Chapter 3

Bifurcation Theory (the minimum)

Continuing the analysis of the previous section we would like to place it on a more systematics ground: we worried only about hyperbolic fixed points; are more complex situations relevant? To answer to such a question it is first necessary to understand its meaning, that is:
what does it mean to be irrelevant?

3.1 Generic Vector fields

By relevant we mean situations which are *typical*. We would like to summarise the content of Section 2 as follows:

Theorem 3.1.1 *We understand the typical local behavior of the solutions of the differential equations*

$$\dot{x} = V(x) \tag{3.1.1}$$

where $V \in \mathcal{C}^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^d)$.

However, to make sense of Theorem 3.1.1 it is necessary to give a technical meaning to the words *behavior, local* and *typical*.

3.1.1 Local behavior

We say that we *understand* the behavior of a vector field in an open set U if it is *equivalent* to a vector field whose associated ODE can be explicitly solved.

Definition 3.1.2 *We say that two vector fields V, W are equivalent in the open set U, if, for each $t > 0$, there exists a homeomorphism $F : U \to U$ such that, calling ϕ_t^V, ϕ_t^W the flows generated by the vector fields, holds $\phi_t^v \circ F = F \circ \phi_t^W$.*

By *local* understanding in a region K we mean that for each point $x \in K$ we are able to consider some neighborhood of x in which we understand the solutions of (3.1.1).[1]

If we could consider only neighborhoods U in which $V(x) \neq 0$ with, at most, the exception of isolated points where the linear part is hyperbolic, then we understand already the local behavior. In fact, either $V(\bar{x}) \neq 0$ and then the flow box Theorem tells us that the field has the same local behavior than a constant vector field; or, if $V(\bar{x}) = 0$, then Grobmann-Hartman Theorem tells us that the field has the same local behavior than its linear part.

Of course, this is not always the case (think of the case $V \equiv 0$), our claim is that the above situation is *typical*.

3.1.2 Typical

Definition 3.1.3 *Given a topological space Ω, we say that a set $A \subset \Omega$ is* generic *if it is open and dense. A set is* typical *if it is the countable intersection of* generic *sets (this is also called a* residual *set).*

Since $\mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d)$ is a Banach space, its topology is determined by the norm.

Problem 3.1 *Prove that the finite intersection of generic set is generic. Prove that a residual set is dense.*

Problem 3.2 *Give an example of a typical set in $\mathbb{R}$ with zero Lebesgue measure.*

Next, for each $K \subset \mathbb{R}^d$, let us define[2]

$$A_K := \{V \in \mathcal{C}^1_{\rm loc}(\mathbb{R}^n, \mathbb{R}^n) \;:\; \forall x \in K,\; V(x) = 0 \text{ implies } \partial_x V \text{ hyperbolic } \}$$

Remark 3.1.4 *In the following we will prove that, for K compact, A_K is generic, hence $A_{\mathbb{R}^d}$ is typical. Note that the same holds for*

$$\{V \in \mathcal{C}^1_{\rm loc}(\mathbb{R}^n, \mathbb{R}^n) \;:\; \forall x \in K,\; V(x) = 0 \textit{ implies } \det(\partial_x V) \neq 0\}.$$

Yet, it is convenient to consider small generic sets (see Problems 3.25, 3.26). This allows to obtain a generic understanding with the least effort.

[1] Note that, if K is compact, then finitely many such neighborhoods will cover K. On the other hand if, for example, $K = \mathbb{R}^d$, then countably many neighborhoods will do the job.

[2] Since our analysis is local, the following can be trivially adapted to the case $\mathcal{C}^1_{\rm loc}(U, \mathbb{R}^n)$, for some open set U. We avoid it to simplify notation.

Problem 3.3 *Prove that, for each compact set $K \subset \mathbb{R}^d$, if $V \in A_K$, then V has only finitely many zeroes in K.*

Problem 3.4 *Prove that, for each compact set $K \subset \mathbb{R}^d$, A_K is open.*

To prove that A_K is generic we need to establish the density, this is not entirely obvious and we need a result of independent interest.

Theorem 3.1.5 (Sard–baby version) *Let $F \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d)$, and $A = \{x \in \mathbb{R}^d \ : \ \det(D_x F) = 0\}$, then $F(A)$ has zero Lebesgue measure.*

PROOF. Let $Q_\delta(x) := \{z \in \mathbb{R}^d \ : \ |x_i - z_i| \leq \delta \quad \forall i \in \{1, \dots, d\}\}$, clearly it suffices to prove that for each $\bar{x} \in \mathbb{R}^d$ the Lebesgue measure of $F(A \cap Q_1(\bar{x}))$ is zero. Now, for each $n \in \mathbb{N}$ and $k \in \{-n, \dots, 0, \dots, n\}^d =: S_n$, let $x_k := \frac{k}{n}$ and $\Delta_k := Q_{1/2n}(\bar{x} + x_k)$. Clearly $Q_1(\bar{x}) \subset \cup_{k \in S_n} \Delta_k$. We will consider only the Δ_k such that $\Delta_k \cap A \neq \emptyset$. For each such Δ_k let us chose $\xi_k \in \Delta_k \cap A$.

Next, consider the function $\Psi : Q_1(\bar{x})^2 \to \mathbb{R}$ defined by

$$\Psi(x, y) := \begin{cases} \frac{\|F(x) - F(y) - D_x F(x-y)\|}{\|x-y\|} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Since $F = \mathcal{C}^1$ we have $\Psi \in \mathcal{C}^0$, hence for each $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\sup_{\|x-y\| \leq n^{-1}} \Psi(x, y) < \varepsilon$$

for each $n > n_\varepsilon$. Since $\xi_k \in A$, there exists $v_k \in \mathbb{R}^d$, $\|v_k\| = 1$, such that $\langle v_k, D_{\xi_k} F w \rangle = 0$ for all $w \in \mathbb{R}^d$. Hence, setting $C = \|DF\|_\infty$ and for n large enough, $F(\Delta_k) \subset \{F(\xi_k) + w + t v_k \in \mathbb{R}^d \ : \ \langle w, v_k \rangle = 0; \ \|w\| \leq C n^{-1}; \ |t| \leq \frac{\varepsilon}{n})\}$.

Thus, calling λ the Lebesgue measure,

$$\lambda(F(\Delta_k)) \leq 4^{d-1} C^{d-1} n^{-d-1} \frac{\varepsilon}{n} = \lambda(\Delta_k) \cdot 4^{d-1} C^{d-1} \varepsilon.$$

Thus

$$\lambda(F(A \cap Q_1(x))) \leq 4^{d-1} C^{d-1} \sum_{k \in S_n} \lambda(\Delta_k) \varepsilon = 4^d C^{d-1} \varepsilon,$$

as announced. □

Problem 3.5 *Use Sard's Theorem to show that, for each compact set $K \subset \mathbb{R}^d$, A_K is dense in $\mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d)$. Prove that $A_{\mathbb{R}^d}$ is typical.*

3.2 Generic families of vector fields

Our next aim is to consider a situation in which the system has a control parameter. That is, it is described by the equations of the type

$$\dot{x} = V(x, \lambda) \tag{3.2.2}$$

where $x \in \mathbb{R}^d$ and $\lambda \in [-2, 2]$ is the parameter that, in principle, can be varied. Now by *local* understanding in a region K we mean that for each point $(\bar{x}, \bar{\lambda}) \in K \times [-1, 1] =: K^1$ we can find a neighborhood of the form $U \times (\lambda - \varepsilon, \lambda + \varepsilon)$ in which we are able to understand the behavior of the solutions of (3.2.2).

