

CHAPTER 4

Global Behavior: simple examples



ifferent local behaviors have been analyzed in the previous chapter. Unfortunately, such analysis is insufficient if one wants to understand the *global* behavior of a Dynamical System. To make precise what we mean by global behavior we need some definitions.

Definition 4.0.2 *Given a Dynamical System (X, ϕ_t) , $t \in \mathbb{N}$ or \mathbb{R}_+ , a set $A \subset X$ is called invariant if, for all t , $\emptyset \neq \phi_t^{-1}(A) \subset A$.*

Essentially, the global understanding of a system entails a detailed knowledge of its invariant set and of the dynamics in a neighborhood of such sets. This is in general very hard to achieve, essentially the rest of this book devoted to the study of some special cases.

Remark 4.0.3 *We start with some simple considerations in the case of continuous Dynamical Systems (this is part of a general theory called Topological Dynamical Systems¹) and then we will address more subtle phenomena that depend on the smoothness of the systems.*

4.1 Long time behavior and invariant sets

First of all let us note that if we are interested in the long time behavior of a system and we look at it locally (i.e. in the neighborhood of a point) then three cases are possible: either the motion leaves the neighborhood

¹Recall that a Topological Dynamical Systems is a couple (X, ϕ_t) where X is a topological space and ϕ_t is a continuous action of \mathbb{R} (or $\mathbb{R}_+, \mathbb{N}, \mathbb{Z}$) on X .

and never returns, or leaves the neighborhood but eventually it comes back or never leaves. Clearly, in the first case the neighborhood in question has little interest in the study of the long time behavior. This is made precise by the following.

Definition 4.1.1 *Given a Dynamical System (X, ϕ_t) , a point $x \in X$ is called wandering if there exists a neighborhood U of x and a $t_0 \geq 1$ such that, for all $t \geq t_0$, $\phi_t(U) \cap U = \emptyset$. A point that is not wandering is called non-wandering. The set of non-wandering points is called $NW(\{\phi_t\})$ or simply NW if no confusion arises.*

Problem 4.1 *If $\phi_t \in C^0$, then the set NW is closed and forward invariant (i.e. $\phi_t(NW) \subset NW$ for each $t \geq 0$). If the ϕ_t are open maps, then NW is also invariant.*

Problem 4.2 *Construct an example of a topological dynamical systems in which the non-wandering set is not invariant.*

Problem 4.3 *Show that if A is invariant, then the sets $\Lambda = \bigcap_{t=0}^{\infty} \phi_t^{-1} \overline{A}$ and $\Omega = \overline{\bigcup_{t=0}^{\infty} \phi_t(A)}$ are non-empty, invariant and, more, $\phi_t^{-1}(\Lambda) = \Lambda$ and $\phi_t^{-1}(\Omega) = \Omega$*

The relevance for the long time behavior is emphasized by the following lemma.

Lemma 4.1.2 *If $K \subset X$ is compact and $K \cap NW = \emptyset$, then for all $x \in K$ there exists T such that $\phi_t(x) \notin K$ for all $t \geq T$. In addition, if K is invariant, then T can be chosen independent of x .*

PROOF. If all the points in K are wandering, then for each $x \in K$ there exists a neighborhood $U(x)$ and a time $t(x)$ such that $\phi_t U(x) \cap U(x) = \emptyset$ for all $t \geq t(x)$. Clearly $\{U(x)\}_{x \in K}$ is an open covering of K , hence we can extract a finite subcover. Let $\{U_i\}_{i=1}^N$ be such a subcover, let $\{t_i\}$ be the corresponding associated times. If $x \in K$ then $x \in U_i$ for some $i \in \{1, \dots, N\}$, and $\phi_t(x) \notin U_i$ for $t \geq t_i$. If $\phi_t(x) \notin K$ for all $t \geq t_i$, then we are done. If there exists $t \geq t_i$ such that $\phi_t(x) \in K$, then $\phi_t(x)$ must belong to another U_j , that will leave forever for $t \geq t_j$. It is then clear that $\phi_t(x)$ cannot remain in K for a time longer than $\sum_i t_i$, nor can the trajectory return form more than N times.

