Chapter 1

The origins: Differential equations

As this book is about Dynamical Systems, let's start by defining the object of study. The concept of Dynamical System is a very general one and it appears in many branches of mathematics from discrete mathematics, number theory, probability, geometry and analysis and has wide applications in physics, chemistry, biology, economy and social sciences.

Probably the most general formulation of such a concept is the action of a monoid over an algebra. Given a monoid $\mathbb{G}$ and an algebra $\mathcal{A}$ the (left)-action of $\mathbb{G}$ on $\mathcal{A}$ is simply a map $f : \mathbb{G} \times \mathcal{A} \to \mathcal{A}$ such that

1. $f(gh, a) = f(g, f(h, a))$ for each $g, h \in \mathbb{G}$ and $a \in \mathcal{A}$;
2. $f(e, a) = a$ for every $a \in \mathcal{A}$, where e is the identity element of $\mathbb{G}$;
3. $f(g, a + b) = f(g, a) + f(g, b)$ for each $g \in \mathbb{G}$ and $a, b \in \mathcal{A}$;
4. $f(g, ab) = f(g, a)f(g, b)$ for each $g \in \mathbb{G}$ and $a, b \in \mathcal{A}$;

In our discussion we will be mainly motivated by physics. In fact, we will consider only the cases in which $\mathbb{G} \in \{\mathbb{N}, \mathbb{Z}, \mathbb{R}_+, \mathbb{R}\}$[1] is interpreted as *time* and $\mathcal{A}$ is given by an algebra of functions over some set X, interpreted as the *observables* of the system.[2] In addition, we will restrict ourselves to situations where the action over the algebra is induced by an action over the set X (this is a map $f : \mathbb{G} \times X \to X$ that satisfies condition 1, 2 above).[3] Indeed, given

[1] Although even in physics other possibilities are very relevant, e.g. in the case of Statistical Mechanics it is natural to consider the action of the space translations, i.e. the groups $\{\mathbb{Z}^d, \mathbb{R}^d\}$ for some $d \in \mathbb{N}$, $d > 1$.

[2] Again other possibilities are relevant, e.g. the case of Quantum Mechanics (in the so called Heisenberg picture) where the algebra of the observable is non commutative and consists of the bounded operators over some Hilbert space.

[3] Again relevant cases are not included, for example all Markov Process where the evolution is given by the action of some semigroup.

an action f of $\mathbb{G}$ on X and an algebra $\mathcal{A}$ of functions on X such that, for all $a \in \mathcal{A}$ and $g \in \mathbb{G}$, $b(\cdot) := a(f(g, \cdot)) \in \mathcal{A}$. It is then natural to define $\tilde{f}(g, a)(x) := a(f(g, x))$ for all $g \in \mathbb{G}$, $a \in \mathcal{A}$ and $x \in X$. It is then easy to verify that $\tilde{f}$ satisfies conditions 1-4 above.

We will call discrete time Dynamical System the ones in which $\mathbb{G} \in \{\mathbb{N}, \mathbb{Z}\}$ and continuous time Dynamical Systems the ones in which $\mathbb{G} \in \{\mathbb{R}_+, \mathbb{R}\}$. Note that, in the first case, $f(n, x) = f(n-1+1, x) = f(1, f(n-1, x))$, hence defining $T : X \to X$ as $T(x) = f(1, x)$, holds $f(n, x) = T^n(x)$.[4] Thus in such a case we can (and will) specify the Dynamical System by writing only (X, T). In the case of continuos Dynamical Systems we will write $\phi_t(x) := f(t, x)$ and call ϕ_t a flow (if the group is $\mathbb{R}$) or a semi-flow (if the group is $\mathbb{R}_+$) and will specify the Dynamical System by writing (X, ϕ_t). In fact, in this notes we will be interested only in Dynamical Systems with more structure i.e. *topological*, *measurable* or *smooth* Dynamical Systems. By topological Dynamical Systems we mean a triplet $(X, \mathcal{T}, T)$, where $\mathcal{T}$ is a topology and T is continuos (if $B \in \mathcal{T}$, then $T^{-1}B \in \mathcal{T}$). By smooth we consider the case in which X has a differentiable structure and T is r-times differentiable for some $r \in \mathbb{N}$. Finally, a measurable Dynamical Systems is a quadruple (X, Σ, T, μ) where Σ is a σ-algebra, T is measurable (if $B \in \Sigma$, then $T^{-1}B \in \Sigma$) and μ is an invariant measure (for all $B \in \Sigma$, $\mu(T^{-1}B) = \mu(B)$).[5]

So far for general definitions that, to be honest, are not so inspiring. Indeed, what characterizes the modern Dynamical Systems is not so much the setting but rather the type of questions that are asked, first and foremost:

- **Which behaviors are visible in nature?** (stability and bifurcation theory).

- **What happens for very long times?** (statistics and asymptotic theory)

The rest of this book will deal in various ways with such questions.

The original motivation for the above setting and for these questions comes from the study of the motion which, after Newton, typically appears as solution of an *ordinary differential equation* (ODE). It is then natural to start with a brief reminder of basic ODE theory.[6]

In section 1.1 I will recall the theorem of existence and uniqueness of the solutions of an ODE. In addition, I will state the Gronwall inequality, a very useful inequality for estimating the growth rate of the solution of an ODE.

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[4]Obviously $T^2(x) = T \circ T(x) = T(T(x))$, $T^3(x) = T \circ T \circ T(x) = T(T(T(x)))$ and so on.

[5]The definitions for continuos Dynamical Systems are the same with $\{\phi_t\}$ taking the place of T.

[6]In fact, also the solutions of a partial differential equation (PDE) may give rise to a Dynamical System, yet the corresponding theory is typically harder to investigate.

Finally, a theorem yielding the smooth dependence of the solutions of an ODE from an external parameter or from the initial conditions is provided.

In section 1.2 is given a very brief account of linear equations with constant coefficients (by discussing the exponential of a matrix) and of Floquet theory. That is the study of the solutions of a linear equation with coefficients varying periodically in time. The basic result being that the asymptotic properties of the solutions can be understood just by looking at the solutions after one period.

Finally, section 1.3 discusses the possibility of qualitative understanding the behavior of the solutions of ODE that cannot be solved explicitly (essentially all the ODEs). The arguments are very naive and are intended only to convince the reader that a) something can be done; b) a more sophisticated theory needs to be developed in order to have a satisfactory picture.

1.1 Few basic facts about ODE: a reminder

Our starting point is the initial Cauchy problem for ODE. That is, given a separable Banach space $\mathcal{B}$,[7] $V \in \mathcal{C}^0_{\text{loc}}(\mathcal{B} \times \mathbb{R}, \mathcal{B})$,[8] and $x_0 \in \mathcal{B}$, find an open interval $0 \ni I \subset \mathbb{R}$ and $x \in \mathcal{C}^1(I, \mathcal{B})$ such that

$$\begin{aligned} \dot{x}(t) &= V(x(t), t) \\ x(0) &= x_0. \end{aligned} \tag{1.1.1}$$

In the next chapters I will be mainly interested in the case $\mathcal{B} = \mathbb{R}^d$, for some $d \in \mathbb{N}$. Thus, the reader uncomfortable with Banach spaces can safely substitute $\mathbb{R}^d$ to $\mathcal{B}$ in all the subsequent arguments. Yet, it is interesting that the theory can be developed for general Banach spaces at no extra cost. For simplicity, in the following we will always assume that all the Banach spaces are separable even if not explicitly mentioned. In essence, this is just a fancy way of saying that much of the following depends only on the Banach

[7] A Banach spaces is a complete normed vector spaces. This means that a Banach space is a vector space V, over $\mathbb{R}$ or $\mathbb{C}$, equipped with a norm $\|\cdot\|$ such that every Cauchy sequence in V has a limit in V. By *separable* we mean that there exists a countable dense set. Check [RS80, Kat66] for more details or [DS88] for a lot more details.