Let us now try to understand the local picture for typical families of vector fields. In analogy with the previous section, for any $K \subset \mathbb{R}^d$, let us consider

$$\begin{aligned}\bar{A}_K := \{V \in \mathcal{C}^1 \ :\forall\,(x,\lambda) \in K^1,\ V(x,\lambda) = 0 \text{ implies}\\ \partial_x V(x,\lambda) \text{ hyperbolic } \}\end{aligned}$$

Problem 3.6 *Prove that if $V \in \bar{A}_K$, then for each $(\bar{x}, \bar{\lambda}) \in K^1$ there exists an open set of the form $U \times (-\varepsilon + \bar{\lambda}, \varepsilon + \bar{\lambda}) =: U \times I$ such that either $V(x, \lambda) \neq 0$ or there exists $X \in \mathcal{C}^1(I, K)$ such that $V(X(\lambda), \lambda) = 0$ for each $\lambda \in K$ and there are no other zeroes in $U \times I$. Then, prove that $\bar{A}_K$ is open.*

Clearly the above situations can be treated exactly as we did in the previous section and are therefore locally understandable. Unfortunately, the above does not exhaust all the possibilities.

Lemma 3.2.1 *For each K with non empty interior $\bar{A}_K$ is not generic.*

Proof. Since $\bar{A}_K$ is open, the problem must be the density. To see this let us consider, for example, the case $d = 1$, a compact set K with interior containing zero and the family

$$V(x, \lambda) = \lambda a + \lambda x + bx^2. \tag{3.2.3}$$

Now let us consider any $W \in \mathcal{C}^1(\mathbb{R} \times [-1, 1], \mathbb{R})$ and look at $\tilde{V}(x, \lambda, \mu) := V(x, \lambda) + \mu W(x, \lambda)$. The claim is that for each μ sufficiently small, then $\tilde{V}(x, \lambda, \mu) \notin \bar{A}_K$. In fact, there exists $(x(\mu), \lambda(\mu)) \in K$ such that both $\tilde{V}(x(\mu), \lambda(\mu), \mu) = 0$ and $\partial_x \tilde{V}(x(\mu), \lambda(\mu), \mu) = 0$. To see this we define the function $F : \mathbb{R}^3 \to \mathbb{R}^2$

$$F(x, \lambda, \mu) := \begin{pmatrix} \lambda a + \lambda x + bx^2 + \mu W(x, \lambda) \\ \lambda + 2bx + \mu \partial_x W(x, \lambda) \end{pmatrix} = \begin{pmatrix} \tilde{V} \\ \partial_x \tilde{V} \end{pmatrix}, \tag{3.2.4}$$

clearly we are looking for $(x(\mu), \lambda(\mu))$ such that $F(x(\mu), \lambda(\mu)) = 0$. Since $F(0, 0, 0) = 0$ we can apply the implicit function theorem provided

$$\begin{pmatrix} 0 & a \\ 2b & 1 \end{pmatrix}$$

is invertible, that is if $ab \neq 0$. We have thus seen that the family has an open neighborhood disjoint from $\bar{A}_K$, hence the latter set cannot be dense. □

Thus, to have a generic situation we need to consider a larger set.

Looking at the above example it is natural to ask that $\partial_\lambda V \neq 0$ if $\det(\partial_x V) = 0$. This is a good idea but it does not suffice to have a nice theory. As we have seen,[3] and we will see later on, it is natural to have some condition on the second derivative. It is then convenient to consider at least $\mathcal{C}^2$ vector fields, since we will see in the following that higher derivatives can play a role, we will consider $\mathcal{C}^r$ vector fields, $r \geq 2$. Accordingly from now on the genericity will be according to the $\mathcal{C}^r$ topology. This would not have changed the previous discussion, see Problem 3.32.

The above can be made precise in many ways. Here is a simple, but not totally satisfactory, possibility. For $K \subset \mathbb{R}^d$ let $K^1 := K \times [-1, 1]$.

$$B_K = \left\{ V \in \mathcal{C}^r_{\text{loc}} \ : \ \forall\, (x, \lambda) \in K^1,\ V(x, \lambda) = 0 \Longrightarrow \text{rank}\,(\partial_x V\ \partial_\lambda V) = d \right\}$$

Let us understand how the vector fields in B_K look like.

Lemma 3.2.2 *If $V \in B_K$ and $V(\bar{x}, \bar{\lambda}) = 0$, then there exists $\varepsilon > 0$ and a neighborhood $U \ni \bar{x}$ such that the set of zeroes of the vector field $V(x, \lambda)$ in $U \times (\bar{\lambda} - \varepsilon, \bar{\lambda} + \varepsilon)$ consists of a smooth curve.*

Proof. First suppose, without loss of generality, that $(\bar{x}, \bar{\lambda}) = (0, 0)$.

If $\det(\partial_x V(0, 0)) \neq 0$, then we can argue as in Problem 3.4. The implicit function theorem yields $\varepsilon > 0$ a neighborhood U of zero and $x \in \mathcal{C}^r([-\varepsilon, \varepsilon], \mathbb{R}^d)$ such that $V(x(\lambda), \lambda) = 0$ are the only zeroes of the vector fields $V(\cdot, \lambda)$, $\lambda \in [-\varepsilon, \varepsilon]$, in U.

On the contrary, if $\det(\partial_x V(0, 0)) = 0$ then the approach based on a direct application of the implicit function theorem fails. The problem is that the curve of the fixed points it is not a graph over λ so one need to change variables before applying the implicit functions theorem, let us see how.

The null space of $\partial_x V(0, 0)$ must have dimension one, otherwise we would have $\text{rank}\,(\partial_x V(0, 0)\ \ \partial_\lambda V(0, 0)) < d$, let $v \in \mathbb{R}^d$, $\|v\| = 1$, be the unique vector such that $\partial_x V(0, 0)v = 0$. Consider the change of variables $(\lambda, x) = F_v(\xi, \tau)$ defined by

$$\begin{aligned} x &= \xi - \tau v \\ \lambda &= \langle \xi, v \rangle. \end{aligned} \tag{3.2.5}$$

It is easy to check that F_v^{-1} is defined by

$$\begin{aligned} \tau &= \lambda - \langle x, v \rangle \\ \xi &= \lambda v + x - \langle x, v \rangle v. \end{aligned}$$

[3] In applying the implicit function theorem to (3.2.4).

