely we can define $\Xi : \mathbb{B}_\beta \to \mathcal{B}_\alpha$, by $\Xi g(x) = g(x)$, for all $x \in \mathbb{R}$.[8] To see the relation between the norms, let us compute the Fourier coefficients
$$[\widehat{\Xi g}]_k = \frac{1}{i}\int_0^1 e^{2\pi i k x} g(x)dx$$

**Problem 4.24** *Show that* $|[\Xi g]_k| \leq e^{-\alpha|k|}\|g\|_\alpha$.

Hence, for $\alpha > \beta$, $\|\Xi\|_{\mathbb{B}_\alpha\to\mathcal{B}_\beta} \leq 2(1-e^{\beta-\alpha})^{-1}$. Note also that we can easily define the inverse: if $g \in \mathcal{B}_\alpha$, then define
$$\Xi^{-1}g(z) = \sum_{k\in\mathbb{Z}} e^{2\pi i k z}\hat{g}_k$$

**Problem 4.25** *Verify that the above is really the inverse of* $\Xi$.

[7] Here we use that, for each $n \in \mathbb{N}$ and $\sigma > 0$,
$$\sup_{k\in N} k^n e^{-\sigma k} \leq \sup_{x\in\mathbb{R}_+} x^n e^{-\sigma x} = \left(\frac{n}{\sigma}\right)^n e^{-n} \leq e^{-1}\sigma^{-n} n!.$$
The last inequality is an application of Stirling formula. If you do not remember it, here is the baby version used above,
$$n! = e^{\sum_{k=1}^n \ln k} \geq e^{\int_1^n \ln x dx} = e^{n\ln n - n + 1} = n^n e^{-n+1}.$$

[8] In other words we simply take the restriction of the function to the real axis.

If $g \in \mathcal{B}_\alpha$, then

$$\|\Xi^{-1}g\|_\alpha \le \sum_{k\in\mathbb{Z}} e^{|k|\alpha}|\hat{g}_k| = |g|_\alpha.$$

Thus $\|\Xi^{-1}\|_{\mathcal{B}_\alpha\to\mathbb{B}_\alpha} \le 1$.

**Problem 4.26** *Show that, for each $\alpha > \beta$, $\alpha-\beta < 2$, setting $h_0 = \Xi^{-1}L_\omega^{-1}\Xi(f-\hat{f}_0)$, holds*

$$\begin{aligned}\|h_0\|_\beta &\le \frac{4\|f-\hat{f}_0\|_\alpha}{C_0(\alpha-\beta)^2}\\ \|h_0'\|_\beta &\le \frac{64\pi}{C_0(\alpha-\beta)^3}\|f-\hat{f}_0\|_\alpha.\end{aligned}$$

The point of the spaces $\mathbb{B}_\alpha$ is that the equation (4.6.6) for $f_1$ reads

$$f_1(x) = h_0(x) - h_0(x+\omega+f_1(x)) + f(x+h_0(x)). \tag{4.6.8}$$

To study such equation in $\mathcal{B}_\alpha$ is highly non trivial, while $\mathbb{B}_\alpha$ is much better suited to estimate the norms of composition of functions.

To study (4.6.8) in $\mathbb{B}_\alpha$ the first step is to verify that it makes sense. Obviously one can see it as the restriction to the real axes of an equation involving functions defined on the complex plane, yet it is necessary to check that the composition is well defined, that is we have to carefully analyze domains and ranges of the various functions. For later use we carry out all the needed estimates in the following Lemma.

**Lemma 4.6.4** *Given functions $f \in \mathbb{B}_\alpha$ and $h \in \mathbb{B}_\beta$, $\alpha > \beta > \alpha/2$ such that, setting $F(z) = z + \omega + f(z)$, we have $\tau(F) = \omega$, $\|f - \hat{f}_0\|_\alpha \le \frac{\alpha-\beta}{2\pi}$ and $h$ satisfies (4.6.5), it follows that $\|h\|_\beta \le \frac{\alpha-\beta}{16\pi}$, $\|h'\|_\beta \le \frac{1}{16}$, $H(z) = z + h(z)$ is invertible, $H^{-1} \in \mathbb{B}_\gamma$, $\gamma \le 2\beta - \alpha$, and there exists a function $f_1 \in \mathbb{B}_\gamma$ with $\|f_1 - \int_{S^1} f_1\|_\gamma \le \frac{1}{2}\|f - \int_{S^1} f\|_\alpha$ satisfying*

$$H^{-1}\circ F\circ H(z) = z + \omega + f_1(z) =: F_1(z).$$

Proof. First of all $H$ is invertible when restricted to the real axis since $H' \ge \frac{1}{2}$. Let $H^{-1}(z) = z + \psi(z)$, clearly

$$\psi(z) = -h(z + \psi(z)).$$

So the inverse is the fixed point of the operator $K(\psi)(z) = -h(z+\psi(z))$ which is well defined on the set $A = \{\psi \in \mathbb{B}_\gamma \;:\; \|\psi\|_\gamma \le \frac{\alpha-\beta}{2\pi}\}$. It is easy to verify that such a fixed point exists and is unique.