[8] Given two Banach spaces $\mathcal{B}_1, \mathcal{B}_2$, an open set $U \subset \mathcal{B}_1$, and $q \in \mathbb{N}$ by $\mathcal{C}^q(U, \mathcal{B}_2)$ we mean the continuous functions from U to $\mathcal{B}_2$ that are q time (Fréchet) differentiable and the q-th differentials are continuous (see Problem 1.18 for a very quick discussion of differentiation in Banach spaces). Such a vector space can be equipped with the norm $\|\cdot\|_{\mathcal{C}^q}$ given by the sup of all its derivatives till the order q included. If we then consider the subset for which such a norm is finte, then we have again a vector space which is, in fact, a Banach space. We will call such a Banach space $\mathcal{C}^q(U, \mathcal{B}_2, \|\cdot\|_{\mathcal{C}^q})$, yet, when no confusion can arise, we will abuse of notation and call it simply $\mathcal{C}^q(U, \mathcal{B}_2)$. By $\mathcal{C}^q_{\text{loc}}(U, \mathcal{B}_2)$ we mean the vector space of the functions $f : U \to \mathcal{B}_2$ such that, for each $u \in U$ and $R > 0$ such that $\overline{B(u, R)} = \{v \in \mathcal{B}_1 \; : \|v - u\| \leq R\} \subset U$, $f \in \mathcal{C}^q(B(u, R), \mathcal{B}_2, \|\cdot\|_{\mathcal{C}^q})$. Note that, $\mathcal{C}^q_{\text{loc}}$ is not a Banach space (in fact, it is a Fréchet space).

structure of $\mathbb{R}^d$, that is on the fact that $\mathbb{R}^d$ is a complete vector space with a norm (e.g. the euclidean norm) and, for example, nowhere is used the fact that $\mathbb{R}^d$ has a finite basis.

On the contrary, in the following chapters I will consider ODE on (finite dimensional) manifolds. Yet, not much extra theory is needed in order to do this, since ODE on manifolds can always be reduced to the case $\mathbb{R}^d$ case. I will briefly comment on this issue in section 1.1.5.

The first problem that comes to mind is

Question 1 *Does the Chauchy problem (1.1.1) always admit a solution? If there exists a solution is it unique?*

To address such an issue it is convenient to consider the equation[9]

$$x(t) = x_0 + \int_0^t V(x(s), s)ds \tag{1.1.2}$$

Problem 1.1 *Show that for each finite open interval $0 \in I \subset \mathbb{R}$, if $x \in \mathcal{C}^1(I, \mathcal{B})$ is a solution of (1.1.1), then it is a solution of (1.1.2). Show that if $x \in \mathcal{C}^0(I, \mathcal{B})$ is a solution of (1.1.2) then $x \in \mathcal{C}^1(I, \mathcal{B})$ and is a solution of (1.1.1).*

1.1.1 Existence and uniqueness

The issue of existence and uniqueness of the solutions of (1.1.1) can be solved by applying the clasical Banach fixed point Theorem (see A.1.1), provided we make a stronger assumption on V.

Theorem 1.1.1 (Existence and Uniqueness theorem for ODE) *For each $V \in \mathcal{C}^1_{\text{loc}}(\mathcal{B} \times \mathbb{R}, \mathcal{B})$ and $x_0 \in \mathcal{B}$ there exists $\delta \in \mathbb{R}_+$ such that there exists a unique solution of (1.1.1) in $\mathcal{C}^1((-\delta, \delta), \mathcal{B})$.*[10]

PROOF. Let $\delta \in (0, 1)$. The reader can verify that the vector space $\mathcal{C}^0([-\delta, \delta], \mathcal{B})$, equipped with the norm $\|u\|_\infty := \sup_{t \in [-\delta,\delta]} \|u(t)\|_{\mathcal{B}}$ is a Banach space.[11] For each $R \geq 0$ let us define the domain $D_R = \{y \in \mathcal{C}^0([-\delta, \delta], \mathcal{B}) : \|y - x_0\|_\infty \leq R\}$ and the operator $K : D_R \to \mathcal{C}^0([-\delta, \delta], \mathcal{B})$ by[12]

$$K(u)(t) := x_0 + \int_0^t V(u(s), s)ds.$$

[9] The most convenient meaning of the integral of a function with values in a Banach space is the *Bochner sense*, which reduces to the usual Lebesgue integral in the case $\mathcal{B} = \mathbb{R}^d$, see [Yos95] for definition and properties. Yet, for our purposes the equivalent of the Riemannian integral suffices and it is defined in the obvious manner. See Problem 1.20 for details.

[10] We equip $\mathcal{B} \times \mathbb{R}$ with the norm $\|(x, t)\| \leq \sup\{\|x\|_{\mathcal{B}}, |t|\}$, where $\|\cdot\|_{\mathcal{B}}$ is the norm of $\mathcal{B}$.

[11] It suffices to remember that the uniform limit of continuous functions is a continuos function.

[12] The meaning of $\mathcal{C}^0(K, \mathcal{B}_2)$ where K is a closed set of $\mathcal{B}_1$ is the usual one.

Let $M_\delta = \sup_{|t|\le\delta}\sup_{u\in D_R}\{\|V(u,t)\| + \|\partial_u V(u,t)\|\}$, note that M_δ is a decreasing function of δ.[13] Then, for each $u \in D_R$ and $|t| \le \delta$, (recall Problem 1.22)

$$\|K(u(t)) - x_0\| \le \delta M_\delta \le R$$

provided we chose $\delta M_\delta \le R$. Thus K maps D_R into D_R. In addition, for each $u, v \in D_R$,

$$\|K(u) - K(v)\|_\infty \le \delta M_\delta \|u - v\|_\infty \le \frac{1}{2}\|u - v\|_\infty,$$

provided we chose $2\delta M_\delta \le 1$. We can then apply Theorem A.1.1 and obtain a unique solution of the equation $Ku = u$ in B_R. This shows the existence and uniqueness of the solution of (1.1.2). The Theorem follows then by remembering Problem 1.1. □

Remark 1.1.2 *Note that in the proof of Theorem A.1.1 one can chose the same δ for an open set of initial condition.*

Remark 1.1.3 *The hypotheses of the above Theorem can be easily weakened to the case of V locally Lipschitz in x and continuous in t, yet only continuity does not suffice for uniqueness as shown by the example*

$$\begin{aligned}\dot{x} &= \sqrt{x}\\ x(0) &= 0.\end{aligned}$$

which has the infinitely many solutions $x_a(t) = 0$ for $t \le a$ and $x_a(t) = \frac{1}{4}(t-a)^2$ for $t \ge a$, $a \in \mathbb{R}$.[14]

Remark 1.1.4 *The restriction to an interval of size δ in Theorem A.1.1 cannot be avoided as shown by the example*

$$\begin{aligned}\dot{x} &= x^2\\ x(0) &= 1.\end{aligned}$$

Its solution $x(t) = (1-t)^{-1}$ is not continuous, nor bounded, for $t = 1$.

We have seen a mechanism whereby the solution cannot be defined for all times, the next Lemma shows that, for $\mathcal{C}^1$ vector fields, the above is the *only* mechanism.[15]

[13] The finiteness of M_δ follows form the definition of $\mathcal{C}^1_{\text{loc}}$ in footnote 8.

[14] If $\mathcal{B}$ is finite dimensional, then $V \in \mathcal{C}^0$ suffices for the existence of a solution. This follows by a direct application of Schauder fixed point Theorem to (1.1.2). For informations on such a fixed point theorem and fixed point theorems in general see [Zei86].

[15] I state the result for positive times, for negative times is the same.