Then define the field $\tilde{V} := V \circ F_v$. Since $F_v(0,0) = 0$, $\tilde{V}(0,0) = 0$. To apply the implicit function theorem in the new variable we need $\partial_\xi \tilde{V}$ to be invertible, but $\partial_\xi \tilde{V}(\xi, \tau) = \partial_x V(x, \lambda) + \partial_\lambda V(x, \lambda) \otimes v$.[4] It follows that $\partial_\xi \tilde{V}(0,0)$ must be invertible, otherwise there would exists $z \in \mathbb{R}^d$ such that, for all $\eta \in \mathbb{R}^d$,

$$0 = \langle z, \partial_\xi \tilde{V}(0,0)\eta \rangle = \langle z, \partial_x V(0,0)\eta \rangle + \langle z, \partial_\lambda V(0,0) \rangle \langle v, \eta \rangle.$$

Choosing $\eta = v$ follows $\langle z, \partial_\lambda V(0,0) \rangle = 0$ and hence $\partial_x V(0,0)^T z = 0$. This would mean that all the column of the rectangular matrix $(\partial_x V(0,0) \;\; \partial_\lambda V(0,0))$ are orthogonal to z contradicting the definition of B_K.

So we can apply the implicit function theorem and obtain (for ξ, τ in a neighborhood of zero) a $\mathcal{C}^1$ function $\xi(\tau)$ such that $\tilde{V}(\xi(\tau), \tau) = 0$, with $\xi'(\tau) = (\partial_x V + \partial_\lambda V \otimes v)^{-1} \partial_x V v$. Then, setting $(x(\tau), \lambda(\tau)) := F(\xi(\tau), \tau)$ we have a $\mathcal{C}^1$ curve and a neighborhood of zero in $\mathbb{R}^{d+1}$ such that $V(x(\tau), \lambda(\tau)) = 0$ and no other zero is present in the neighborhood. Note that

$$x'(\tau) = \frac{dx(\tau)}{d\tau} = -(\partial_x V + \partial_\lambda V \otimes v)^{-1} \partial_\lambda V. \tag{3.2.6}$$

To conclude, we note that if $\lambda(\tau)$ were invertible, then we could have parametrized the curve as $(x(\lambda), \lambda)$ without the above change of coordinates. It is then natural to investigate the points for which $\frac{d\lambda}{d\tau} = 0$.

$$\frac{d\lambda}{d\tau} = \langle \xi'(\tau), v \rangle = \langle (\partial_x V + \partial_\lambda V \otimes v)^{-1} \partial_x V v, v \rangle. \tag{3.2.7}$$

Since $\partial_x V(0,0)v = 0$, $\frac{d\lambda}{d\tau}(0) = 0$. □

Problem 3.7 *Show that $B_{\mathbb{R}^d}$ is typical.*

We have thus a typical set, yet it contains behaviors that we have never analyzed: equilibrium points with derivative having a one dimensional kernel and equilibrium points with no kernel but non hyperbolic derivative. It would be convenient if we could limit the appearance of such situations to a bare minimum. To do this systematically would require the development of a formalism beyond the present goals. Yet, for the case of one parameter families it is still possible to do it naively, provided one is willing to put up with some boring computations.

Definition 3.2.3 *Given $V \in \mathcal{C}^r$ we call a point $(\bar{x}, \bar{\lambda}) \in \mathbb{R}^{d+1}$ such that $V(\bar{x}, \bar{\lambda}) = 0$ and $V(\cdot, \bar{\lambda}) \notin A_{\overline{U}}$, for a neighborhood U of $\bar{x}$, a* bifurcation point*. Let $(\bar{x}, \bar{\lambda})$ be a bifurcation point, we call such point* non degenerate,

[4] Given two vectors $v, w \in \mathbb{R}^d$, by $v \otimes w$ we mean the matrix with elements $(v \otimes w)_{ij} = v_i w_j$.

if $\operatorname{rank}\big(DV(\bar{x},\bar{\lambda})\big) = d-1$, $\langle w, D^2V(v,v)\rangle \neq 0$ *where* v, w *are such that* $DVv = DV^Tw = 0$. *We call the the bifurcation point* regular *if it is non degenerate or if* $\det\big(DV(\bar{x},\bar{\lambda})\big) \neq 0$ *but there are two eigenvalues with zero real part and* $\operatorname{Tr}\left(\Pi_0\left[\frac{d}{d\lambda}A(\lambda)\right]\Pi_0\right) \neq 0$ *where* Π_0 *is the eigenprojector on the eigenspace associated to the above eigenvalues and* $A(\lambda) = \partial_x V(x(\lambda),\lambda)$, *where* $x(\lambda)$ *is determined by* $V(x(\lambda),\lambda) = 0$.

The idea is then to define the new sets

$$\tilde{B}_K = \{V \in B_K \;:\; \text{all the bifurcation points are regular}\}.$$

Let us show that the elements of $\tilde{B}_K$ enjoy a nice characterization.

Lemma 3.2.4 *In* $\tilde{B}_K$ *the bifurcation points are isolated.*

PROOF. Let us start analyzing the case of non degenerate bifurcation points, suppose without loss of generality that the bifurcation point is at $(0,0)$. By Lemma 3.2.2 we know that the zeroes of V lie on a curve $(x(\tau),\lambda(\tau))$, with the derivative with respect to τ given by (3.2.6), (3.2.7). We know that there exists unique normalized vectors w, v such that $\partial_x V(0,0)v = [\partial_x V(0,0)]^T w = 0$. In addition, (3.2.7) can be written has

$$\frac{d\lambda}{d\tau} = \langle \xi'(\tau), v\rangle = \langle \partial_x V v, (\partial_x V^T + v\otimes\partial_\lambda V)^{-1}v\rangle.$$

Also note that $(\partial_x V(0,0)^T + v\otimes\partial_\lambda V(0,0))w = v\langle\partial_\lambda V(0,0), w\rangle$,[5] and that $(\partial_x V(0,0) + \partial_\lambda V(0,0)\otimes v)v = \partial_\lambda V(0,0)$ and remember that we have a bifurcation point iff $\frac{d\lambda}{d\tau} = 0$. If the bifurcation point it is not isolated, then it must be that also the second derivative at zero is zero.[6] To show that we have an isolated point it thus suffices to compute the second derivative and show that it is not zero at zero. Since for $\tau = 0$ we have $\partial_x V v = 0$, remembering (3.2.6) we have

$$\frac{d^2\lambda}{d\tau^2}(0) = \sum_i \langle \partial_{x_i}\partial_x V v, (\partial_x V^T + v\otimes\partial_\lambda V)^{-1}v\rangle x_i' = \sum_i \frac{\langle \partial_{x_i}\partial_x V v, w\rangle v_i}{\langle \partial_\lambda V w\rangle},$$

which is different form zero iff $\langle \partial_x^2 V(v,v), w\rangle \neq 0$.

We are left with the case $\det(\partial_x V(\bar{x},\bar{\lambda})) > 0$ but with an eigenvalue which has a zero real part. This means that we have two purely imaginary eigenvalues. Let Π_0 be the eigenprojection associated to such two eigenvalues. By perturbation theory (see Appendix C) it follows that there exists a projector family $\Pi(x,\lambda)$ such that $\Pi\partial_x V = \partial_x V\Pi$ and $\Pi(\bar{x},\bar{\lambda}) = \Pi_0$.