If K is invariant then it follows that if $x \notin K$ then $\phi_t x \notin K$ for all $t \geq 0$. Thus once a point exits K it can never come back. The above argument then show that each point must exit forever in a time at most $\sum_i t_i$. \square

Corollary 4.1.3 *If $K \subset X$ is compact and invariant, then either there exists $n \in \mathbb{N}$ such that $T^{-n}K = \emptyset$ or $NW \cap K \neq \emptyset$.*

PROOF. If $NW \cap K = \emptyset$, then Lemma 4.1.2 imply that there exists $n \in \mathbb{N}$ such that $T^n K \cup K = \emptyset$, hence $T^{-n}K = \emptyset$. \square

To see the connection to long time behavior and invariant sets we need an extra definition

Definition 4.1.4 *Given a topological Dynamical System (X, ϕ_t) , $t \in I \in \{\mathbb{R}, \mathbb{Z}, \mathbb{R}_+, \mathbb{N}\}$, and $x \in X$ we call $\omega(x)$ (the ω -limit set of x) the accumulation points of the set $\cup_{t \geq 0} \{\phi_t(x)\}$. If t belongs to \mathbb{R} or \mathbb{Z} , then the α -limit set is defined analogously with $t \leq 0$.*

Problem 4.4 *Show that the ω -limit sets are closed sets such that $\phi_t(\omega) = \omega$ (hence if ϕ_t is invertible then the omega limits are invariant).*

Theorem 4.1.5 *For each $x \in X$ we have $\omega(x) \subset NW$. In addition, if X is a proper metric space,² then for each $z \in X$ either holds $\lim_{t \rightarrow \infty} d(\phi_t(x), z) = \infty$, or $\lim_{t \rightarrow \infty} d(\phi_t(x), NW) = 0$.*

PROOF. Let $x \in X$. If $z \in \omega(x)$, then for each neighborhood U of z we have $\{t_n\} \subset \mathbb{R}_+$ such that $\phi_{t_n}(x) \in U$. Thus there exists t and $s \geq t + 1$ such that $\phi_{t-s}U \cap U \supset \{\phi_s(x)\} \neq \emptyset$. Hence $z \in NW$.

Let us come to the second part of the Theorem. If the two alternatives do not hold, then there exists a compact set (a closed ball) that contains infinitely many points of the orbit of x all at a finite distance from NW . This implies that the orbit has an accumulation point (hence an element of $\omega(x)$) not in NW contradicting the first part of the Theorem. \square

²That is, a distance d is defined and the base for the topology is made of the sets $B_r(x) = \{y \in X : d(x, y) < r\}$ (this is called a *metric space*). A *proper* metric space is one in which all the closed balls $\{y \in X : d(x, y) \leq r\}$ are compact.

In particular the above Theorem shows that all the interesting long time dynamical behavior happens in a neighborhood of the non-wandering set.

Problem 4.5 *Given a discrete topological dynamical system (X, T) , let $A = NW(T)$. Since A is forward invariant, one can consider the restriction S of T to A . Find an example in which $NW(S)$ is strictly smaller than A .*

Definition 4.1.6 *Given a Dynamical System (X, ϕ_t) , a point $x \in X$ is called recurrent if $x \in \omega(x)$. The set of recurrent points is called $R(\{\phi_t\})$, or simply R if no confusions arises.*

Problem 4.6 *Consider a linear system $\dot{x} = Ax$. Show that if A is hyperbolic, then $NW = \{0\}$.*

Problem 4.7 *Consider a saddle-node bifurcation in one dimension. Show that in a small neighborhood of the bifurcation point, when two fixed points x_1, x_2 are present, $NW = \{x_1, x_2\}$. Show that this may not be the case in higher dimensions.*