Note that the function $f_1$ must satisfy equation (4.6.8). To solve (4.6.8) we must look for a fixed point for the operator

$$\mathcal{K}(\varphi)(z) = h(z) - h(z+\omega+\varphi(z)) + f(z+h(z))$$

on the set $A = \{\varphi \in \mathbb{B}_\gamma \;:\; \|\varphi - \hat f_0\|_\gamma \le \frac{1}{4}\|f-\hat f_0\|_\alpha\}$. Note that the composition of functions is well defined, hence so is $\mathcal{K}$.

Let us check that $\mathcal{K}(A) \subset A$.

$$\begin{aligned}\mathcal{K}(\varphi)(z) - \hat f_0 &= h(z) - h(z+\omega) + h(z+\omega) - h(z+\omega+\varphi(z)) + f(z+h(z)) - \hat f_0\\ &= f(z+h(z)) - f(z) + h(z+\omega) - h(z+\omega+\varphi(z)).\end{aligned}$$

Thus, using the estimate in Problem 4.35 and recalling (4.6.2),

$$\|\mathcal{K}(\varphi)-\hat f_0\|_\gamma \le \|f'\|_\beta\|h\|_\gamma + \|h'\|_\beta\|\varphi\|_\gamma \le \frac{1}{8}\|f\|_\alpha + \frac{1}{16}\|\varphi-\hat f_0\|_\gamma + \frac{1}{16}|\hat f_0| \le \frac{1}{4}\|f\|_\alpha.$$

In addition, if $\varphi, \tilde\varphi \in A$, then

$$\|\mathcal{K}(\varphi) - \mathcal{K}(\tilde\varphi)\|_\gamma \le \|h'\|_\beta\|\varphi-\tilde\varphi\|_\gamma \le \frac{1}{16}\|\varphi-\tilde\varphi\|_\gamma.$$

Thus, by the usual contraction argument, there exists $f_1 \in A$ such that $\mathcal{K}(f_1) = f_1$. On the other hand $F_1$ is conjugated to $F$ and hence it has rotation number $\omega$. Thus (4.6.3) implies $|\int_{S^1} f_1| \le \|f_1 - \int_{S^1} f_1\|_\gamma$ and

$$\left\|f_1 - \int_{S^1} f_1\right\|_\gamma \le \left\|f_1 - \int_{S^1} f_0\right\|_\gamma + \left|\int_{S^1} f_0 - f_1\right| \le \frac{1}{2}\|f-\hat f_0\|_\alpha.$$

□

Since we need to restrict the domain several time it is convenient to do it in a systematic fashion. Let $\rho_k := e^{-k\tau}\alpha$, and apply Lemma 4.6.4 with $\beta = \rho_2$ and $\gamma = \rho_4$. A simple computation shows that the condition on $\beta, \gamma$ are satisfied if $e^{-\tau} \ge \frac{2}{3}$. Then, setting $\varepsilon = \|f - \hat f_0\|_\alpha$, Lemma 4.6.4 applies provided $\varepsilon \le \min\{\frac{\tau\alpha}{\pi e}, \frac{C_0\tau^3\alpha^3}{128e^3\pi}\}$.[9] We then choose $\tau_0 = \alpha^{-1}C_0^{-\frac{1}{3}}\varepsilon^{\frac{1}{3}}$. Hence $\min\{\frac{\tau\alpha}{\pi e}, \frac{C_0\tau^3\alpha^3}{128e^3\pi}\} = \frac{C_0\tau^3\alpha^3}{128e^3\pi}$ provided $\varepsilon \le 10^3e^3\sqrt{C_0}$.

We now implement an iterative procedure by setting: $f_0 = f$,

$$\begin{aligned}&h_n(z+\omega) - h_n(z) = f_n(z)\,, \quad H_n(z) = z + h_n(z)\,, \quad F_n(z) = z+\omega+f_n(z),\\ &H_n^{-1}\circ F_n \circ H_n(z) = z + \omega + f_{n+1}(z).\end{aligned}$$

[9] Just use Problem 4.26 and the fact that $1 - e^{-x} = \int_0^x e^{-y}dy \ge e^{-1}x$, for $x \in (0,1)$, to check the hypotheses of the Lemma.

In addition, we set $\alpha_0 = \alpha$, $\alpha_{n+1} = e^{-4\tau_n}\alpha_n$, $\varepsilon_{n+1} = \frac{\varepsilon_n}{2}$ and $\tau_n = \alpha_n^{-1}C_0^{-\frac{1}{3}}\varepsilon_n^{\frac{1}{3}}$. Note that this choices imply that Lemma 4.6.4 can be applied at each stage of the iteration. Now, if $\alpha_n \geq \frac{1}{2}\alpha_0$, holds $\varepsilon_n = 2^{-n}\varepsilon$, $\tau_n \leq 2\alpha_0^{-1}2^{-n/3}C_0^{-\frac{1}{3}}\varepsilon^{\frac{1}{3}}$. This implies $\alpha_n = \alpha_0 e^{-4\sum_{k=0}^{n-1}\tau_k} \geq e^{-40\alpha_0^{-1}C_0^{-\frac{1}{3}}\varepsilon^{\frac{1}{3}}}\alpha_0$ which is always larger than $\alpha_0/2$ provided $\varepsilon \leq C_0\left[\frac{\alpha_0 \ln 2}{40}\right]^3$. Note that all our condition on $\varepsilon$ are satisfied if $\varepsilon \leq \frac{1}{5}C_0\alpha_0^3 10^{-5}$.