Lemma 1.1.5 *In the hypotheses of Theorem 1.1.1, if* $x \in \mathcal{C}^1_{\text{loc}}((-\underline{\delta}, \delta), \mathcal{B})$ *is a solution of* (1.1.1) *for some* $\underline{\delta}, \delta > 0$*, and if there exists* $M > 0$ *such that* $\sup_{t \in [0,\delta)} \|x(t)\| \le M$*, then there exists* $\bar{\delta} > \delta$ *and* $\bar{x} \in \mathcal{C}^1((-\underline{\delta}, \bar{\delta}), \mathcal{B})$ *that solves* (1.1.1) *(i.e. the solution can be extended for longer times).*

Proof. Let $\{t_n\}$ be any sequence that converges to δ, then

$$\|x(t_n) - x(t_m)\| \le \int_{t_n}^{t_m} \|V(x(s), s)\| ds \le |t_n - t_m| \sup_{\|z\| \le M} \sup_{s \in [0,\delta)} \|V(z, s)\|.$$

Thus $\{x(t_n)\}$ is a Cauchy sequence and admits a limit $x_* \in \mathcal{B}$ such that

$$x_* = \lim_{n \to \infty} x(t_n) = \lim_{t \to \delta} x(t) = x_0 + \int_0^\delta V(x(s), s) ds.$$

We can then consider the equation

$$y(t) = x_* + \int_0^t V(y(s), s + \delta) ds.$$

By Theorem 1.1.1 there exists δ_1 and $y \in \mathcal{C}^1((-\delta_1, \delta_1), \mathcal{B})$ which satisfy the above equation. Let then $\bar{\delta} = \delta + \delta_1$ and define

$$\bar{x}(t) := \begin{cases} x(t) & \text{fot all } t \in (-\underline{\delta}, \delta) \\ y(t - \delta) & \text{fot all } t \in [\delta, \bar{\delta}). \end{cases}$$

Clearly $\bar{x} \in \mathcal{C}^0((-\underline{\delta}, \bar{\delta}), \mathcal{B})$ and, for $t \in [\delta, \bar{\delta})$ holds true

$$\begin{aligned} \bar{x}(t) &= x_* + \int_\delta^t V(\bar{x}(s), s) ds = x_0 + \int_0^\delta V(\bar{x}(s), s) ds + \int_\delta^t V(\bar{x}(s), s) ds \\ &= x_0 + \int_0^t V(\bar{x}(s), s) ds. \end{aligned}$$

Thus, again remembering Problem 1.1, the Lemma follows. □

Remark 1.1.6 *Applying repeatedly Lemma 1.1.5 it follows that there exists a maximal open interval* $J \subset \mathbb{R}$ *such that the Cauchy problem* (1.1.1) *has a unique solution belonging to* $\mathcal{C}^1_{\text{loc}}(J, \mathcal{B})$*.*

1.1.2 Growald inequality

We have seen that the escape (growth) to infinity is the only obstruction to enlarging the domain of the solution.[16] The question remains: how large the maximal interval J in Remark 1.1.6 can be?

[16] Of course, this is the case only for regular vector fields. For other possibilities think of the case of collisions among planets.

To understand better how the solution of an ODE can grow we need a technical but extremely useful Lemma.

Lemma 1.1.7 (Integral Gronwald inequality) *Let $L, T \in \mathbb{R}_+$ and $\xi, f \in \mathcal{C}^0([0,T],\mathbb{R})$. If, for all $t \in [0,T]$,*

$$\xi(t) \le L \int_0^t \xi(s)\, ds + f(t),$$

then

$$\xi(t) \le f(t) + L \int_0^t e^{L(t-s)} f(s)\, ds.$$

Proof. Let us first consider the case in which $f \equiv 0$. In this case the Lemma asserts $\xi(t) \le 0$. Indeed, since ξ is a continuos function there exists $t_* \in [0,(2L)^{-1}] \cap [0,T] =: I_1$ such that $\xi(t_*) = \sup_{t \in I_1} \xi(t)$. But then,

$$\xi(t_*) \le L \int_0^{t_*} \xi(s)\, ds \le \xi(t_*) L t_* \le \frac{1}{2}\xi(t_*)$$

which implies $\xi(t_*) \le 0$ and hence $\xi(t) \le 0$ for each $t \in I_1$. If $I_1 = [0,T]$, then we are done, otherwise letting $t_1 := (2L)^{-1}$ we have

$$\xi(t) \le L \int_{t_1}^t \xi(s)\, ds$$

and we can make the same argument as before in the interval $[t_1, 2t_1]$. Iterating we have $\xi(t) \le 0$ for all $t \in [0,T]$.

To treat the general case we reduce it to the previous one. Let

$$\zeta(t) := \xi(t) - f(t) - L \int_0^t e^{L(t-s)} f(s)\, ds.$$

Then

$$\begin{aligned}
\zeta(t) &\le L \int_0^t \xi(s)\, ds - \int_0^t L e^{L(t-s)} f(s)\, ds \\
&= L \int_0^t \zeta(s)\, ds + L \int_0^t \left\{ f(s) ds + L \int_0^s e^{L(s-\tau)} f(\tau) d\tau \right\} \\
&\quad - \int_0^t L e^{L(t-s)} f(s)\, ds.
\end{aligned}$$

Next, notice that

$$\begin{aligned}
\int_0^t ds L \int_0^s e^{L(s-\tau)} f(\tau)\, d\tau &= L \int_0^t d\tau f(\tau) \int_\tau^t ds e^{L(s-\tau)} \\
&= \int_0^t f(s)\{e^{L(t-s)} - 1\} ds.
\end{aligned}$$

Thus,

$$\zeta(t) \leq L \int_0^t \zeta(s)\, ds.$$

We have then reduced the problem to the previous case which implies that it must be $\zeta(t) \leq 0$ from which the Lemma follows. □

Let us see the usefulness of the above Lemma in a concrete example. Let $L(\mathcal{B}, \mathcal{B})$ be the Banach space of the linear bounded operators from $\mathcal{B}$ to $\mathcal{B}$.[17]

Lemma 1.1.8 *For each $A \in \mathcal{C}^1_{\text{loc}}(\mathbb{R}, L(\mathcal{B}, \mathcal{B}))$, consider the Cauchy problem*

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) \\ x(0) &= x_0. \end{aligned}$$

If $\|A(t)\| \leq L$ for all $0 \leq t \leq T \in \mathbb{R}$, then $\|x(t)\| \leq e^{Lt}\|x_0\|$ for all $0 \leq t \leq T$. In particular, the solution is defined on all $\mathbb{R}$.

Proof. If we write the equation in the equivalent integral form we have

$$\|x(t)\| \leq \|x_0\| + \int_0^t \|A(s)x(s)\|\, ds \leq \|x_0\| + L \int_0^t \|x(s)\|\, ds.$$

Setting $\xi(t) := \|x(t)\|$ and applying Lemma 1.1.7 the Lemma follows.

Since, for all $T \in \mathbb{R}_+$, $\sup_{t \leq T} \|A(t)\| < \infty$, the the Lemma follows. □

Problem 1.2 *Explain why Lemma 1.1.8 does not apply to the following setting: $\mathcal{B} = \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ and*

$$\dot{x}(t, z) = \alpha(z, t)\partial_z x(t, z),$$

for some $\alpha \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$, $\alpha(z, T + t) = \alpha(z, t)$, $T > 0$. Compare with Problem 1.24.

1.1.3 Flows

In this section we analyze the case in which the vector field is time independent and grows at most linearly.

Lemma 1.1.9 *Given $V \in \mathcal{C}^1_{\text{loc}}(\mathcal{B}, \mathcal{B})$, if there exists $L, M \geq 0$ such that $\|V(x)\| \leq L\|x\| + M$, then the solution of (1.1.1) exists for all times and for all initial conditions.*

[17] The norm of $L \in L(\mathcal{B}, \mathcal{B})$ is given by $\|L\| := \sup_{\substack{v \in \mathcal{B} \\ \|v\|=1}} \|Lv\|$. If $\mathcal{B} = \mathbb{R}^d$, then $L(\mathcal{B}, \mathcal{B})$ is just the vector space of the $d \times d$ matrices.