Problem 4.8 *Consider the ODE $\dot{x} = \begin{pmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{pmatrix} x$, $\omega_0 > 0$. Show that $NW = \mathbb{R}^2$, while for each $x \in \mathbb{R}^2$ holds $\omega(x) = \{z \in \mathbb{R}^2 : \|z\| = \|x\|\}$.*

Problem 4.9 *In the case of the Hopf bifurcation in two dimensions when the fixed point O is repelling, and hence the periodic orbit γ is attracting, show that (in a neighborhood of O for the bifurcation parameter small enough) $NW = \{O\} \cap \gamma$.*

Remark 4.1.7 *We have thus seen examples in which the ω -limit sets can be a point or a periodic orbit, **do other possibilities exists?***

This question is going to lead us in a long journey.

4.2 Poincaré-Bendixon

See [HS74].

4.3 Equations on the Torus

As we have seen a generic family of vector fields in \mathbb{R}^2 can have a very limited choice of bounded invariant sets: either a fixed point and the associated stable and unstable manifolds, or (by Poincaré-Bendixon) a periodic orbit. Yet one can have a differential equation on different manifolds, notably the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

Problem 4.10 Consider the vector fields $V(x) = \omega \in \mathbb{R}^2$ on \mathbb{T}^2 and show that the orbit of the associated flow can be everywhere dense.

The above problem shows that on \mathbb{T}^2 it is possible to have a new ω -limit set: \mathbb{T}^2 itself! Can such a situation take place for an open set of vector fields? To understand the situation it is useful to generalize the setting of Problem 4.10.

Definition 4.3.1 A closed non self-intersecting curve $\gamma \in C^r(S^1, \mathbb{T}^2)$, $r \geq 1$, is called a global (cross) section for the flow associated to V if

a) γ' is always transversal to V .³

b) for each $x \in \mathbb{T}^2$ there exists $t \in \mathbb{R}_+$ such that $\phi_t(x) \in \gamma$.

Given a cross section γ we can define the return time $\tau : \gamma \rightarrow \mathbb{R}_+$ as the first $t > 0$ such that $\phi_t(x) \in \gamma$ and the Poincaré map $f : \gamma \rightarrow \gamma$ as $f(x) = \phi_{\tau(x)}(x)$.

Problem 4.11 Show that if $\gamma \in C^r(S^1, \mathbb{T}^2)$ is a global cross section and f is the associate Poincaré map, then $f \in C^r$ and (γ, f) is a Dynamical Systems that describe the dynamics when it returns to γ .

Lemma 4.3.2 (Siegel) Let $V \in C^r(\mathbb{T}^2, \mathbb{R}^2)$ be a nowhere zero vector field. If the associated flow has no periodic orbits, then there exists a global section γ . In addition, if $f : \gamma \rightarrow \gamma$ is the Poincaré map associated to the flow, then $f \in C^r(\gamma, \gamma)$.

PROOF. The (nice) idea is to construct a section close to an orbit. Let ϕ_t be the flow associated to the vector field V . Let $x \in NW$ and consider an open segment, of length less than $1/2$, Σ , $x \in \Sigma$, transversal

³That is, the vectors $\{\gamma'(t), V(\gamma(t))\}$ span \mathbb{R}^2 for all $t \in S^1$.

to the vector field (similar to the construction in the Flow Box Theorem 2.1.1). Since x is non-wandering and due to Theorem 2.1.1, there exists $z \in \Sigma$, $z \neq x$, and $t \in \mathbb{R}$ such that $\phi_t(z) \in \Sigma$, this being the first return to Σ . Since there are no periodic orbits $z \neq \phi_t(z)$.

We will construct a global section close to $\{\phi_s(z)\}_{s=0}^t \cup \Sigma$. Note that the closed curve that one obtains joining z to $\phi_t(z)$ along Σ cannot be homotopic to a point. Otherwise the curve would have an interior homeomorphic to a disk in \mathbb{R}^2 from which the orbits cannot escape either in the future or the past. By Poincaré–Bendixon this would imply the existence of a periodic orbit contrary to the hypothesis. To properly explain the construction it is convenient to introduce a flow box type system of coordinates near such an orbit.