We have thus a sequence of changes of variables $H_n(z) = z + h_n(z)$, the next question is if it exists $H(z) = \lim_{n\to\infty} H_0 \circ H_1 \circ \cdots \circ H_n(z)$. It suffices to prove that the sequence is uniformly bounded on $D_{\alpha_0/2}$

$$
\begin{aligned}
|H_0 \circ H_1 \circ \cdots \circ H_n(z) - z| &\leq \sum_{k=0}^{n}\|h_k\|_{\alpha_k} \leq \sum_{k=0}^{n}\frac{e^2\varepsilon_k}{C_0\tau_k^2\alpha_k^2} \\
&\leq \sum_{k=0}^{\infty}2^{-k/3}\varepsilon^{\frac{1}{3}}e^2C_0^{-\frac{1}{3}} \leq \varepsilon^{\frac{1}{3}}5e^2C_0^{-\frac{1}{3}}
\end{aligned}
$$

Similarly it follows that the $H_n$ form a Chauchy sequence, hence they have a limit $H \in \mathbb{B}_{\alpha_0/2}$ with $\|\mathrm{id} - H\|_{\alpha_0/2} \leq \varepsilon^{\frac{1}{3}}5e^2C_0^{-\frac{1}{3}}$. From this it follows also (see Problem 4.35)

$$
\|1 - H'\|_{\alpha_0/4} \leq \frac{40\pi e^2\varepsilon^{\frac{1}{3}}}{\alpha_0 C_0^{\frac{1}{3}}} \leq \frac{1}{2}, \tag{4.6.9}
$$

provided $\varepsilon \leq 10^{-10}C_0\alpha^3$. Hence $H$ is invertible and this concludes the proof.
□

### 4.6.2 Smooth KAM theory

The final question is: do similar results hold assuming less smoothness? The answer is yes, yet to explore optimal results it is not an easy task. Here we content ourselves with a partial, but enlightening, result.

**Theorem 4.6.5** *For each $r > 4$,*[10] *if $\tau(F) = \omega$, $\|f - \hat{f}_0\|_{\mathcal{C}^r} \leq 10^{-17}C_0(r-4)^9$ and $\omega > 0$ satisfies*

$$
\left|\omega - \frac{p}{q}\right| \geq \frac{C_0}{q^2}
$$

*for each $p, q \in \mathbb{N}$, then there exists $\mathfrak{h} \in \mathcal{C}^1$ such that, setting $\mathcal{H}(x) = x + \mathfrak{h}(x)$, $\mathcal{H}$ is invertible and*

$$
\mathcal{H}^{-1} \circ F \circ \mathcal{H}(x) = x + \omega.
$$

[10] In fact, by a more sophisticated proof, $r > 3$ suffices [Her83].

Proof. The basic idea is to write $f = \hat{f}_0 + \sum_{m=0}^{\infty} \tilde{f}_m$ where

$$\tilde{f}_m(x) = \sum_{e^{am} \le |k| < e^{a(m+1)}} \hat{f}_k e^{2\pi i k x}$$

and $a > 1$ is a parameter to be chosen later.[11] Then one can apply Theorem 4.6.1 one $\tilde{f}_m$ at a time. Indeed, let $\alpha_m = b(m+1)e^{-a(m+1)}$, for some $a, b > 0$ to be chosen later, where then

$$\begin{aligned}
\|\tilde{f}_m\|_{\alpha_m} &\le \sum_{e^{am} \le |k| < e^{a(m+1)}} |\hat{f}_k| e^{\alpha_m |k|} \le \sum_{e^{am} \le |k| < e^{a(m+1)}} |f|_{\mathcal{C}^r} (2\pi)^{-r} |k|^{-r} e^{\alpha_m |k|} \\
&\le \sum_{e^{am} \le k < e^{a(m+1)}} 2|f|_{\mathcal{C}^r} (2\pi)^{-r} e^{-ram} e^{b(m+1)} \\
&\le 2|f|_{\mathcal{C}^r} e^{-(ar-a-b)m+a}.
\end{aligned}$$

If $|f|_{\mathcal{C}^r}$ is small enough, we can apply Theorem 4.6.1 to $\tilde{f}_0$. Indeed, let $\tilde{F}_0(z) = z + \omega + \xi_0 + \tilde{f}_0(z)$, then $\xi_0 - \|\tilde{f}_0\|_\infty \le \tau(F_0) - \omega \le \xi_0 + \|\tilde{f}_0\|_\infty$, so there exists $|\xi_0| \le \|\tilde{f}_0\|_\infty$ such that $\tau(F_0) = \omega$. Hence, there exist $\tilde{h}_0$ such that, setting $\tilde{H}_0(z) = z + \tilde{h}_0(z)$ and $\tilde{F}_0(z) = z + \xi_0 + \tilde{f}_0(z)$,

$$\tilde{H}_0^{-1} \circ \tilde{F}_0 \circ \tilde{H}_0(z) = z + \omega =: R_\omega(z).$$

The obvious next step is to compute $\mathbb{f}_1$ such that, for each $n \in \mathbb{N}$,

$$\tilde{H}_0^{-1} \circ \left( R_\omega + \sum_{k=0}^{1} \tilde{f}_k \right) \circ \tilde{H}_0(z) = z + \omega + \mathbb{f}_1(z).$$

This is possible if $|f|_{\mathcal{C}^r}$ is small enough. We can then try to iterate the above procedure by applying Theorem 4.6.1 to $\mathbb{f}_1$ and so on.