[5] Remark that this implies $w = \alpha(\partial_x V(0,0)^T + v\otimes\partial_\lambda V(0,0))^{-1}v$ and $\alpha = \langle\partial_\lambda V(0,0), w\rangle \neq 0$ otherwise the vector field would not belong to B_K.

[6] Just compute the limit defining the derivative along a sequence of zeroes of λ that converge to zero.

Problem 3.8 *Show that a two by two real matrix has purely imaginary eigenvalues iff its trace is zero and the determinant positive.*

Then we have $\mathrm{Tr}(\Pi_0 \partial_x V(\bar{x}, \bar{\lambda})\Pi_0) = 0$. Now, let $x(\lambda)$ be the curve of the zeroes of V, then

$$\frac{d}{d\lambda} \mathrm{Tr}(\Pi \partial_x V \Pi)\bigg|_{\lambda=0} = \mathrm{Tr}\left(\Pi_0 \left[\frac{d}{d\lambda}\partial_x V\right] \Pi_0\right)$$

since $\Pi^2 = \Pi$ implies $\Pi\left(\frac{d}{d\lambda}\Pi\right)\Pi = 0$.[7] This concludes the argument. $\square$

Problem 3.9 *Show that $\tilde{B}_K$ is still generic.*

Thus, to achieve a typical local understanding of the behavior of one parameter families of vector fields we have to worry only about families with, at most, one regular bifurcation point. Let us suppose, without loss of generality, that the regular bifurcation point is at $(0,0)$, then by Taylor expansion

$$V(x,\lambda) = a(\lambda) + A(\lambda)x + \frac{1}{2}\langle x, B(\lambda), x\rangle + R(x,\lambda), \tag{3.2.8}$$

where B is a vector of $d \times d$ symmetric matrices and $a(0) = 0$, $R(0,\lambda) = \partial_x R(0,\lambda) = \partial_x^2 R(0,\lambda) = 0$.

Due to the previous discussion we need to consider only the following cases

a) $A^T(0)$ has one, and only one, zero eigenvalue w and $\langle w, a'(0)\rangle \neq 0$;

b) $A(0)$ has two purely imaginary conjugated eigenvalues.

3.3 One dimension

In the one dimensional case (b) cannot take place. Then in (3.2.8) we have $a = a'(0) \neq 0$, $A(0) = 0$, $c = B(0) \neq 0$. Then $V(x,\lambda) = 0$ has no solutions if $ac > 0$, while for $ac < 0$ there are the two solutions $x = \pm\sqrt{-\frac{\lambda b}{B}} + \mathcal{O}(\lambda)$. We have therefore the generic picture: either two points collide and kill each other or there is a creation of two zeroes of the vector field.

Problem 3.10 *Study the solutions of*

$$\dot{x} = \frac{B}{2}x^2 + g(x)$$

near zero when $g(0) = g'(0) = g''(0) = 0$.

[7] Indeed, $\mathrm{Tr}(\Pi' \partial_x V \Pi) = \mathrm{Tr}(\Pi\Pi' \partial_x V \Pi) = \mathrm{Tr}((\Pi\Pi'\Pi \partial_x V) = 0$. Analogously, $\mathrm{Tr}(\Pi \partial_x V \Pi') = 0$.

Problem 3.11 *Prove that the two equilibrium points of the vector field* (3.2.8) *are one attractive and the other repulsive.*

The above scenario is called a *saddle-node* bifurcation.

A natural question is if there exists a simpler standard form of the above bifurcation. Indeed, we can try to kill some of the terms in 3.2.8 by a change of variable.

Problem 3.12 *Show that with a change of variables of the type* $x = \alpha\lambda + \rho z$, *one can change the vector field* (3.2.8) *to the from* $\tilde{V}(z, \lambda) = \lambda + bz^2 + \mathcal{O}(\lambda^2) + o(z^2)$.

The above is the *normal form* of the saddle node bifurcation. This type of reduction can be made for each bifurcation and gives rise to the large field of normal form theory which, unfortunately, goes beyond the scopes of the present notes.

3.4 Two dimensions

3.4.1 A zero eigenvalue

In this case the vector field must have the form (possibly after a linear change of variable to put $\partial V_x(0,0)$ in diagonal form)

$$V(x, \lambda) = \begin{pmatrix} 0 & 0 \\ 0 & \nu \end{pmatrix} x + b\lambda + \frac{1}{2}\begin{pmatrix} \langle x, B_1 x\rangle \\ \langle x, B_2 x\rangle \end{pmatrix} + \lambda C x + \mathcal{O}(\lambda^2) + o(\|x\|^2), \quad (3.4.9)$$

with $b_1, B_1 \neq 0$. It easy to show that the scenario is exactly the same than in the one dimensional case. We leave the details to the reader.

3.4.2 Two purely imaginary eigenvalues: Hopf bifurcation

In this case the vector field must have the form (possibly after a translation and a linear change of variable to put $\partial V_x(0,0)$ in chosen form, see Problem 3.33)

$$V(x, \lambda) = Ax + R(x, \lambda), \quad (3.4.10)$$

with $A = \begin{pmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{pmatrix}$ for some $\omega_0 > 0$, $R(0,0) = \partial_x R(0,0) = 0$ and $\mathrm{Tr}(\partial_{xx} R(0,0) A^{-1} \partial_\lambda R(0,0) - \partial_{x\lambda} R(0,0)) \neq 0$.

In the above situation no new fixed point can appear, yet one expects something to happen. We will see that, depending on λ, a *periodic orbit* circling the fixed point is created. This is called an *Hopf bifurcation.*

To see how such an orbit is created some work is needed. To minimize it, we start by performing some changes of variables that reduces the ODE to a simpler one.

Problem 3.13 *Show that, with a change of coordinates of the type* $x = \xi + \alpha(\lambda)$*, the remainder* R *in* (3.4.10) *can be made to satisfy* $R(0, \lambda) = 0$*, for each* λ *small enough,* $\partial_\xi R(0, 0) = 0$ *and* $\mathrm{Tr}(\partial_{\xi\lambda} R(0, 0)) \neq 0$.

Problem 3.14 *Show that with a further change of variables* $x = D(\mu)z$, $\lambda = \mu\rho(\mu)$ *one can put* (3.4.10) *in the form*

$$\dot{z} = [\omega(\mu)J + \mu\mathbb{1}]\, z + R(z, \mu), \text{ where } J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{3.4.11}$$

with $\omega(0) = \omega_0$ *and* $R(0, \mu) = \partial_z R(0, \mu) = 0$.

Problem 3.15 *Find the solutions of* (3.4.11) *in the case* $R \equiv 0$.

Given that the solutions of the linear part of (3.4.11) rotate around zero almost in circles, it may occur the idea to treat the problem in polar coordinates. In fact this point of view is quite advantageous and we will adopt it. The reader who wants to appreciate the advantages of this choice is invited to try to do the following analysis in Euclidean coordinates.