For $s \in [-1/2, 1/2]$ let $\varphi(s) = z + s(x - z)\|x - z\|^{-1}$. Clearly $\varphi(0) = z$, $\varphi(\|x - z\|) = x$, and holds $\varphi([-1/2, 1/2]) \supset \Sigma$. Next, for each $y \in \Sigma$ let $s \in [-1/2, 1/2]$ be the unique number such that $y = \varphi(s)$ and $\tau(s) = \inf\{t > 0 : \phi_t(y) \in \Sigma\}$ be the first return time to the section. By Theorem 2.1.1 and Corollary 1.1.14 there exists $\delta \in (0, 2\|x - z\|)$ such that $\tau \in \mathcal{C}^r([-\delta, \delta], \mathbb{R}_+)$. For $A := \{(s, t) \in \mathbb{R}^2 : s \in [-\delta, \delta], t \in [0, \tau(s)]\}$ let us define the map $\Xi : A \rightarrow \mathbb{T}^2$ by $\Xi(s, t) = \phi_t(\varphi(s))$. Note that this map is \mathcal{C}^r and invertible (provided δ is chosen small enough), hence it can be used as a change of coordinates. Note that this are essentially the coordinates used in the flow box theorem, only now they are used in a long neighborhood of an orbit.

The next step is to understand how the orbit comes back. Indeed, if we use standard flow box coordinates (s', t') in a neighborhood of Σ , then $(s, t) = (s', t')$ for $t \geq 0$ but for t close to $\tau(s)$ we are again in the neighborhood of Σ corresponding to $t' < 0$. The change of coordinates can then be described by the function θ such that $\phi_{\tau(s)}\varphi(s) = \varphi(\theta(s))$. Then (s, t) corresponds to $(\theta(s), t - \tau(s))$.

Problem 4.12 Let $\tau_0 = \tau(0)$, then $\langle x - z, \frac{d}{ds}\phi_{\tau_0}(\varphi(s))|_{s=0} \rangle > 0$.⁴

The above problem means simply that $\theta' > 0$.

To conclude we must analyze two possibilities: either $\phi_{\tau}z$ is closer to x than z or vice versa. The two cases are treated exactly in the same way so we discuss only the first, that is $\theta(0) > 0$. We can then

⁴This is really a consequence of the fact that the torus is orientable, yet it can be proven directly in several ways.

chose $\varepsilon \in (0, \delta)$ such that $\theta(-\varepsilon) > 0$. Consider a line $(\varepsilon - 2\varepsilon\tau_0^{-1}t, t)$, $t \in [0, \tau_0]$, obviously it is always transversal to the flow. If we look at it in the standard flow box coordinates in a neighborhood of Σ we see that it start as a decreasing curve and, since $\theta' > 0$, it reappears (for $t' < 0$) as a still decreasing curve. It is then easy to see that it can be smoothly deformed, in a neighborhood of Σ , into a closed curve that is always transversal to the flow. We have thus constructed a smooth transversal section it remains to show that it is global.

Problem 4.13 *Consider a piecewise smooth closed curve Γ in \mathbb{T}^2 . Show that $\mathbb{T}^2 \setminus \Gamma$ is either disconnected (and one connected component is isomorphic to an open set in \mathbb{R}^2) or it is isomorphic to a cylinder.*

If the above section would not be global, then there would be trajectories that stay forever in a set (either a piece of \mathbb{R}^2 or a cylinder) to which Poincaré-Bendixon applies. But this would imply the presence of a periodic orbit, contrary to the assumption. \square

Problem 4.14 *Show that, in the setting of the above theorem, the condition $f' \neq 0$ is generic.*

It is important to notice that, given a topological Dynamical System (M, f) and a function $\tau \in C^0(M, \mathbb{R}_+ \setminus \{0\})$ (called *roof function*) one can always see them as a Poincaré section and a return time of a flow. The resulting object is called a *suspension* or *standard flow* and is constructed as follows.