To this end we set up the following iterative scheme: $\mathbb{f}_0 = \tilde{f}_0$, $\mathcal{H}_{-1} = \mathrm{id}$. For $k \in \mathbb{N}_0$ let $F_k(z) = z + \omega + \varsigma_k + \hat{f}_0 + \sum_{j=0}^{k} \tilde{f}_j(z)$, $\tau(F_k) = \omega$, $\int_{S^1} \mathbb{f}_k = 0$

$$\mathbb{F}_k(z) = z + \omega + \xi_k + \mathbb{f}_k(z)\,; \quad \tau(\mathbb{F}_k) = \omega \tag{4.6.10}$$

$$H_k^{-1} \circ \mathbb{F}_k \circ H_k(z) = z + \omega\,; \quad H_k(z) = z + h_k(z) \tag{4.6.11}$$

$$\mathcal{H}_k = \mathcal{H}_{k-1} \circ H_k \tag{4.6.12}$$

$$\mathcal{H}_k^{-1} \circ F_{k+1} \circ \mathcal{H}_k = \mathbb{F}_{k+1}. \tag{4.6.13}$$

Note that, for each $k \in \mathbb{N}_0$,

$$\begin{aligned}
\mathcal{H}_k^{-1} \circ F_k \circ \mathcal{H}_k(z) &= H_k^{-1} \circ \mathcal{H}_{k-1}^{-1} \circ F_k \circ \mathcal{H}_{k-1} \circ H_k(z) \\
&= H_k^{-1} \circ \mathbb{F}_k \circ H_k = R_\omega.
\end{aligned} \tag{4.6.14}$$

[11] This choice (a la Panley Wiener) for the decomposition of $f$ is not optimal, yet it makes the latter computations simpler.

The rest of the proof consists in a rather tedious verification that the induction is well posed and in estimating the norms of the objects involved.

Let us assume by induction that there exists $B > 1$ such that, for each $k \in \mathbb{N}$ and $j < k$, $\|\mathbb{f}_j\|_{\alpha_j/2} \le B\|\tilde{f}_j\|_{\alpha_j}$. In addition, we write $\mathcal{H}_k(z) = z + \mathfrak{h}_k(z)$ and, setting $3\delta := a(r-4) - b$, assume that

$$\begin{aligned}
\|\mathfrak{h}_{k-1}\|_{\alpha_{k-1}/4} &\le 10^{-3}\sum_{j=0}^{k-1} e^{-\delta j}\alpha_j =: 10^{-3}A_{k-1}\\
\|\mathfrak{h}'_{k-1}\|_{\alpha_k/8} &\le \frac{1}{4} - \frac{1}{2k+1}.
\end{aligned}$$

Note that this is obviously true for $k = 0$. Remark that Theorem 4.6.1 implies that there exists a solution $h_k \in \mathbb{B}_{\alpha_k/4}$ to (4.6.11) provided $\|\mathbb{f}_k\|_{\alpha_k/2} \le C_\star \alpha_k^3$, with $C_\star = C_0 10^{-11}$. Under the above hypotheses,

$$\begin{aligned}
\|\mathbb{f}_k\|_{\alpha_k/2} &\le B\|\tilde{f}_k\|_{\alpha_k} \le 2B|f|_{\mathcal{C}^r} e^{-(ar-a-b)k+a}\\
&\le 2B|f|_{\mathcal{C}^r} b^{-3}(k+1)^{-3} e^{-3\delta k+4a}\alpha_k^3 \le C_\star \delta^6 e^{-3\delta k}\alpha_k^3 \le C_\star \delta^6 e^{-3\delta k}\alpha_k^3
\end{aligned}$$

provided $\delta > 0$ and $|f|_{\mathcal{C}^r} \le \frac{1}{2}C_\star B^{-1} b^3 e^{-4a}\delta^6$. Thus, by Theorem 4.6.1,

$$\|h_k\|_{\alpha_k/4} \le 3C_0^{-\frac{1}{3}}\|\mathbb{f}_k\|_{\alpha_k/2}^{\frac{1}{3}} \le 3C_0^{-\frac{1}{3}} C_\star^{\frac{1}{3}}\delta^2 e^{-\delta k}\alpha_k \le 10^{-3}\delta^2 e^{-\delta k}\alpha_k.$$

Moreover (see Problem 4.35)

$$\|h'_k\|_{\alpha_k/8} \le 16\pi\|h_k\|_{\alpha_k/4}\alpha_k^{-1} \le 4\cdot 10^{-2}\delta^2 e^{-\delta k} < 1/4.$$