The polar coordinates can be written as $x = \rho v(\theta)$, where $\rho \in \mathbb{R}_+$, $\theta \in \mathbb{R}$ and $v(\theta) := (\cos\theta, \sin\theta)$.

Remark 3.4.1 *Note that such a change of coordinates is singular for* $\rho = 0$. *In addition, it is not globally one-one. Yet, to consider* θ *in the universal cover of* S^1 *rather than in* S^1 *will be very useful in the following.*

If we substitute such coordinates in (3.4.11), we obtain

$$\dot{\rho}v(\theta) + \rho n(\theta)\dot{\theta} = \mu\rho v(\theta) + \omega(\mu)\rho n(\theta) + R(\rho v(\theta), \mu),$$

where $n(\theta) := (-\sin\theta, \cos\theta)$. That is

$$\begin{aligned} \dot{\rho} &= \mu\rho + \langle v(\theta), R(\rho v(\theta), \mu)\rangle =: \mu\rho + a(\theta, \rho, \mu) \\ \dot{\theta} &= \omega_\mu + \rho^{-1}\langle n(\theta), R(\rho v(\theta), \mu)\rangle =: \omega(\mu) + b(\theta, \rho, \mu), \end{aligned} \tag{3.4.12}$$

where $a(\theta, 0, \mu) = \partial_\rho a(\theta, 0, \mu) = b(0, \mu) = 0$. In addition, note for later use that, $\partial_\rho^2 a(\theta, 0, 0)$ and $\partial_\rho b(\theta, 0, 0)$ are homogeneous trigonometric polynomials of degree three, while $\partial_\rho^3 a(\theta, 0, 0)$ and $\partial_\rho^2 b(\theta, 0, 0)$ are of degree four. By Problem 3.35 it follows that we can write $a(\theta, \rho, \mu) = a_0(\theta, \mu)\rho^2 + a_1(\theta, \rho, \mu)\rho^3$ and $b(\theta, \rho, \mu) = b_0(\theta, \mu)\rho + b_1(\theta, \rho, \mu)\rho^2$. Finally, the reader can easily verify that $a \in \mathcal{C}^r$, while $b \in \mathcal{C}^{r-1}$.

Note that the equation (3.4.12) is well defined also for $\rho = 0$ but in such a case, instead of a fixed point, it has the periodic orbit $(\rho(t), \theta(t)) = (0, \omega_0 t)$. Thus in polar coordinates for $\rho = 0$ we have a rotation, this captures the behavior of the system much better than the fixed point in Euclidean coordinates.

Problem 3.16 *Solve (3.4.12) in the case $b = 0$, $a = \rho^2$. Do it for $b = 0$, $a = \mu\rho^2 + \rho^3$.*

Since for small ρ we have $\dot{\theta} > 0$, it is convenient to use θ rather than t to parameterize the motion (here is now evident the advantage of using the universal cover of S^1). Calling again ρ the distance from the origin as a function of θ we have

$$\frac{d\rho}{d\theta} = \frac{\mu\rho + a(\theta, \rho)}{\omega + b(\theta, \rho)} =: \frac{\mu}{\omega}\rho + \beta(\theta, \mu)\rho^2 + \gamma(\theta, \rho, \mu)\rho^3, \tag{3.4.13}$$

where

$$\begin{aligned}
\beta(\theta, \mu) &= \omega^{-1} a_0(\theta, \mu) - \mu\omega^{-2} b_0(\theta, \mu) \\
\gamma(\theta, 0, \mu) &= \mu\omega^{-3} b_0^2 + a_0 b_0 \omega^{-2} - \mu b_1 \omega^{-2} + a_1 \omega^{-1}.
\end{aligned}$$

Note, that $\beta(\theta, 0)$ is a trigonometric homogeneous polynomial of third degree while $\gamma(\theta, 0, 0)$ is the sum of two monomial, one of degree four and one of degree six.

It is now convenient to perform a last change of variables: $\rho = \nu r$, $\mu = \pm\nu^2$, $\nu \geq 0$.[8] Under such changes of variables (3.4.13) becomes

$$\frac{dr}{d\theta} = \pm\frac{\nu^2}{\omega(\pm\nu^2)} r + \beta(\theta, \pm\nu^2)\nu r^2 + \nu^2\gamma(\theta, \nu r, \pm\nu^2) r^3, \tag{3.4.14}$$

Remark 3.4.2 *The reader my wonder what is going on: if the coefficients would not depend on θ, then the periodic orbit would be circular and would correspond to a zero in the above vector field. Such a zero would occur for $r = \mathcal{O}(\nu^{-1}\beta\gamma^{-1})$, thus it seems that I have just done the wrong scaling. The point is that the above naive analysis is correct only if we consider the average (with respect to θ) of the coefficients, but the average of β is zero! This is a very simple instance of a general theory called* averaging.

Remark 3.4.3 *In the following we will choose the case in which $\mu > 0$, hence the change of variable with the plus is selected. The computations for $\mu < 0$ are completely analogous and are left to the reader.*

[8] In fact, we have two different changes of variable according to the sign of μ.

Let us call $r(\theta, \xi, \nu)$ the solution of (3.4.14) with initial condition ξ and parameter ν.

Problem 3.17 *Prove that, for each $\theta \in [0, 2\pi]$ the function $r(\theta, \cdot, \cdot)$ are $\mathcal{C}^{r-1}$.*

We are finally ready to prove the existence of a periodic orbit. Clearly, an orbit is periodic if and only if $r(0, \xi, \nu) = r(2\pi, \xi, \nu)$. In other words, if we look at the motion only when it crosses the $\{\theta = 0 \mod 2\pi\}$ line, then we see the orbit always at the same point. We have thus another instance of a *Poincaré section*.

In concrete, if we consider the map $S : \mathbb{R}_+^2 \to \mathbb{R}_+$ defined by $S(\xi, \nu) := r(2\pi, \xi, \nu)$, then the periodic orbits of the flow correspond to the fixed points of the maps $S(\cdot, \nu)$.[9]

Our last task is thus to study such a maps. The right idea is to develop them in power series of ν. Note that $r(\theta, \xi, 0)$ satisfies the Cauchy problem

$$\begin{aligned}&\frac{dr}{d\theta} = 0\\ &r(0, \xi, 0) = \xi.\end{aligned}$$

Thus $S(\xi, 0) = \xi$. To compute the derivative we must compute $\eta := \partial_\nu r(\theta, \xi, \nu)$. Such a derivative satisfies the equation obtained by differentiating (3.4.14) (see Theorem 1.1.12)

$$\begin{aligned}\frac{d\eta}{d\theta} &= \frac{2\nu}{\omega} r - \frac{2\nu^3 \omega'}{\omega^2} r + \frac{\nu^2}{\omega}\eta + \beta r^2 + 2\nu r \eta \beta + 2\nu^2 r^2 \partial_{\nu^2}\beta \\ &\quad + 2\nu\gamma r^3 + 3\nu^2 r^2 \eta\gamma + 2\nu^3 r^3 \partial_{\nu^2}\gamma + \nu^3 r^3 (r + \nu\eta)\partial_{\nu r}\gamma \\ \eta(0, \xi, \nu) &= 0.\end{aligned} \tag{3.4.15}$$

Setting $\nu = 0$ in the above equation yields $\eta(\theta, \xi, 0) = \xi^2 \int_0^\theta \beta(\varphi, 0) d\varphi$. Accordingly, $\partial_\nu S(\xi, 0) = 0$ (see Problem 3.36).