PROOF. We argue by contradiction. Choose any initial condition $x_0 \in \mathcal{B}$ and let $I(x_0) = (-\delta_-(x_0), \delta_+(x_0))$ be the maximal interval on which the solution is defined. If $\delta_+(x_0) < \infty$, then for each $t \leq \delta_+(x_0)$

$$\|x(t)\| \leq \|x_0\| + L \int_0^t \|x(s)\| ds + Mt.$$

Thus Gronwald inequality implies

$$\|x(t)\| \leq e^{Lt} \left\{ \|x_0\| + ML^{-1} \right\}$$

for $t \in [0, \delta_+(x_0))$. Then, by Lemma 1.1.5, the solution can be extended, contrary to the assumption that $(-\delta_-(x_0), \delta_+(x_0))$ was the maximal interval. A similar argument holds for negative t. □

For each $x_0 \in \mathcal{B}$ and $t \in \mathbb{R}$ let $x(t, x_0)$ be the solution of (1.1.1) at time t.

Lemma 1.1.10 *For each V as in Lemma 1.1.9, setting $\phi_t(x_0) := x(t, x_0)$, $\phi_{-t} = \phi_t^{-1}$ for $t \geq 0$, we have that $(\mathcal{B}, \phi_t)$, $t \in \mathbb{R}$, is a Dynamical System.*

PROOF. All we need to prove is that ϕ_t is an action of $\mathbb{R}$ on $\mathcal{B}$. First of all note that ϕ_t is indeed invertible. If not then there would be $x, x' \in \mathcal{B}$ such that $\phi_t(x) = \phi_t(x')$. But then the uniqueness of the solutions of the ODE implies $x = x'$. Moreover it is easy to check that $\phi_{-t}(x_0) = x(-t, x_0)$. Finally, $\phi_t(\phi_s(x)) = \phi_{t+s}(x)$. □

Remark 1.1.11 *We have thus proved that a large class of vector fields gives rise to* flows.

1.1.4 Dependence on a parameter

Having established the existence and uniqueness of the solution, the next natural questions present itself.

Question 2 *How do the solutions depend on the initial condition? How do the solutions depend on a change of the vector field?*

To discuss such issues it is convenient to analyze first the second question. More precisely, given $V \in \mathcal{C}^2_{\text{loc}}(\mathcal{B} \times \mathbb{R} \times \mathbb{R}^d, \mathcal{B})$ we consider the Chauchy problem

$$\begin{aligned} \dot{x}(t) &= V(x(t), t, \lambda) \\ x(0) &= x_0. \end{aligned} \tag{1.1.3}$$

Clearly the solution will depend on the parameter λ. The question is then: calling $x(t, \lambda)$ the solution of (1.1.3), for a given $t \in \mathbb{R}$ what can we say about the function $x(t, \cdot)$?

For simplicity let us consider the case $V \in \mathcal{C}^2(\mathcal{B} \times \mathbb{R} \times \mathbb{R}^d, \mathcal{B})$, the general case is similar and is left to the reader.

Theorem 1.1.12 (Smooth dependence on a parameter) *Given two Banach spaces $\mathcal{B}, \mathcal{B}_1$, let $V \in \mathcal{C}^2(\mathcal{B} \times \mathbb{R} \times \mathcal{B}_1, \mathcal{B})$. Let $X(t, x_0, \lambda)$ be the unique solution of (1.1.3), then $X(t, x_0, \cdot) \in \mathcal{C}^1_{\rm loc}(\mathcal{B}_1, \mathcal{B})$.*

Proof. For each $x_0 \in \mathcal{B}$ consider the ODE for $\xi \in \mathcal{C}^1_{\rm loc}(\mathbb{R} \times \mathcal{B}_1, L(\mathcal{B}_1, \mathcal{B}))$

$$\begin{aligned} &\dot\xi(t,\lambda) = \partial_x V(X(t,x_0,\lambda),t,\lambda)\cdot\xi(t,\lambda) + \partial_\lambda V(X(t,x_0,\lambda),t,\lambda) \\ &\xi(0,\lambda) = 0. \end{aligned} \tag{1.1.4}$$

We claim that $\xi(t) = \partial_\lambda X(t, x_0, \lambda)$.[18] To verify the claim it suffices to prove that there exists $C > 0$ such that, for $h \in \mathcal{B}_1$ small enough, if $\zeta(t, h, \lambda) := X(t, x_0, \lambda+h) - X(t, x_0, \lambda) - \xi(t)h$, then $\|\zeta(t, h)\| \le C\|h\|^2$. By Taylor formula we have[19]

$$\begin{aligned} \dot\zeta(t,h) &= V(X(t,x_0,\lambda+h),t,\lambda+h) - V(X(t,x_0,\lambda),t,\lambda) \\ &\quad - \partial_x V(X(t,x_0,\lambda),t)\cdot\xi(t)h - \partial_\lambda V(X(t,x_0,\lambda),t,\lambda)h \\ &= \partial_x V(X(t,x_0,\lambda),t)\cdot\zeta(t,h) + R(t) \end{aligned} \tag{1.1.5}$$

where, in the last line, we have used

$$\begin{aligned} &V(X(t,x_0,\lambda+h),t,\lambda) - V(X(t,x_0,\lambda),t,\lambda) \\ &= \partial_x V(X(t,x_0,\lambda),t,\lambda)\cdot(X(t,x_0,\lambda+h),t,\lambda) - X(t,x_0,\lambda)) \\ &\quad + \mathcal{O}(\|X(t,x_0,\lambda+h),t,\lambda) - X(t,x_0,\lambda)\|^2), \end{aligned}$$

and

$$\begin{aligned} \|R(t)\| &\le C\left(\|X(t,x_0,\lambda+h) - X(t,x_0,\lambda)\|^2 + \|h\|^2\right) \\ &\le 2C(\|\zeta(t,h)\|^2 + (1+\|\xi(t)\|^2)\|h\|^2). \end{aligned}$$

with $C = \|V\|_{\mathcal{C}^2}$. Note that $\zeta(0) = 0$. We can then conclude by using Lemma 1.1.7. Indeed such a Lemma applied to (1.1.4) implies $\|\xi(t)\| \le e^{C_1 t}$, for some $C_1 > 0$. Next, let $T > 0$ be the maximal time such that $\|\zeta(t, h)\| \le 1/2$ and $e^{2C_1 T} \le 2$. Then, for $t \le T$, (1.1.5) yields

$$\|\zeta(t,h)\| \le \int_0^t 2C\|\zeta(s)\|ds + 3\|h\|^2$$

and Lemma 1.1.7, again, implies the announced estimate. $\square$

Problem 1.3 *Prove the analogous of Theorem 1.1.12 when $V \in \mathcal{C}^1_{\rm loc}$.*

[18] If $\mathcal{B} = \mathbb{R}^d$ e $\mathcal{B}_1 = \mathbb{R}^m$ then ξ is just a $d \times m$ matrix.

[19] Note that we cannot Taylor expand $X(t, x_0, \lambda + h)$ with respect to h, since we do not know yet that X is differentiable with respect to λ.

Corollary 1.1.13 (Smooth dependence on initial conditions) *Let* $V \in \mathcal{C}^2(\mathcal{B} \times \mathbb{R}, \mathcal{B})$*. For each* $x_0 \in \mathcal{B}$ *let* $X(t, x_0)$ *be the unique solution of* (1.1.1)*. Then, for each* $t \in \mathbb{R}$*,* $X(t, \cdot) \in \mathcal{C}^1_{\text{loc}}(\mathcal{B}, \mathcal{B})$ *and* $\xi = \partial_{x_0} X$ *is a solution of*

$$\begin{aligned} \dot{\xi}(t) &= \partial_x V(X(t, x_0), t) \cdot \xi(t) \\ \xi(0) &= \mathbb{1}. \end{aligned} \tag{1.1.6}$$

PROOF. Set $z = x - x_0$ and consider the resulting equation

$$\begin{aligned} \dot{z} &= V(z + x_0, t) =: \bar{V}(z, t, x_0) \\ z(0) &= 0. \end{aligned}$$

One can then consider x_0 as an external parameter, applying Theorem 1.1.12 yields the result. □

1.1.5 ODE on Manifolds–few words

Let us remind that a *topological manifold* is a second countable Hausdorff space which is locally homeomorphic to Euclidean space. A *chart* over a topological manifold M is a pair (U, ϕ) such that $U \subset M$ is an open set and $\phi : U \to \mathbb{R}^n$, for some $n \in \mathbb{N}$, is an homeomorphism between U and the open set $\Phi(U)$. An *atlas* on a topological manifold is a countable collection of charts $\{(U_\alpha, \phi_\alpha)\}$. We say that an atlas is $\mathcal{C}^k$ if $\phi_\alpha \circ \phi_\beta^{-1}$ is $\mathcal{C}^k$ when is defined. We say that two $\mathcal{C}^k$ atlas are equivalent if their union is a $\mathcal{C}^k$ atlas. A $\mathcal{C}^k$ manifold is a topological manifold equipped with an equivalence class of $\mathcal{C}^k$ atlas (often called a *differentiable structure*).