By (4.6.12) it follows

$$\mathfrak{h}_k(z) = h_k(z) + \mathfrak{h}_{k-1}(z + h_k(z)),$$

which is well posed in $\mathbb{B}_{\alpha_k/4}$ provided $a \ge 2$, since this implies that $\alpha_k(1 + 4\cdot 10^{-3}\delta^2 e^{-\delta k}) \le \alpha_{k-1}$. Moreover $\|\mathfrak{h}_k\|_{\alpha_k/4} \le 10^{-3}A_k$

$$\|\mathfrak{h}'_k\|_{\alpha_k/8} \le \left[\frac{1}{4} - \frac{1}{2k+1}\right](1 + 4\cdot 10^{-2}\delta^2 e^{-\delta k}) + 4\cdot 10^{-2}\delta^2 e^{-\delta k} \le \frac{1}{4} - \frac{1}{2k+2}.$$

Equation (4.6.13), also recalling (4.6.14), is equivalent to

$$\begin{aligned}
\tilde{\mathbb{f}}_{k+1}(z) &= \mathbb{f}_{k+1}(z) + \xi_{k+1}\\
\tilde{\mathbb{f}}_{k+1}(z) &= \varsigma_{k+1} - \varsigma_k + \mathfrak{h}_k(z+\omega) - \mathfrak{h}_k(z + \omega + \tilde{\mathbb{f}}_{k+1}(z))\\
&\quad + \tilde{f}_{k+1}(x + \mathfrak{h}_k(x)).
\end{aligned} \tag{4.6.15}$$

Since $\mathcal{H}_k$ is invertible this implies that $\tilde{\mathfrak{f}}_{k+1}$ is well defined on the real line. This implies that

$$\begin{aligned}\mathbb{F}_{k+1}(x) =& x+\omega+\varsigma_{k+1}-\varsigma_k+\mathfrak{h}_k(x+\omega)-\mathfrak{h}_k(x+\omega+\tilde{\mathfrak{f}}_{k+1}(x))+\tilde{f}_{k+1}(x+\mathfrak{h}_k(x))\\ =:& x+\omega+g(x).\end{aligned}$$

By induction it follows that, for all $q\in\mathbb{N}$,

$$\mathbb{F}_{k+1}^q(z)=z+n\omega+\sum_{j=0}^{n-1}g(\mathbb{F}_{k+1}^j(x)).$$

As usual remark that $\mathbb{F}_{k+1}^q(z)-z$ cannot be an integer since otherwise we would have a periodic point and we would have $\tau(F_{k+1})=\frac{p}{q}\in\mathbb{Q}$, contrary to the hypothesis. It follows that for each $q\in\mathbb{N}$ there exists $p\in\mathbb{N}$ such that, for all $x\in\mathbb{R}$,

$$p-1\le x+q\omega+\sum_{j=0}^{q-1}g(\mathbb{F}_{k+1}^j(x))\le p.$$

Since, $\tau(F)=\omega$ it follows

$$\left|\frac{1}{q}\sum_{j=0}^{q-1}g(\mathbb{F}_{k+1}^j(x))\right|\le\frac{1}{q}$$

hence, by the arbitrariness of $q$,

$$|\varsigma_{k+1}-\varsigma_k|\le\frac{1}{4}\|\tilde{\mathfrak{f}}_{k+1}\|_\infty+\|\tilde{f}_{k+1}\|_\infty$$

Using the above estimate in equation (4.6.15) yields $\|\mathfrak{f}_{k+1}\|_\infty\le 4\|\tilde{f}_{k+1}\|_\infty$, hence

$$|\varsigma_{k+1}-\varsigma_k|\le 2\|\tilde{f}_{k+1}\|_\infty. \tag{4.6.16}$$

To obtain an estimate of the $\|\cdot\|_{\alpha_{k+1}}$ norm of $\tilde{\mathfrak{f}}_{k+1}$ from equation (4.6.15) we consider the operator $\mathcal{K}:D\to\mathbb{B}_{\alpha_{k+1}/2}$, where

$$D=\{\varphi\in\mathbb{B}_{\alpha_{k+1}/2}\;:\;\|\varphi\|_{\alpha_{k+1}/2}\le\frac{1}{2}B\|\tilde{f}_{k+1}\|_{\alpha_{k+1}}\},$$

defined by

$$\mathcal{K}(\varphi)=\varsigma_{k+1}-\varsigma_k+\mathfrak{h}_k(z+\omega)-\mathfrak{h}_k(z+\omega+\varphi(z))+\tilde{f}_{k+1}(x+\mathfrak{h}_k(x)).$$

The operator is well defined if $e^{-a}\le\frac{1}{4}$ and $|f|_{\mathcal{C}^r}\le e^{-2a}\frac{b}{4B}$. Moreover $\mathcal{K}(D)\subset D$ provided $B\ge 8$. By the usual contraction theorem it follows

$\|\tilde{\mathbb{f}}_{k+1}\|_{\alpha_{k+1}/2} \leq \frac{1}{2}B\|\tilde{f}_{k+1}\|_{\alpha_{k+1}}$. Thus $\|\mathbb{f}_{k+1}\|_{\alpha_{k+1}/2} \leq \|\tilde{\mathbb{f}}_{k+1} - \int_{S^1}\tilde{\mathbb{f}}_{k+1}\|_{\alpha_{k+1}/2} \leq B\|\tilde{f}_{k+1}\|_{\alpha_{k+1}}$, whereby concluding the induction.