To conclude we need to compute the second derivative at $\nu = 0$. Setting $\zeta(\theta, \xi) = \partial_\nu \eta(\theta, \xi, 0)$ and differentiating (3.4.15), yields

$$\begin{aligned}&\frac{d\zeta}{d\theta} = \frac{2}{\omega_0}\xi + 4\beta\xi\eta(\theta, \xi, 0) + 2\gamma(\theta, 0, 0)\xi^3\\ &\zeta(0, \xi, 0) = 0.\end{aligned}$$

which yields

$$\zeta(\theta, \xi) = \frac{2\theta}{\omega_0}\xi + 4\xi \int_0^\theta \beta(\varphi, 0)\eta(\varphi, \xi, 0) d\varphi + 2\xi^3 \int_0^\theta \gamma(\varphi, 0, 0) d\varphi.$$

[9] I mean the non trivial ones, since zero is always a trivial fixed point by construction.

Next, note that $\frac{d\eta(\varphi,\xi,0)}{d\varphi} = \xi^2\beta(\varphi,0)$, hence

$$\int_0^\theta \beta(\varphi,0)\eta(\varphi,\xi,0)d\varphi = \frac{\eta(\theta,\xi,0)^2}{2\xi^2} = \frac{\xi^2}{2}\left(\int_0^\theta \beta(\varphi,0)d\varphi\right)^2.$$

Thus, setting $\bar\gamma = \int_0^{2\pi}\gamma(\varphi,0,0)d\varphi$, we have[10]

$$S(\xi,\nu) = (1+\frac{2\pi}{\omega_0}\nu^2)\xi + \xi^3\bar\gamma\nu^2 + \nu^3\xi\Gamma(\xi,\nu) \tag{3.4.16}$$

To study the solution of $S(\xi,\nu) = \xi$ for $\nu \neq 0$ and $\xi \neq 0$ it is convenient to introduce the function $F(\xi,\nu) = \nu^{-2}\xi^{-1}(S(\xi,\nu)-\xi) = \frac{2\pi}{\omega_0} + \xi^2\bar\gamma + \nu\Gamma(\xi,\nu)$.

If $\bar\gamma > 0$, then $F(\xi,0)$ has no solutions different from zero and the same must hold for small ν.

If $\bar\gamma < 0$, then $\xi_0 = \sqrt{-\frac{2\pi}{\omega_0\bar\gamma}}$ is the only positive solution of $F(\xi,0) = 0$. We can then apply the implicit function theorem since $F(\xi_0,0) = 0$ and

$$\partial_\xi F(\xi_0,0) = \frac{2\pi}{\omega_0} + 3\xi^2\bar\gamma = -\frac{4\pi}{\omega_0} \neq 0.$$

As a conclusion we have a unique $\xi(\nu) = \xi_0 + \mathcal{O}(\nu)$ such that $S(\xi(\nu),\nu) = \xi(\nu)$ for $\nu \neq 0$.

Problem 3.18 *Compute, in terms of the Tailor coefficients of V, what it means $\bar\gamma = 0$ and shows that it is not possible for $V \in \widetilde{B}_{\mathbb{R}^2}$.*

3.5 The Hamiltonian case

It is important to note that non generic situations may appear due to symmetries or other type of constraints. To give an example of such a situation let us consider an Hamiltonian vector field, that is a vector field of the type $V(x,p) = (\partial_p H, -\partial_x H)$ for some function $H(x,p)$. In this case

$$DV = \begin{pmatrix} \partial_{xp}H & \partial_{pp}H \\ -\partial_{xx}H & -\partial_{xp}H \end{pmatrix}.$$

Note that the trace of DV is always zero. Hence if $V(x,p) = 0$ and $\det DV \neq 0$ either the fixed point is hyperbolic or is has two purely imaginary eigenvalues. This means that having two purely imaginary eigenvalues is generic for Hamiltonian vector fields, contrary to the general ones. Analogously the situation for a one parameter family, already when $(x,p) \in \mathbb{R}^2$ is more complex.

[10]Since $S(0,\nu) = 0$, the coefficient of ν^3 must have the form $\xi\Gamma$.

For example, at a generic bifurcation point the vector field will have two, not one, zero eigenvalues.

In fact, for mechanical systems, the Hamiltonian has often the form $H(x,p) = \frac{1}{2}p^2 + U(x)$, for some function U. Hence, $V(x,p) = (p, -\partial_x U)$, which means that the zeroes of the vector field are the critical points of U. Let us discuss Hamiltonian systems in which the Hamiltonian is of the above type.

We start with the so called *one degree of freedom*, i.e. $x, p \in \mathbb{R}$.

Problem 3.19 *Show that if U has a minimum, then the fixed point is a* center*, while if U has a maximum, then the corresponding fixed point is hyperbolic.*

We have thus a new phenomena: a center that is stable under small perturbations!

Let us consider the case in which a one parameter family of potentials $U(x,\lambda)$ has a degenerate minimum at zero, i.e. $U(0,\lambda) = 0, \partial_x U(0,\lambda) = 0, \partial_x^2 U(0,0) = 0$. This means that $U(x) = \lambda x^2 + a(\lambda,x)x^3$ and

$$V(x,\lambda) = (p, 2\lambda x + a_1(x,\lambda)x^2)$$

Problem 3.20 *Show that in the above family we have the collision of two fixed point (a center and a saddle) that collide and exchange type.*

This means that the zeroes of the vector fields are $p = 0$, $x(\lambda) = 0$ and $x(\lambda) \sim -\frac{2\lambda}{a_1(0,0)}$. We then have a new phenomena: two fixed point that cross and exchange type.[11]

Even more singular situations may happen if more constraints are present. Consider, for example the above situation when, for some reason, the Hamiltonian is constrained to being symmetrical: $H(x,p) = H(-x,p)$. Then it would have the form $U(x) = \lambda x^2 + a(\lambda,x)x^4$.

Problem 3.21 *Show that in the above case one has one fixed point that evolves into three fixed points. Moreover show that if when only one fixed point is present, the fixed point is unstable, then of the three fixed point two are unstable and one stable. This is called a* peach fork *bifurcation.*

Next let us consider the case of two degree of freedom, i.e. $x, p \in \mathbb{R}^2$. Limited to the case of a minimum. In such a situation, at the point of minimum, we have

$$\partial_x V(x,p) = \begin{pmatrix} 0 & \mathbb{1} \\ -\partial_x^2 U & 0 \end{pmatrix}. \tag{3.5.17}$$

where $\partial_x^2 U$ is a positive symmetric matrix, let ω_1^2, ω_2^2 be its eigenvalues.

[11] Hence the set of fixed points no longer forms a smooth curve in the x, λ space.