Consider the set $\tilde{\Omega} = \{(x, s) \in M \times \mathbb{R}_+ : s \in [0, \tau(x)]\}$ with the topology induced by $M \times \mathbb{R}_+$ equipped with the product topology.

Problem 4.15 *Consider the relation $(x, s) \sim (y, t)$ iff $x = y$ and $s = t$ or $s = \tau(x)$, $t = 0$ and $y = f(x)$ or $t = \tau(y)$, $s = 0$ and $x = f(y)$. Prove that it is an equivalence relation.*

One can then consider the space of the equivalence classes $\Omega = \tilde{\Omega} / \sim$ with the induced topology, this is the space on which the flow is defined: let $t \leq \inf \tau$, define

$$\phi_t(x, s) = \begin{cases} (x, s + t) & \text{if } t < \tau(x) - s \\ (f(x), t + s - \tau(x)) & \text{if } t \geq \tau(x) - s \end{cases}$$

and extend ϕ_t by the group property.

Theorem 4.3.3 *Let $V \in \mathcal{C}^2(\mathbb{T}^2, \mathbb{R}^2)$ be a nowhere zero generic vector field with no periodic orbits. Then for each point $y \in \mathbb{T}^2$, $\omega(y) = \mathbb{T}^2$.*

PROOF. By Lemma 4.3.2 we have a smooth global section γ with a Poincaré map g . Let $h : S^1 \rightarrow \gamma$ be a parametrization of γ . If we set $f = h^{-1} \circ g \circ h$, we can consider the return map as \mathcal{C}^2 map on the unit circle such that $f' \neq 0$ at each point. Note that a periodic point for the map f corresponds to a periodic orbit for the flow, hence f cannot have periodic orbits. The claim follows then by Lemma 4.5.2 in which it is proven that a smooth circle map with no periodic orbits has dense orbits. \square

The final natural question is:

In the hypotheses of Theorem 4.3.3, is it possible to conjugate the flow to a rigid rotation of the torus, and, if yes, to which one?

Motivated by the above question and results we will now study orientation preserving circle maps. It turns out to be interesting and helpful to study their properties in relations to their increasing smoothness.

4.4 Circle maps: topology

Here, and in the following, we study a Dynamical System (S^1, f) where f is a homeomorphism of S^1 (i.e. f is invertible and $f(S^1) = S^1$).

We start with some facts that follow from the simple hypothesis of continuity.

First of all note that one can lift the map f to the universal cover \mathbb{R} of the circle, that is defining $\pi : \mathbb{R} \rightarrow S^1$ as $\pi(x) = x \bmod 1$, it is possible to find $F \in \mathcal{C}^0(\mathbb{R}, \mathbb{R})$ such that

$$f \circ \pi = \pi \circ F.$$

Problem 4.16 *Construct explicitly such an F . Show that $F(x+1) = F(x) + 1$.*

Problem 4.17 *If there exists $L > 0$ such that $-L \leq a_{m+n} \leq a_n + a_m + L$ for all $n, m \in \mathbb{N}$, then the limit $\lim_{n \rightarrow \infty} \frac{a_n}{n}$ exists.*

Lemma 4.4.1 *Let $f : S^1 \rightarrow S^1$ an homeomorphism and $F \in \mathcal{C}^0(\mathbb{R}, \mathbb{R})$ a lift of f . Then the limit*

$$\tau(f) := \lim_{|n| \rightarrow \infty} \frac{F^n(x)}{n} \pmod{1}$$

exists and is independent both from the point and the lift.