The last thing we must prove is that the change of coordinate $\mathcal{H}_n$ is convergent. Note that

$$|\mathcal{H}_n'(x)| \leq \prod_{k=0}^{n} \|\tilde{H}_k'\|_{\frac{\alpha_k}{8}} \leq \prod_{k=0}^{n} e^{4\cdot 10^{-2}\delta^2 e^{-\delta k}} \leq e^{4\cdot 10^2\delta}.$$

It is then easy to see that the $\mathcal{H}_n$ form a Chauchy sequence in $\mathcal{C}^1$. The theorem follows by collecting all the above inequalities and setting $B = 8$, $a = 2$, $b = (r-4)/3$ and recalling the condition $|f|_{\mathcal{C}^r} \leq \frac{1}{2}C_\star B^{-1}b^3e^{-4a}\delta^6$. $\square$

## Problems

**4.27.** If $M$ is a $\mathcal{C}^r$ manifold, $f \in \mathcal{C}^r(M, M)$ is a diffeomorphism and $\tau \in \mathcal{C}^r(M, (0,\infty))$, show that the associated suspension flow is defined on a $\mathcal{C}^r$ manifold and is $\mathcal{C}^r$.

**4.28.** Consider the Dynamical System $([0,1], T)$ where

$$T(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor = \frac{1}{x} \mod 1$$

($\lfloor a \rfloor$ is the integer part of $a$). This is called the *Gauss map*. Prove that for each $x \in \mathbb{Q} \cap [0,1]$ holds $\lim_{n\to\infty} T^n(x) = 0$.

**4.29.** Prove that any infinite continuous fraction of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

with $a_i \in \mathbb{N}$ defines a real number.

**4.30.** Prove that, for each $a \in \mathbb{N}$,

$$x = \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \ddots}}} = \frac{-a + \sqrt{a^2+4}}{2}.$$

**4.31.** Prove that, for all $s > 2$, for Lebesgue almost all numbers $x \in [0,1]$ there exists $C > 0$ such that[12]

$$\left| x - \frac{p}{q} \right| \geq \frac{C}{q^s}$$

for all $p, q \in \mathbb{N}$.

**4.32.** Let $f_a(x) = \frac{1}{a+x}$. Given a sequence $[a_0, a_1, \dots, a_n]$ show that

$$f_{a_0} \circ \cdots \circ f_{a_n}(x) = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \ddots \cfrac{1}{a_n + x}}} = \frac{p_n + p_{n-1}x}{q_n + q_{n-1}x},$$

where $p_{n+1} = a_{n+1}p_n + p_{n-1}$ and $q_{n+1} = a_{n+1}q_n + q_{n-1}$, $p_{-1} = 0$, $q_{-1} = 1$, $p_0 = 1$, $q_0 = a_0$. In addition, show that, for all $n \in \mathbb{N}$, $p_n q_{n-1} - q_n p_{n-1} = (-1)^n$ and decude that $p_n, q_n$ have no common divisor different from one. Finally, verify that

$$f_{a_0} \circ \cdots \circ f_{a_n}(x) - f_{a_0} \circ \cdots \circ f_{a_{n+1}}(x) = \frac{(-1)^{n+1}[x^2 + a_{n+1}x - 1]}{(q_n + q_{n-1}x)(q_{n+1} + q_n x)}.$$

**4.33.** Let $\omega \in [0,1)$. Show that there exists infinitely many $p, q \in \mathbb{N}$ such that

$$\left| \omega - \frac{p}{q} \right| \leq \frac{1}{q^2}.$$

**4.34.** Let $\omega \in [0,1)$ have the continuous fraction expansion given by $[a_0, a_1 \dots]$. Suppose that $\inf_n a_n > 0$ and $\sup_n a_n < \infty$.[13] Show that there exists a constant $c > 0$ such that for all $p, q \in \mathbb{N}$

$$\left| \omega - \frac{p}{q} \right| \geq \frac{c}{q^2}.$$

**4.35.** For each $\varphi \in \mathbb{B}_\alpha$ and $\beta < \alpha$ show that $\|\varphi'\|_\beta \leq \frac{2\pi\|\varphi\|_\alpha}{\alpha - \beta}$.

**4.36.** Let us consider an holomorphic function $f : U \subset \mathcal{C} \to \mathcal{C}$ where $U$ is an open set containing zero. Assume that $f(0) = 0, f'(0) = e^{2\pi i \omega}$. Prove that, if $\omega$ is Diophantine, then it is possible to find an open set $D \subset U$ on which $f$ is conjugated to the map $f_\omega(z) = e^{2\pi i \omega} z$.

[12] The composition below is often called *iterated function system*, it can be naturally viewed as a time dependent dynamical system.

[13] Such numbers $\omega$ are called of *constant type*.

# Hints to solving the Problems

4.2 Consider a system $([0,1],T)$ such that $T$ is piecewise linear, it has an unstable fixed point at $x_0$ and an attracting fixed point at $z \in (0,x_0)$ so that the set $[z,x_0]$ is forward invariant. Finally arrange so that $T(0) = x_0$ and $T(x) \le x_0$ for $x$ near zero.