Although most often we will be concerned with manifolds embedded in some $\mathbb{R}^d$, also other possibilities will be relevant. Let us consider two examples.

Problem 1.4 *Show that* $\mathbb{R}^d$ *is a* $\mathcal{C}^\infty$ *manifold.*[20]

Problem 1.5 *Let* $f \in \mathcal{C}^k(\mathbb{R}^d, \mathbb{R})$*, and consider* $M = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} \, : \, y = f(x)\}$*. Consider the atlas consisting of the chart* (M, ϕ) *where* $\phi(x, y) = x$*. This is a* $\mathcal{C}^\infty$ *manifold.*

Problem 1.6 *Check that* $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ *is a* $\mathcal{C}^\infty$ *manifold.*

Given two differentiable manifolds ($\mathcal{C}^k$ manifolds with $k \geq 1$) M_1, M_2 and a map $f : M_1 \to M_2$ we say that $f \in \mathcal{C}^r(M_1, M_2)$, $r \leq k$, if for each atlas

[20] Note that, contrary to $\mathcal{C}^k$, $\mathcal{C}^\infty$ is not a Banach space (there is no good norm). It is possible to give to it the structure of a Fréchet space [RS80], but we will refrain from such subtleties. We just consider $\mathcal{C}^\infty = \cap_{n \in \mathbb{N}} \mathcal{C}^n$ as a vector space.

$\{(U_\alpha, \phi_\alpha)\}$ of M_1 and atlas $\{(V_\beta, \psi_\beta)\}$ of M_2, holds true $\psi_\beta \circ f \circ \phi_\alpha^{-1} \in \mathcal{C}^r$ on their domains of definition.

Given a differentiable manifold M and $x \in M$, we say that two curves $\gamma_1, \gamma_2 \in \mathcal{C}^1((-1,1), M)$, such that $\gamma_1(0) = \gamma_1(0) = x$, are equivalent at x if for each chart (U, ϕ) such that $x \in U$ holds true $(\phi \circ \gamma_1)'(0) = (\phi \circ \gamma_2)'(0)$. A *tangent vector* at x is an equivalence class of curves.

Problem 1.7 *Show that if M is localy homeorphic to $\mathbb{R}^d$, then the set of tangent vectors at any $x \in M$ form canonically a d dimensional vector space.*[21]

We will use $\mathcal{T}_x M$ to designate the *tangent space* at x, that is the set of the tangent vectors at x. The tangent bundle is the disjoint union of the tangent spaces, i.e. $\mathcal{T}M = \cup_{x \in M} \{x\} \times \mathcal{T}_x M$. Finally, a *vector field* is a section of the tangent bundle, i.e. $\tilde{V} : M \to \mathcal{T}M$ such that $\tilde{V}(x) = (x, V(x))$, $V(x) \in \mathcal{T}_x M$. Form now on, with a slight abuse of notation, we will identify $\tilde{V}$ with V. Also, given $f \in \mathcal{C}^1(M_1, M_2)$, since the image of a $\mathcal{C}^1$ curve is a $\mathcal{C}^1$ curve, ve have naturally defined a map $f_* : \mathcal{T}M_1 \to \mathcal{T}M_2$.

Problem 1.8 *If $f \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^n)$ discuss the relation between f_* and the derivative Df.*

We have finally the language to define O.D.E. on manifolds, in fact the Cauchy problem is exactly given again by (1.1.1), only now V is a, possibly time dependent, $\mathcal{C}^1$ vector field.

Problem 1.9 *Suppose that x_0 belongs to some chart (U, ϕ), show that the solution of*

$$\begin{aligned} \dot{x} &= V(x,t) \\ x(0) &= x_0 \end{aligned}$$

for a sufficiently small time can be obtained by the solution of an appropriate O.D.E. in $\phi(U)$.

Problem 1.10 *Given a finite atlas $\{(U_\alpha, \phi_\alpha)\}$, show that there exists a smooth partition of unity subordinated to the atlas, that is a collections $\{\varphi_\alpha\} \in \mathcal{C}^\infty(M, \mathbb{R})$ such that $\sum_\alpha \varphi_\alpha = 1$ and $\operatorname{supp} \varphi_\alpha \subset U_\alpha$.*

Problem 1.11 *Given a smooth vector field V consider*

$$\begin{aligned} \dot{x} &= V(x) \\ x(0) &= x_0 \end{aligned} \tag{1.1.7}$$

[21] If (U, ϕ) is a chart containing x, and γ_1, γ_2 two curves, think of the curves $\gamma_\lambda(t) = \gamma_1(\lambda t)$ and $\phi^{-1}(\phi(\gamma_1(t)) + \phi(\gamma_2(t)) - \phi(x))$.

with $x_0 \in U_\alpha$ for some element of an atlas $\{(U_\alpha \phi_\alpha)\}$. Let $z_\alpha(t)$ be the solution of

$$\begin{aligned}\dot z_\alpha &= (\phi_\alpha)_* V(z_\alpha)\\ z_\alpha(0) &= \phi_\alpha(x_0)\end{aligned}$$

and suppose that $\phi_\alpha^{-1}(z(1)) \in U_\beta$. Consider then the solution of

$$\begin{aligned}\dot z_\beta &= (\phi_\beta)_* V(z_\beta)\\ z_\beta(1) &= \phi_\beta(\phi_\alpha^{-1}(z_\alpha(1))).\end{aligned}$$

Show that there exists $t_1 > 1$ such that

$$\begin{aligned}x(t) &= \phi_\alpha^{-1}(z_\alpha(t)) \quad \text{for } t \in [0,1]\\ x(t) &= \phi_\beta^{-1}(z_\beta(t)) \quad \text{for } t \in (1,t_1)\end{aligned}$$

is a solution of (1.1.7) in the time interval $[0, t_1)$.

Remark 1.1.14 *We have seen that the theory of ODE on manifolds can be reduced locally to the case of $\mathbb{R}^d$. Yet, the reader should be aware that the global properties of the solutions can be very different. We will comment at length on this issue later on.*

1.2 Linear ODE and Floquet theory

Let us briefly discuss the simplest possible differential equation: the affine ones. We restrict ourselves to the case $\mathcal{B} = \mathbb{R}^d$ for some $d \in \mathbb{N}$ since we will use some spectral theory which is substantially more complex in the general case.

1.2.1 Linear equations

Consider

$$\begin{aligned}\dot x &= Ax\\ x(0) &= x_0.\end{aligned} \tag{1.2.8}$$

Problem 1.12 *Show, by induction, that for each $n \in \mathbb{N}$ the solution of (1.2.8) satisfies*

$$x(t) = \sum_{k=0}^{n} \frac{1}{k!} A^k t^k x_0 + \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n A^{n+1} x(t_n).$$

Taking the limit for $n \to \infty$ in the above expression one readily obtains $x(t) = \sum_{n=0}^{\infty} \frac{1}{n!} A^n t^n x_0$. That this is a solution can be verified directly inserting this formula in (1.2.8) (and noticing that the series and the series obtained by deviating term by term are uniformly convergent). By the standard analytic functional calculus for matrices (and operators, see Appendix C) we can thus write $x(t) = e^{At} x_0$. The above discussion provides a general solution for all equations of the type (1.2.8).