PROOF. Applying Problem 4.17 to the sequence $F^n(x)$ the existence of the limit follows. The other assertions depend on the already mentioned equality $F(x+1) = x+1$. \square

Lemma 4.4.2 *Show that $\tau(f) \in \mathbb{Q}$ if and only if f has a periodic orbit.*

PROOF. If $f^q(x) = x$ and F is a lift then it must be $F^q(x) = x + p$ for some $p \in \mathbb{N}$. This immediately implies $F^{kq}(x) = x + kp$ and hence $\tau(f) = \frac{p}{q} \in \mathbb{Q}$. On the other hand, if $\tau(f) = \frac{p}{q} \in \mathbb{Q}$, we have $\tau(f^q) = p \pmod{1} = 0$. It thus suffices to prove that $\tau(f) = 0$ implies f has a fixed point. Let us do a proof by contradiction: we suppose that f has no fixed points. Note that this is the same than saying that $G(\mathbb{R}) \cap \mathbb{Z} = \emptyset$ where $G(x) = F(x) - x$. Since G is continuous this implies $\max G - \min G < 1$. Let $\alpha = \min G$, $\beta = \max G$. Note that, by properly choosing the lift F , one can insure that $[\alpha, \beta] \subset (0, 1)$. Then

$$F^n(x) = G(F^{n-1}(x)) + F^{n-1}(x) \geq \alpha + F^{n-1}(x) \geq n\alpha$$

hence $\tau(f) \geq \alpha$, analogously $\tau(f) \leq \beta$ which contradicts $\tau(f) = 0$. \square

Problem 4.18 *Given $f \in \mathcal{C}^0(S^1, S^1)$, for any interval $I \subset S^1$, if $f(I) \subset I$, then f has a fixed point in I .*

Problem 4.19 *If $\tau(f) \notin \mathbb{Q}$, then for each $n \in \mathbb{N} \setminus \{0\}$ and $x, y \in S^1$, $\{f^k(y)\}_{k \in \mathbb{N}} \cap [x, f^n(x)] \neq \emptyset$.*

Problem 4.20 *If $\tau(f) \notin \mathbb{Q}$, then for each $x \in S^1$ there exist infinitely many $n \in \mathbb{Z}$ such that $\{f^k x\}_{|k| < n} \cap [x, f^n x] = \emptyset$.*

Lemma 4.4.3 *For any homomorphism $f : S^1 \rightarrow S^1$ with $\tau(f) \notin \mathbb{Q}$ and any $x, y \in S^1$ holds $\omega(x) = \omega(y)$.*

PROOF. If $z \in \omega(x)$, then there exists $\{n_j\}$ such that $\lim_{j \rightarrow \infty} f^{n_j}(x) = z$. But then Problem 4.19 implies that for each $j \in \mathbb{N}$ there exists $k_j \in \mathbb{N}$ such that $f^{k_j}(y) \in [f^{n_j}(x), f^{n_{j+1}}(x)]$. Clearly $\lim_{j \rightarrow \infty} f^{k_j}(y) = z$, thus $z \in \omega(y)$. Reversing the role of x and y the Lemma follows. \square

Problem 4.21 Let f be a homeomorphism of S^1 with irrational rotation number show that for each $\varepsilon > 0$ there exists a homeomorphism f_ε , $\|f - f_\varepsilon\|_\infty \leq \varepsilon$, with $\tau(f_\varepsilon) \in \mathbb{Q}$.

Problem 4.22 Let f_λ be a one parameter family of homeomorphisms such that $\tau(f_0) < \tau(f_1)$. Suppose that $\tau(f_\lambda)$ is increasing, what can you say on the possible intervals in which it is not strictly increasing?

4.5 Circle maps: differentiable theory

In this section we assume $f \in \mathcal{C}^2(S^1, S^1)$ and $\ln f' \in \mathcal{C}^1(S^1, \mathbb{R})$.⁵

Lemma 4.5.1 If $\tau(f) \notin \mathbb{Q}$ and $x_0 \notin \omega(x_0)$, then

$$\sum_{n=0}^{\infty} (f^n)'(x_0) < \infty.$$

PROOF. Let $U(x_0) \ni x_0$ be the largest open interval not intersecting $\omega(x_0)$, call $K(x_0)$ its closure. First of all we see that the invariance of the ω -limit set implies $\{f^n(\partial K(x_0))\}_{n=1}^{\infty} \subset \omega(x_0)$. This implies that either $f^n K(x_0) \cap K(x_0) = \emptyset$ or $f^n K(x_0) \supset K(x_0)$ but the latter would imply the existence of a fixed point for f^n , which is impossible, hence all the sets $\{f^n K(x_0)\}_{n \in \mathbb{Z}}$ must be disjoint. We can now conclude thanks to a typical distortion estimate: let $K_n(x_0) := f^n(K(x_0))$, then, setting $D := \left| \frac{f''}{f'} \right|_\infty$,