4.10 The equation $\dot{x} = \omega = (\omega_1,\omega_2)$ on $\mathbb{T}^2$ has the solution $x(t) = (x_1(t),x_2(t)) = x_0 + \omega t \mod 1$. If one looks at the flow only at the times $\tau_n = n\omega_1^{-1}$, then $x(n\tau) = x_0 + (0,\alpha n) \mod 1$ where $\alpha := \frac{\omega_2}{\omega_1}$. One can then consider the circle map $f : S^1 \to S^1$ defined by $f(z) = z + \alpha \mod 1$. Clearly, if the orbits of such a map are dense in $S^1$ the original flow will be dense in $\mathbb{T}^2$. The density follows in the case $\alpha \notin \mathbb{Q}$. In fact this implies that $f$ has no periodic orbits. Then $\{f^n(0)\}$ is made of distinct points and contains a converging subsequence (by compactness) hence for each $\varepsilon > 0$ exists $\bar{n} \in \mathbb{N}$ such that $|z - f^{\bar{n}}(z)| \le \varepsilon$, that is $f^{\bar{n}}$ is a rotation by less than $\varepsilon$. Hence the orbit $\{f^{k\bar{n}}(z)\}$ enters in the $\varepsilon$-neighborhood of each point of $S^1$.

4.13 Suppose that there exists $\varphi(r,s)$, $\varphi \in \mathcal{C}^0([0,1]\times\mathbb{T}^1,\mathbb{R})$, such that $\varphi(1,\cdot)$ is a parametrization of $\Gamma$ and $\varphi(0,s) = y$ for some fixed $y \in \mathbb{T}^2$ (i.e. $\Gamma$ is homotopic to $y$).

4.12 First of all notice that if $\xi(t)$ is the derivative with respect to the initial condition and $\xi(0) = \lambda V(x(0))$, for some $\lambda$, then $\xi(t) = \lambda V(x(t))$ for all $t$. Define then $\omega(x,y) = x_1y_2 - x_2y_1$ and verify that $x, y \ne 0$ and $\omega(x,y) = 0$ imply that there exists $\lambda \in \mathbb{R}$ such that $x = \lambda y$.[14] This means that $\omega(\xi(t), V(x(t)))$ cannot change sign. Hence the result.

4.17 Let $\liminf_{n\to\infty} \frac{a_n}{n} = a > -\infty$, then for each $\varepsilon, m > 0$ exists $\bar{n} \in \mathbb{N}$, $\bar{n} > m$, such that $|a_{\bar{n}} - a\bar{n}| \le \varepsilon\bar{n}$. Let $l \in \mathbb{N}$, $l > \bar{n}$, and write $l = k\bar{n} + r$, $r < \bar{n}$, then

$$\begin{aligned} a - \varepsilon \le \frac{a_l}{l} \le \frac{ka_{\bar{n}} + kL + a_r}{l} &\le \frac{k\bar{n}(a+\varepsilon) + kL + a_r}{l} \\ &= a + \varepsilon + \frac{L}{m} + \frac{a_r}{l}. \end{aligned}$$

From which the claim follows.

4.18 Stetting $I = [a,b]$ note that $g(x) = f(x) - x$ has a zero in $I$.

[14] By the way, $\omega$ is a symplectic form and its existence implies that the manifold is orientable.

4.19 This is the same than saying $\bigcup_{k\in\mathbb{N}} f^{-k}[x, f^n(x)] = S^1$. Argue by contradiction. Consider $f^{-kn}[x, f^n(x)]$, this are contiguous intervals. If they do not cover all $S^1$, then their length must go to zero and $f^{-kn}x$ must have a limit, call it $z$. Then

$$z = \lim_{k\to\infty} f^{-kn}(x) = \lim_{k\to\infty} f^{-kn}(f^n(x)) = f^n(z).$$

Hence $f$ must have a periodic point contradicting $\tau(f) \notin \mathbb{Q}$.

4.26 For the second inequality use Problem 4.35.

4.28 If $x = \frac{p_0}{q_0}$, $p_0 \le q_0$, then $q_0 = k_1 p_0 + p_1$, with $p_1 < p_0$, and $T(x) = \frac{p_1}{p_0}$. Let $q_1 = p_0$ and go on noticing that $p_{i+1} < p_i$.[15]

4.29 Note that if you fix the first $n$ $\{a_i\}$, this corresponds to specifying which elements of the partition $\{[\frac{1}{i+1}, \frac{1}{i}]\}$ are visited by the trajectory of $\{T^i x\}$, $T$ being the Gauss map. By the expansivity of the map readily follows that $x$ must belong to an interval of size $\lambda^{-n}$ for some $\lambda > 1$.

4.30 Note that $T(x) = x$, where $T$ is the Gauss map. Study periodic continuous fractions of period two.

4.31 To see it consider the sets $I_{p,q} := [\frac{p}{q} - Cq^{-s}, \frac{p}{q} - Cq^{-s}]$. If $p \le q$, then $I_{p,q} \subset [0, 1]$. Clearly if $\alpha \notin I_{p,p}$ for all $q \ge p \in \mathbb{N}$, then $\alpha$ satisfies the Diophantine condition. But $\sum_{q\ge p} |I_{p,q}| \le C \sum_{q=1}^{\infty} q^{-s+1}$ which converges provided $s > 2$ and can be made arbitrarily small by choosing $C$ small. Accordingly, almost all numbers are Diophantine for any $s > 2$.