Problem 3.22 *Show that the eigenvalues of $\partial_x V$, at the fixed point, are $\pm\omega_i$.*

Another surprise: a stable situation with four imaginary eigenvalue (an higher dimensional center).

Problem 3.23 *Consider the linear equation (obtained by the matrix (3.5.17) after a change of variables)*

$$\begin{aligned}\dot{x} &= p\\ \dot{p} &= \begin{pmatrix} -\omega_1^2 & 0 \\ 0 & -\omega_2^2 \end{pmatrix} x\end{aligned}$$

Show that $p_i^2 + x_i^2$ are invariant of the motion, i.e. the motion takes place on two-dimensional tori.

Remark 3.5.1 *Contrary to the case of one degree of freedom, in which the conservation of the Hamiltonian implies that the center is stable for the full motion, in higher dimension it is not clear if the center is stable or not for the full dynamics. Indeed this is a rather complex matter at present not completely clarified. Part of the answer is the subject of the so called KAM theory. We will discuss some aspects of KAM theory in the following.*

Problems

3.24. Compute $\tilde{V} = V \circ F$ where V is given by (3.2.3) and F by (3.2.5), i.e. $F(\xi,\tau) = (\xi - \tau, \xi)$. Show, by direct computation, that $\tilde{V}(\xi,\tau) = 0$ has solution $\xi(\tau) = -\frac{b}{a}\tau^2 + \mathcal{O}(\tau^3)$.

3.25. Prove that the set $\{A \in GL(n,\mathbb{R}) \ : \ \det(A) \neq 0\}$ is generic with respect to the topology induced by the norm.

3.26. Prove that the set $\{A \in GL(n,\mathbb{R}) \ : \ A \text{ is hyperbolic}\}$ is generic.

3.27. Prove that $\{A \in \mathcal{C}^0([-1,1], GL(n,\mathbb{R})) \ : \ \operatorname{rank}(A(\lambda)) \geq n-1 \ \forall \lambda \in [-1,1]\}$ is generic.

3.28. Prove that the set $\{A \in GL(n,\mathbb{R}) \ : \ A \text{ is hyperbolic and has only simple eigenvalues}\}$ is generic (i.e. Jordan blocks are atypical).

3.29. Show that if $A \in GL(2,\mathbb{R})$ and its eigenvalues have zero real part, then $\operatorname{Tr}(A) = 0$.

3.30. If $A \in \mathcal{C}^1([-1,1], GL(n,\mathbb{R}))$ and $\Pi \in \mathcal{C}^1([-1,1], GL(n,\mathbb{R}))$ is an eigen-projector, show that $\frac{d}{d\lambda}\operatorname{Tr}(\Pi A) = 2\operatorname{Tr}(\Pi \frac{d}{d\lambda} A)$.

3.31. Show that the set $\{A \in \mathcal{C}^1([-1,1], GL(n,\mathbb{R}))\ :$ at most two eigenvalues have zero real part$\}$ is generic.

3.32. Prove that the set

$$A_K := \{V \in \mathcal{C}^r(\mathbb{R}^n, \mathbb{R}^n)\ : V(x) = 0 \text{ implies } \partial_x V \text{ hyperbolic } \forall\, x \in K\}$$

is generic in the $\mathcal{C}^r$ topology.

3.33. Show that any matrix $A \in GL(2,\mathbb{R})$ with two eigenvalue with zero trace and positive determinant is conjugate to a matrix of the form

$$\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$$

for some $\omega > 0$.

3.34. Let $f \in \mathcal{C}^r(\mathbb{R}^{d+1})$ and write the elements of $\mathbb{R}^{d+1}$ as $(\xi_1, \dots, \xi_d, t)$. If $f(\xi, 0) = \partial_t^k f(\xi, 0) = 0$ for all $k \leq s < r$, then there exists $g \in \mathcal{C}^{r-s}$ such that $f(\xi, t) = t^s g(\xi, t)$.

3.35. Let $f \in \mathcal{C}^r(\mathbb{R}^{d+1})$ and write the elements of $\mathbb{R}^{d+1}$ as $(\xi_1, \dots, \xi_d, t)$. Then, for all $s < r$, there exists $g \in \mathcal{C}^{r-s}$ such that $f(\xi, t) = \sum_{k=0}^{s-1} f^k(\xi, 0) t^k + t^s g(\xi, t)$.[12]

3.36. Show that if $p(\theta)$ is a product of an odd number of functions equal either to $\sin\theta$ or $\cos\theta$, then $\int_0^{2\pi} p(\theta) = 0$.

Hints to solving the Problems

3.3 Let $\bar{x} \in K$ such that $V(\bar{x}) = 0$. Then, by assumption $D_{\bar{x}}V$ is invertible, so $V(\bar{x} + \xi) = 0$ can be written as

$$D_{\bar{x}}V^{-1}(D_{\bar{x}}V\xi - V(\bar{x} + \xi)) = \xi.$$

Since $D_{\bar{x}}V\xi - V(\bar{x} + \xi) = o(\|\xi\|)$, it follows that the above equation has the unique solution $\xi = 0$ in a sufficiently small neighborhood of zero. Hence there exists a neighborhood of $\bar{x}$ in which there are no other zeroes. Next, for each point in K consider a neighborhood as follows: if the V is different from zero at such a point, then consider a neighborhood for which the vector field is different from zero. If the vector field is zero at the point then consider the above neighborhood in which the point is the only zero. In such a way we have a covering of K, we can then extract a finite subcover hence proving the statement.

[12] Essentially this is Taylor formula where one controls the smoothness of the remainder. This issue is relevant in the applications, but often not investigated in standard textbooks.

3.4 Let $V \in A_K$ and $\{x_i\}_{i=1}^M$ be the zeroes of V. Then for each vector field $W \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d)$, $\|W\| \leq 1$, consider the family $V(x, \mu) := V(x) + \mu W(x)$. For each $i \in \{1, \dots, M\}$, use the implicit function theorem to show that there exists $\varepsilon_i, \delta_i > 0$ and $X_i \in \mathcal{C}^1([-\varepsilon_i, \varepsilon_i], \mathbb{R}^n) \to \mathbb{R}^d$, $X_i(x_i) = 0$, such that $V(X_i(\mu), \mu) = 0$ and $V(x, \mu) = 0$, $\|x - x_i\| \leq \delta_i$, $|\mu| \leq \varepsilon_i$ implies that $x = X_i(\mu)$. Verify (using perturbation theory) that, for μ small enough $\partial_x V(X(\mu), \mu)$ is hyperbolic. Next, set $\delta = \min \delta_i$ and $\rho := \inf_{|x - x_i| \geq \delta} \|V(x)\|$. Clearly $V(x, \mu) \neq 0$ if $|x - x_i| \geq \delta$ and $\mu < \rho$. Hence a neighborhood of V of size $\min\{\varepsilon_i, \rho\}$ belongs to A_K, hence A_K is open.