In reality life it is not that simple: if one has a concrete matrix A and wants to compute e^{At}, this may be quite unpleasant. A general strategy, although not necessarily the simplest one, is to perform a linear change of variables $x = Uz$. Then $\dot{z} = U^{-1}AUz$, and U is chosen so that $\Lambda = U^{-1}AU$ is in Jordan normal form. Then

$$x(t) = Uz(t) = Ue^{\Lambda t} z_0 = Ue^{\Lambda t} U^{-1} x_0.$$

It suffices then to know how to take exponentials of Jordan blocks, and this can be computed by using the defining series.

Problem 1.13 *Compute $e^{\Lambda t}$ for*

$$\Lambda = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \; ; \quad \Lambda = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} \; ; \quad \Lambda = \begin{pmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix}.$$

Another, equivalent, point of view is to look for solutions of the type $x(t) = e^{at} v$, substituting in the first of (1.2.8) one obtains $av = Av$. Thus, as we know already, each eigenvalue of A provides a solution of (1.2.8) (ignoring the initial condition). If there exists real eigenvectors $\{v_i\}_{i=1}^d$ which span all $\mathbb{R}^d$ then one can write the general solution, depending on d parameters α_i, as $x(t) = \sum_{i=1}^d \alpha_i v_i e^{a_i t}$, where a_i is he eigenvalue associated to the eigenvector v_i. One can then satisfy the initial condition by solving $x_0 = \sum_{i=1}^d \alpha_i v_i$. The same can be done is the eigenvectors are complex, by working in $\mathbb{C}^d$ instead then $\mathbb{R}^d$. If Jordan blocks are present one can look for solutions of the form $x(t) = \sum_{k=0}^p \frac{1}{(p-k)!} t^k e^{at} v_k$, compare this formula with your solution of Problem 1.13.

Remark 1.2.1 *Note that if the matrix A does not have eigenvalues with zero real part, then (by spectral decomposition) one can write $\mathbb{R}^d = V_- \oplus V_+$, where $AV_\pm = V_\pm$ and A restricted to V_- has eigenvalues with negative real part while on V_+ has eigenvalues with positive real part. Hence if $x_0 \in V_-$ it will hold $\lim_{n\to\infty} x(t) = 0$, and if $x_0 \in V_+$ it will hold $\lim_{n\to\infty} \|x(t)\| = \infty$. Accordingly if $x_0 \notin V_-$ we can write it as $x_0 = x_- + x_+$, where $x_\pm \in V_\pm$. Hence $\lim_{n\to\infty} \|x(t)\| = \infty$ and the trajectory will escape to infinity while getting exponentially close to the subspace V_+. This is our first* long time *result.*

A slightly more complex situation is given by

$$\begin{aligned} \dot{x} &= Ax + b(t) \\ x(0) &= x_0, \end{aligned} \tag{1.2.9}$$

where $b \in \mathcal{C}^0(\mathbb{R}, \mathbb{R}^d)$. The solution of (1.2.9) is given by[22]

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}b(s)ds. \tag{1.2.10}$$

1.2.2 Floquet theory

Let us consider the simplest case of a linear time dependent equation: there exists a continuous function $A \in \mathcal{C}^0_{\text{loc}}(\mathbb{R}, L(\mathcal{B}, \mathcal{B}))$ and $T \in \mathbb{R}_+$ such that, for all $t \in \mathbb{R}$, $A(t+T) = A(t)$. More precisely, let $\Phi(x_0, s, t)$ be the solution of the Cauchy problem[23]

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) \\ x(s) &= x_0. \end{aligned} \tag{1.2.11}$$

Problem 1.14 *Verify the following facts for each $x_0, y_0 \in \mathcal{B}$ and for each $a, b, t, s, \tau \in \mathbb{R}$*

- $\Phi(ax_0 + by_0, s, t) = a\Phi(x_0, s, t) + b\Phi(y_0, s, t)$,
- $\Phi(x_0, s, t) = \Phi(\Phi(x_0, s, \tau), \tau, t)$,
- $\Phi(x_0, s + T, t + T) = \Phi(x_0, s, t)$.

By the first property of Problem 1.14 there exists $K \in \mathcal{C}^1_{\text{loc}}(\mathbb{R}^2, L(\mathcal{B}, \mathcal{B}))$ such that $\Phi(x_0, s, t) = K(s, t)x_0$, the second property implies that $K(\tau, t)K(s, \tau) = K(s, t)$, the third that $K(s + T, t + T) = K(s, t)$. The next step is the first occurrence in this book of a very simply but very powerful idea to analyze dynamical systems: a Poincaré section. Essentially the idea consist in looking at the system only at specially selected moments in time. In this case it is convenient to look at $t \in \{nT\}_{n \in \mathbb{Z}}$. That is, we want to investigate $\Phi(x_0, 0, nT) =: F(x_0, n)$.

Lemma 1.2.2 *The couple $(\mathbb{R}^d, F)$ is a discrete Dynamical System.*

[22] To obtain it just look for a solution of the form $x(t) = e^{At}z(t)$ and deduce the differential equation for z.

[23] The solution is well defined for all times by Lemma 1.1.9.

PROOF. We have to show that F is an action of $\mathbb{Z}$ on $\mathbb{R}^d$. Let $f(x_0) := F(x_0, 1)$.

$$\begin{aligned}F(x_0,n) &= \Phi(x_0,0,nT) = \Phi(\Phi(x_0,0,(n-1)T),(n-1)T,nT)) \\ &= \Phi(\Phi(x_0,0,(n-1)T),0,T)) = f(\Phi(x_0,0,(n-1)T)) = f^n(x_0).\end{aligned}$$

In addition, note that the uniqueness of the solutions of the ODE implies that if $f(x_0) = 0$, then $x_0 = 0$. Now, by construction, $f(x_0) = K(0,T)x_0$, thus $K(0,T)$ is an invertible matrix. Hence $F(x_0,-n) = f^{-n}(x_0)$ for all $n \in \mathbb{N}$. □

By using the functional calculus (see Problem C.19) one can define $B := T^{-1} \ln K(0,T)$, so $e^{BT} = K(0,T)$. Let us now consider $P(t) := K(0,t)e^{-Bt}$.

$$\begin{aligned}P(t+T) &= K(0,t+T)e^{-B(t+T)} = K(T,t+T)K(0,T)K(0,T)^{-1}e^{-Bt} \\ &= K(0,t)e^{-Bt} = P(t).\end{aligned}$$

We have just proven the following result.

Theorem 1.2.3 (Floquet theorem) *The solutions of the equation (1.2.11) can be written as $x(t) = P(t)e^{Bt}K(s,0)x_0$ where $P(t+T) = P(t)$ is periodic.*

Note that the matrix B can be complex valued. This can be avoided at a little extra cost.

Problem 1.15 *Prove that the solutions of the equation (1.2.11) can be written as $x(t) = P(t)e^{Bt}x_0$ where B is real and $P(t+2T) = P(t)$ is periodic of period $2T$.*

Note that Theorem 1.2.3 implies that the long time behavior is completely contained in the eigenvalues of the matrix B often called *floquet exponents*.

Problem 1.16 *Find the solutions of*

$$\dot{x} = a(t)Ax$$

where $a \in \mathcal{C}^0(\mathbb{R},\mathbb{R})$ is periodic of period T and A is a fixed matrix.

Problem 1.17 *Given a fixed matrix A and a function at matrix values $B(t)$ of period T, consider the equation $\dot{x} = (A + \varepsilon B(t))x$, $\varepsilon \in \mathbb{R}$. Show that, for ε small enough, calling ν_i the Floquet exponents and setting $\lambda_i = e^{\nu_i}$ (often called* Floquet multiplier*), the λ_i are ε-close to the eigenvalues of A.*

1.3 Qualitative study of ODE

The previous discussion has shed some light on the behavior of linear ODE, unfortunately the interesting ODE are typically non linear. Although some nonlinear ODE can be solved explicitly (see any ODE book for examples) typically this is not possible, hence the need of a qualitative theory. As for the qualitative study of functions this can be done quite naively in one dimension, while higher dimensions requires some non trivial theory. Let us see such a naive qualitative theory for ODE via few examples.