4.32 By induction.

4.33 The result is trivial for rational numbers. By Problem 4.29, $\omega = \lim_{n\to\infty} f_{a_0} \circ \cdots \circ f_{a_n}(0)$. Moreover, $f_a([0, \infty)) \subset [0, a^{-1}]$. Thus for each $n \in \mathbb{N}$ there exists $x_n \in [0, a_{n+1}^{-1}]$ such that $\omega = f_{a_0} \circ \cdots \circ f_{a_n}(x_n)$. Thus, be the monotonicity of the $f_a$ it follows that either $\omega \in [f_{a_0} \circ \cdots \circ$

[15] This is nothing else that the *Euclidean algorithm* to find the greatest common divisor of two integers [Euc78, Elements, Book VII, Proposition 1 and 2]. The greatest common divisor is clearly the last non-zero $p_i$. This provides also a remarkable way of writing rational numbers: *continuous fractions*

$$\frac{p_0}{q_0} = \cfrac{1}{k_1 + \cfrac{1}{k_2 + \ddots + \cfrac{1}{k_n}}}.$$

$f_{a_n}(0), f_{a_0}\circ\cdots\circ f_{a_{n+1}}(0)]$ or $\omega\in[f_{a_0}\circ\cdots\circ f_{a_{n+1}}(0), f_{a_0}\circ\cdots\circ f_{a_n}(0)]$. One can then use the equalities of Problem 4.32 to conclude all the rationals $f_{a_0}\circ\cdots\circ f_{a_n}(0)$ satisfy

$$|\omega - f_{a_0}\circ\cdots\circ f_{a_n}(0)| \leq \frac{1}{a_{n+1}q_n^2}.$$

You did not like this argument? Here is an interesting alternative. Problem 4.32 implies that

$$f_{a_0}\circ\cdots\circ f_{a_n}(0) = \sum_{k=0}^{n}\frac{(-1)^k}{q_k q_{k-1}}.$$

Since the odd and even partial sum of an alternating series form monotone sequences that converge to the limit from opposite sides, it follows that

$$\begin{aligned}|\omega - f_{a_0}\circ\cdots\circ f_{a_n}(0)| &\leq |f_{a_0}\circ\cdots\circ f_{a_n}(0) - f_{a_0}\circ\cdots\circ f_{a_{n+1}}(0)|\\ &\leq \frac{1}{a_{n+1}q_n^2}.\end{aligned}$$

4.34 As we have argued at the end of the hint of Problem 4.34, $\omega\in[f_{a_0}\circ\cdots\circ f_{a_n}(0), f_{a_0}\circ\cdots\circ f_{a_{n+1}}(0)] =: I_n$. Note that if $q < q_n$ then

$$\left|\frac{p}{q}-\frac{p_n}{q_n}\right| \geq \frac{1}{q_n q}\ ; \quad \left|\frac{p}{q}-\frac{p_n}{q_n}\right| \geq \frac{1}{q_{n+1} q}.$$

But $|I_n| = \frac{1}{q_n q_{n+1}}$ so it cannot contain any rational number with denominator strictly less than $q_n$. Accordingly, $\frac{p}{q}\notin I_n$ and thus $|\omega-\frac{p}{q}|\geq \frac{1}{q_{n+1}q} > \frac{1}{q_{n+1}q_n}$. In other words the fraction determined by $[a_0,\dots,a_n]$ are the best approximation of $\omega$ among all the numbers with denominator smaller than $q_n$. Since,

$$\begin{aligned}|\omega - f_{a_0}\circ\cdots\circ f_{a_n}(0)| &\geq |f_{a_0}\circ\cdots\circ f_{a_n}(0) - f_{a_0}\circ\cdots\circ f_{a_{n+2}}(0)|\\ &\geq \frac{1}{(a_{n+1}+2)q_n^2}.\end{aligned}$$

the result follows by simple computations.

4.35 Since $\varphi$ is holomorphic by Rienmann formula we have

$$\varphi'(z) = \frac{1}{2\pi i}\int_\gamma \frac{\varphi(\zeta)}{(z-\zeta)^2}d\zeta$$

where $\gamma$ is a simple closed curve in $D_\alpha$ surrounding $z \in D_\beta$. For $\gamma$ we chose the curve $\{z + \frac{\alpha-\beta}{2\pi} e^{i\theta}\}_{\theta \in [0,2\pi]}$. Hence

$$\|\varphi'\|_\beta \le \frac{1}{2\pi} \int_0^{2\pi} \frac{2\pi |\varphi|_\alpha}{\alpha - \beta} d\theta = \frac{2\pi |\varphi|_\alpha}{\alpha - \beta}.$$

4.36 Mimic Theorem 4.6.1.

## Notes

Lemma 4.3.2 is due to Siegel [Sie45], see [NZ99] for a detailed treatment of flows on surfaces. A detailed treatment of circle rotations can be found in [Her83, Her86]. A general treatment of KAM theory for Hamiltonian Systems, with an emphasis on concrete applications, can be found in [CC95].