3.5 If $Z_K = \{z \in K \ : \ \det(D_x V) = 0\}$, then $V(Z_K)$ is a zero measure set by Sard's Theorem. Let $Z \subset \mathbb{R}^d$ be a zero measure set and, for each $v \in \mathbb{R}^d$, define $Z(v) = \{z \in \mathbb{R}^d \ : \ z - v \in Z\}$. Show that for each $\varepsilon > 0$ there exists $v \in \mathbb{R}^d$, $\|v\| \leq \varepsilon$, such that $0 \notin Z(v)$. Given $V \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d)$, use this to show that for each $\varepsilon > 0$ there exists $v \in \mathbb{R}^d$, $\|v\| = 1$ such that $V_\varepsilon(x) := V(x) + \varepsilon v$ has the property that $\det(D_x V_\varepsilon) = \det(D_x V) = 0$ implies $V_\varepsilon(x) \neq 0$. An application of the implicit function theorem then shows that the zeroes of V_ε are isolated. Finally, construct $\tilde{V}_\varepsilon$, $\|V_\varepsilon - \tilde{V}_\varepsilon\|_{\mathcal{C}^1} \leq \varepsilon$, such that the zeroes are unchanged but the derivative is hyperbolic, hence $\tilde{V}_\varepsilon \in A_K$. This last step can be performed locally so it suffices to show how to perform it around one single point. First of all note that, by continuity, there exists $\alpha > 0$ such that $V_\varepsilon(x) = 0$ implies $\|(D_x V_\varepsilon)^{-1}\| \leq \alpha^{-1}$. Next, let $x_0 \in K$ such that $V_\varepsilon(x_0) = 0$. Then $V_\varepsilon(x) = D_{x_0} V_\varepsilon(x - x_0) + o(x - x_0)$. Thus there exists $\delta > 0$ such that, for all $\|x - x_0\| \leq \delta$,[13]

$$\|V_\varepsilon(x)\| \geq \frac{\alpha}{2} \|x - x_0\|.$$

Finally, consider the vector field $\tilde{V}_t(x) = V_\varepsilon(x) + t(x - x_0)\varphi(x - x_0)$. Where $\varphi \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R})$ is some fixed function such that the support of φ is contained in the ball of radius δ, $\varphi(0) = 1$, $\nabla\varphi(0) = 0$ and $\|\varphi\|_\infty \leq 1$. Then

$$\|\tilde{V}_t(x)\| \geq (\frac{\alpha}{2} - t)\|x - x_0\|$$

so if $t < \frac{\alpha}{2}$, the filed $\tilde{V}_t(x)$ has the same zeroes than V_ε. Moreover, $D_{x_0}\tilde{V}_t = D_{x_0} V_\varepsilon + t\mathbb{1}$ which is hyperbolic and

$$\|V_\varepsilon - \tilde{V}_t\|_{\mathcal{C}^1} \leq 2t\delta + t\|\varphi\|_{\mathcal{C}^1}$$

which can be made smaller than ε by choosing t sufficiently small.

[13] Note that, by the uniform continuity of the derivative on K, δ can be chosen independent on the point.

3.7 It suffices to show that B_K is generic for each compact $K \subset \mathbb{R}^d$. The openness comes from the fact that a small perturbations cannot change the condition on the rank. For the density, consider the set $\Omega := \{(x, \lambda) \in K^1 : \operatorname{rank}(\partial_x V \ \partial_\lambda V) < d\}$. Using the same strategy as in Theorem 3.1.5 show that $V(\Omega)$ has zero Lebesgue measure.[14] This means that, for each $\varepsilon > 0$ there exists $v \in \mathbb{R}^d$, $\|v\| \leq \varepsilon$, such that for each $(x, \lambda) \in K^1$ such that $V(x, \lambda) = -v$ holds $\operatorname{rank}(\partial_x V \ \partial_\lambda V) = d$. We can then consider the vector field $V_\varepsilon = V + v$ and argue as in the first part of Problem 3.5.

3.13 We know from the discussion Lemma 3.2.2 that there exists $x(\lambda)$ such that $V(x(\lambda), \lambda) = 0$, we can then set $\alpha(\lambda) = x(\lambda)$. We get then the wanted equation with the new remainder given by $R(\xi + x(\lambda), \lambda) - R(x(\lambda), \lambda)$. The other properties of R are obtained by direct computation.

3.14 Remember that the change of variable must be performed on the equation $\dot{x} = V(x, \lambda)$, so the vector field changes as $D^{-1}V(Dz)$. In addition, since $\partial_\xi R(0, 0) = 0$, Problem 3.35 implies that we can write $\partial_\xi R(0, \lambda) = C(\lambda)\lambda$ for some $\mathcal{C}^{r-1}$ matrix C. Choose $D(\lambda) = D_0(\lambda)(\mathbb{1} + D_1(\lambda))$. Since we do not want to change the form of $\partial_x V$ at first order in λ we impose $[D_0, A] = 0$. Show that this implies $D_0(\lambda) = \begin{pmatrix} 1 & -a(\lambda) \\ a(\lambda) & 1 \end{pmatrix}$. Show that one can choose a such that

$$D_0^{-1} \partial_x V(0, \lambda) D_0 = A + \lambda H(\lambda)$$

with $H_{11} = H_{22} \geq 0$. Note then that $H_{ii}(0) \neq 0$ since $\operatorname{Tr} H(0) \neq 0$ by hypothesis. Next, choose $D_1 = \begin{pmatrix} 0 & 0 \\ 0 & \lambda b(\lambda) \end{pmatrix}$. Show that b can be chosen so that

$$(\mathbb{1} + D_1)^{-1} D_0^{-1} \partial_x V(0, \lambda) D_0 (\mathbb{1} + D_1) = A + \lambda \tilde{H}(\lambda)$$

with $\tilde{H}_{ii} = H_{ii}$ and $\tilde{H}_{12} = -\tilde{H}_{21}$. The problem is then solved by choosing ρ.

3.30 Using a "dot" to mean differentiation holds $\frac{d}{d\lambda} \operatorname{Tr}(\Pi A) = \operatorname{Tr}(\dot{\Pi} A + \Pi \dot{A})$. If B is the portion of spectrum associated to $\Pi(0)$ and γ a curve surrounding it and no other part of the spectrum, then

$$\dot{\Pi}(0) = \frac{1}{2\pi i} \int_\gamma (z - A(0))^{-1} \dot{A}(0) (z - A(0))^{-1} dz$$

[14] In fact, this is nothing else than another special case of the general Sard Theorem.

Thus

$$\begin{aligned}\operatorname{Tr}(\dot{\Pi}A) &= \operatorname{Tr}(\Pi\dot{A}) + \frac{1}{2\pi i}\int_\gamma z\operatorname{Tr}\Big((z-A(0))^{-1}\dot{A}(0)(z-A(0))^{-1}\Big)\,dz\\ &= \operatorname{Tr}(\Pi\dot{A}) + \frac{1}{2\pi i}\int_\gamma z\operatorname{Tr}\Big((z-A(0))^{-1}\dot{A}(0)\Big)\,dz = \operatorname{Tr}(\Pi\dot{A}).\end{aligned}$$

Notes

The present discussion is intended only to give a flavor of the subject and of how it can be systematically developed. For a more complete (and advanced) treatment of bifurcation theory see [Arn83, CH82].