1.3.1 The one dimensional case

This situation is very similar to the study of the graph of a function of one variable. Indeed to draw the graph one studies the first derivative and here the first derivative is specified by the equation. Let us consider a couple of simple examples. Consider

$$\begin{aligned}\dot{x} &= e^{-x^2} + x - 2 = V(x)\\ x_0 &= 0.\end{aligned}$$

One cannot integrate the function $V(x)^{-1}$ (which would yield an explicit solution of the ODE), yet from the equation follows that there exists a close to 2 such that $\dot{x}$ is negative if $x \leq a$ and positive otherwise. This implies that the solution starts to be decreasing and keeps decreasing forever.

Next, consider

$$\begin{aligned}\dot{x} &= 1 - 2tx\\ x_0 &= a.\end{aligned}$$

Such an equation cannot be solved by separation of variables, yet the above arguments still apply. In particular, for $t \geq 0$, we have $\dot{x}(t) < 0$ iff $x(t) > \frac{1}{2t}$. On the other hand if $x(t) > \frac{1}{2t}$ it will be so forever. In fact, consider $g(t) = x(t) - \frac{1}{2t}$, then $g'(t) = \dot{x}(t) + \frac{1}{2t^2}$. So if $g(t_*) = 0$, then $g'(t_*) > 0$ hence for $t < t_*$ one has $g(t) < 0$. Thus the solution will increase until it will intersect the curve $\frac{1}{2t}$ and then it will start decreasing but always staying above such a curve. Accordingly, for $t \geq t_*$ we can write $x(t) = \frac{1+\alpha(t)}{2t}$ with $\alpha \geq 0$. Then $\dot{x}(t) = -\alpha(t)$, that is

$$\frac{1}{2t} \leq x(t) = \frac{1}{2t_*} - \int_{t_*}^{t} \alpha(s)ds \tag{1.3.12}$$

moreover $-\frac{1+\alpha(t)}{2t^2} + \frac{\dot{\alpha}(t)}{2t} = -\alpha(t)$

$$\dot{\alpha}(t) = -(2t - \frac{1}{t})\alpha(t) + \frac{1}{t}$$

which means that either $\alpha(t) \le \frac{1}{2t^2-1}$ or it is decreasing. But if it is decreasing it must decrease to zero otherwise (1.3.12) would be falso for large t. Accordingly it must be $\lim_{t\to\infty} \alpha(t) = 0$.

1.3.2 Autonomous equations in two dimensions

In this case the basic idea is to consider one component as a function of the other and in this way reduce to the previous case. Let us see some examples.

Van Der Pol equation

Consider the equation

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= (1-3x^2)y - x. \end{aligned} \tag{1.3.13}$$

Clearly $(0,0)$ is the unique zero of the vector field. If we linearise (1.3.13) around zero we have

$$\frac{d}{dt}(x,y) = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The matrix has eigenvalues $\lambda_\pm = \frac{1\pm\sqrt{3}i}{2}$ hence the fixed point is repelling and the solutions spiral away from it.

The next question is if a similar motion takes place also far away from the origin. To this end we want to forget the time dependence and concentrate only on the shape of the trajectories. Thus we can represent trajectories on the xy plane. Indeed, apart from the point $(0,0)$, either $\dot{x}$ or $\dot{y}$ are different from zero. In the first case one can locally invert $x(t)$ and write $y(x) = y(t(x))$. When this is possible one obtains

$$\frac{dy}{dx} = 1 - 3x^2 - \frac{x}{y},$$

which can be studied as in the previous examples. With a bit of work one can see that the trajectory spirals around zero, but exactly how?

To better understand the behaviour of the solution we introduce a "Lyapunov" like function.

$$L(x,y) = 2(x - x^3 - y)^2 + (x-y)^2 + 3x^2.$$

If $(x(t), y(t))$is a solution of (1.3.13), then a direct computation yields

$$\frac{d}{dt}L(x(t),y(t)) = x^2\left[6 - x^2 - 3(x-y)^2 - 3y^2\right].$$

Hence L is decreasing outside an ellipse. Since $2ab \leq a^2 + b^2$,[24]

$$\begin{aligned} L(x,y) &= 3(x-y)^2 - 4(x-y)x^3 + 2x^6 + 3x^2 \geq (x-y)^2 + 3x^2 \\ &= 4x^2 - 2xy + y^2 \geq 2x^2 + \frac{1}{2}y^2. \end{aligned}$$

Hence the level sets $K_\alpha = \{(x,y) \in \mathbb{R}^2 \ : \ L(x,y) \leq \alpha\}$ are contained in the ellipses $\{(x,y) \in \mathbb{R}^2 \ : \ 2x^2 + \frac{1}{2}y^2 \leq \alpha\}$ and hence are compact.

Hence far away from the origin the trajectory spirals inwardly. It follows then, by the continuity with respect to the initial data, that there exists an $a_* \geq 0$ such that the corresponding solution is a periodic orbit.

Lotka-Volterra equation

$$\begin{aligned} \dot{x} &= ax - Ax^2 - \lambda xy \\ \dot{y} &= -dy + \lambda xy. \end{aligned}$$

This equation is meant to describe the evolution of two populations one feeding on the other (predator-prey). It also has periodic solutions, try to prove it using qualitative methods.

Second order in one dimension

Consider the equation

$$\begin{aligned} &\ddot{x} = -\gamma\dot{x} + \frac{x^2}{1+x^4} \\ &x(0) = 0; \quad \dot{x}(0) = v. \end{aligned}$$

Setting $(z,w) = (x,\dot{x})$, we can write it as

$$\begin{aligned} \dot{z} &= w \\ \dot{w} &= -\gamma w + \frac{z^2}{1+z^4} \end{aligned}$$

which is the type discussed above.

Clearly if we consider still higher dimensional cases the above naive approach cannot help us very much, hence the need of a more sophisticated theory.

[24] It follows from $(a-b)^2 \geq 0$.

Problems

1.18. Given two Banach spaces $\mathcal{B}_1, \mathcal{B}_2$ and a function$f : \mathcal{B}_1 \to \mathcal{B}_2$ we can define the partial derivative at $x \in \mathcal{B}_1$ in the direction $v \in \mathcal{B}_1$ (Gâteaux derivative) by

$$\partial_v f(x) = \lim_{h\to 0} h^{-1} \left[f(x+hv) - f(x) \right],$$

if the limit exists. On the other hand we say that f is Fréchet differentiable at x if there exists $A \in L(\mathcal{B}_1, \mathcal{B}_2)$ (the space of the continous linear operators from $\mathcal{B}_1$ to $\mathcal{B}_2$) such that

$$\lim_{h\to 0} \frac{\|f(x+h) - f(x) - Ah\|}{\|h\|} = 0,$$

and A is called the Fréchet differential at of f at x (often written $Df(x)$). Show that if f is Fréchet differentiable at zero, then it is continuous and Gâteaux differentiable.

1.19. Let $f \in \mathcal{C}^0(\mathcal{B}_0, \mathcal{B}_1)$ and $g \in \mathcal{C}^0(\mathcal{B}_1, \mathcal{B}_2)$ such that f is Fréchet differentiable at $x \in \mathcal{B}_0$ and g is Fréchet differentaible at $f(x) \in \mathcal{B}_1$. Show that $g \circ f \in \mathcal{C}^0(\mathcal{B}_0, \mathcal{B}_2)$ is Fréchet differentaible at x and that $D(g \circ f)(x) = Dg(f(x)) \cdot Df(x) \in L(\mathcal{B}_0, \mathcal{B}_2)$. Of course, this is nothing else than a glorified version of the *chain rule*.

1.20. Given a compact interval $I \subset \mathbb{R}$, a Banach space $\mathcal{B}$, and a continuous function $f \in \mathcal{C}^0(I, \mathcal{B})$, shows that one can define the equivalent of the Riemannian integral.

1.21. Prove the fundamental theorem of calculus in this setting. That is, for $f \in \mathcal{C}^1(\mathcal{B}_1, \mathcal{B}_2)$ let $Df(x) \in L(\mathcal{B}_1, \mathcal{B}_2)$ be the Fréchet differential at $x \in \mathcal{B}_1$, then for each $x, y \in \mathcal{B}_1$

$$f(y) = f(x) + \int_0^1 Df(x + t(y-x)) \cdot (x-y)dt.$$

1.22. Show that, for all $f \in \mathcal{C}^0([a,b], \mathcal{B})$,

$$\left\| \int_a^b f(t)dt \right\| \le \int_a^b \|f(t)\|dt.$$

1.23. Study the solutions of the following equations for all possible initial conditions and $p \in \mathbb{N}$

$$\dot{x} = |x|^p$$
$$\dot{x} = x(\ln|x|)^p$$

1.24. Let $K \in \mathcal{C}^1(\mathbb{R} \times [0,1])$. Show that the equation

$$\begin{aligned} \partial_t u(t,s) &= \int_0^1 K(t+s,\tau)u(t,\tau)^2 d\tau \\ u(0,s) &= s^2. \end{aligned}$$

has a unique continuos solution for t small enough.

1.25. Under the same hypotheses of Problem 1.17 show that if $\int_0^T B(s)ds = 0$ and the eigenvalues of A have all multiplicity one, then the Floquet multiplier differ from the eigenvalues of e^{AT} only of order ε^2.

1.26. Study the equation

$$(1+x)y\dot{y} + (x+y^2) = 0.$$

1.27. Study the equation (Bernoulli)

$$\dot{y} + p(x)y = q(x)y^n.$$

1.28. Study the equation

$$\ddot{x} = -\gamma\dot{x} - x^3.$$

Hints to solving the Problems

In this section, and in the parallel sections in later chapters, I give hints for the solution of some of the Problems.

It is a very good idea to try very hard to solve the problems *before* looking at the hints: it is impossible to appreciate the solution if one has no feeling for the difficulties in the problem. The only way I know to get such a feeling is to *seriously* try to solve it.

Also, keep in mind that I suggest one way to proceed, often other ways are possible and maybe better.

1.1 The proof is the same as the standard proof for the case $\mathcal{B} = \mathbb{R}^d$. However to see this you have to do Problems 1.18 and 1.20 to understand exactly what the derivate and integral mean in this more general case.

1.12 For $n = 0$ it is just (1.1.2). To verify it for any n it suffices to show that

$$\int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n 1 = \frac{t^n}{(n+1)!}.$$

This follows since the domain of integration is $D = \{x \in [0,t]^{n+1} : t_{n+1} \leq t_n \leq \cdots \leq t\}$. On the other hand, for each permutation σ of the set $\{1, \ldots, n+1\}$ the sets $D_\sigma = \{x \in [0,t]^{n+1} : t_{\sigma_{n+1}} \leq t_{\sigma_n} \leq \cdots \leq t\}$ have the same measure, all the D_σ are disjoint and the union of all of them gives $[0,t]^{n+1}$.

1.15 First notice that if a matrix has no eigenvalues on the negative axis then the contour γ in C.3.2 can be taken symmetric around the real axis and, by using C.3.2 with the standard definition of ln with a cut on the negative real axis, this defines $\ln K(0,T)$ with real entries (since the formula for his complex conjugate is the same). In general use the spectral decomposition to write $K(0,T) = C + D$ where $\sigma(C) \cap \mathbb{R}_- = \emptyset$ and $\sigma(D) \subset \mathbb{R}_-$. Then $\sigma(D^2) \subset \mathbb{R}_+$, hence $B = \frac{1}{T} \ln C + \frac{1}{2T} \ln D^2$ is real and $e^{2BT} = C^2 + D^2 = K(0,T)^2$. The rest of the argument is as before.

1.17 Show that the solution satisfies

$$x(t) = e^{At} x_0 + \varepsilon \int_0^t e^{A(t-s)} B(s) x(s) ds.$$

and apply the perturbation theory in Appendix C.

1.20 Let $I = [a,b]$. Since the function is continuos, it is uniformly continuous, hence for $\varepsilon > 0$ there exists $\delta > 0$ such that, for each partition $\xi = \{[x_0, x_1], \ldots, [x_{n-1}, x_n]\}$, $x_0 = a, x_n = b$, $x_{n+1} - x_n \leq \delta$, holds $\sup_{z,y \in [x_{n+1}, x_n]} \|f(z) - f(y)\| \leq \varepsilon$. Accordingly, for each choice of $z_n, y_n \in [x_{n+1}, x_n]$ we have

$$\left\| \sum_{k=0}^{n-1} f(z_k)(x_{k+1} - x_k) - \sum_{k=0}^{n-1} f(y_k)(x_{k+1} - x_k) \right\| \leq \varepsilon.$$

By similar arguments one can compare the sum defined on one partition with the sum defined on a finer partition. Finally sum on different partitions can be compared with the sum on the coarser partition finer of both. This shows that all sufficiently fine partitions yield the same approximate value, hence one can consider the partitions $\xi_n = \{[a + i\frac{b-a}{n}, a + (i+1)\frac{b-a}{n}]\}_{i=0}^{n-1}$ and define

$$\int_I f(t)dt := \lim_{n \to \infty} \sum_{i=0}^{n-1} f(a + i\frac{b-a}{n}) \frac{b-a}{n}.$$

By the above discussion this is equivalent to the same limit taken along any other partition the diameter of which elements tend uniformly to zero.

1.24 Consider the Banach space $\mathcal{B} = \mathcal{C}^0([0,1], \mathbb{R})$. Then $u(t, \cdot) \in \mathcal{B}$ and one can apply Theorem 1.1.1.

1.25 By Problem 1.17 we know that the solution at time T is given by the matrix $D(\varepsilon) := e^{AT}\left[\mathbb{1} + \varepsilon \int_0^T e^{-As}B(s)e^{As}ds\right]$. By the results in Appendix C it follows that, for ε small enough, the eigenvalues of $D(\varepsilon)$ are still simple and analytic on ε. Thus, let $\lambda(\varepsilon)$ one of such eigenvalues and $\Pi(\varepsilon)$ the associated eigenprojector. We have $D(\varepsilon)\Pi(\varepsilon) = \lambda(\varepsilon)\Pi(\varepsilon)$. Differentiating yields $\dot{D}(\varepsilon)\Pi(\varepsilon) + D(\varepsilon)\dot{\Pi}(\varepsilon) = \dot{\lambda}(\varepsilon)\Pi(\varepsilon) + \lambda(\varepsilon)\dot{\Pi}(\varepsilon)$. Multiplying on the right by $\Pi(\varepsilon)$, since $\Pi(\varepsilon)D(\varepsilon) = D(\varepsilon)\Pi(\varepsilon)$, we have

$$\Pi(\varepsilon)\dot{D}(\varepsilon)\Pi(\varepsilon) = \dot{\lambda}(\varepsilon)\Pi(\varepsilon).$$

Since $\Pi(\varepsilon)v = \langle a(\varepsilon), v\rangle b(\varepsilon)$ for some vectors a, b analytic in ε, $\dot{\lambda}(\varepsilon) = \langle a(\varepsilon), \dot{D}(\varepsilon)b(\varepsilon)\rangle$. We can now apply such a general formula to our specific case:

$$\begin{aligned}\langle a(0), \dot{D}(0)b(0)\rangle &= \langle a(0), e^{AT}\int_0^T e^{-As}B(s)e^{As}b(0)ds\rangle \\ &= \langle a(0), e^{AT}\int_0^T e^{-As}B(s)e^{As}b(0)ds\rangle \\ &= \lambda(0)\int_0^T \langle a(0), B(s)b(0)\rangle ds = 0.\end{aligned}$$

Notes

This chapter is super condensed and has no pretension to exhaust the theory of ODE. If one wants to have a better understanding of the field and some ideas of how an ODE can be solved in special cases better consult [HS74, Arn92, CL55].