$$\begin{aligned} 1 &> \sum_{n \in \mathbb{N}} |K_n(x_0)| = \sum_{n \in \mathbb{N}} \int_{K(x_0)} (f^n)'(x) dx = \sum_{n \in \mathbb{N}} (f^n)'(x_0) \int_{K(x_0)} \frac{(f^n)'(x)}{(f^n)'(x_0)} dx \\ &\geq \sum_{n \in \mathbb{N}} (f^n)'(x_0) \int_{K(x_0)} e^{-\sum_{k=0}^{n-1} |\ln f'(f^k(x)) - \ln f'(f^k(x_0))|} dx \\ &\geq \sum_{n \in \mathbb{N}} (f^n)'(x_0) \int_{K(x_0)} e^{-\sum_{k=0}^{n-1} D |K_k(x_0)|} dx \geq |K(x_0)| e^{-D} \sum_{n \in \mathbb{N}} (f^n)'(x_0). \end{aligned}$$

⁵These hypotheses can be slightly weakened, see [HK95].

□

Lemma 4.5.2 *If $\tau(f) \notin \mathbb{Q}$, then, for all $x \in S^1$, $\omega(x) = S^1$.*

PROOF. We use the same notation as in Lemma 4.5.1. If the Lemma is false then there exists $x \in S^1$ such that $\omega(x) \neq S^1$. But by Lemma 4.4.3 all the omega limit sets are equal, hence there exists $x_0 \in S^1$ such that $x_0 \notin \omega(x_0)$. Note that if there exists $n \in \mathbb{N}$, $n \neq 0$, such that $f^n(x_0) \in K(x_0)$ then, by the invariance of $\omega(x_0)$, it must be $f^n(x_0) \neq \partial K(x_0) \subset \omega(x_0)$ and then Problem 4.19 implies that there are infinitely many k such that $f^k(x_0) \in [x_0, f^n(x_0)] \subset K(x_0)$, but this is impossible since such an interval does not contain accumulation points of the forward trajectory. Thus, for each $n \in \mathbb{Z}$, $n \neq 0$, $f^n(x_0) \notin K(x_0)$, accordingly there exist $\delta > 0$ such that each interval $[x_0, f^n(x_0)]$ has length at least δ .

Next, choose $L > 0$, by Lemma 4.5.1 there exists $m \in \mathbb{N}$ such that $(f^n)'(x_0) < L^{-1}$, for all $n > m$. We can then apply Problem 4.20 to find an $|n| > m$ such that $\{f^k x\}_{|k| < n} \cap [x_0, f^n(x_0)] = \emptyset$. Suppose $n < 0$ and let $J_- = [x_0, f^n(x_0)]$, then for each $k \in \{1, \dots, -n-1\}$, $f^k J_- = [f^k x_0, f^{n+k} x_0]$, since the extreme of such an interval do not belong to J it follows that $f^k J_- \cap J_- = \emptyset$ (otherwise the first would be contained in the second and there would be a fixed point). Thus, setting $J = [x_0, f^{|n|}(x_0)]$, for all $k \in \{1, \dots, -n-1\}$, holds $f^k J \cap J = \emptyset$. The same result follows, setting $J_- = [x_0, f^{-n}(x_0)]$, for $n > 0$. Finally we conclude with another distortion argument

$$\begin{aligned} |f^{-|n|}J| &= \int_J (f^{-|n|})'(x) dx = \frac{1}{(f^{|n|})'(x_0)} \int_J \frac{(f^{|n|})'(f^{-|n|}(f^{|n|}(x_0)))}{(f^{|n|})'(f^{-|n|x})} dx \\ &\geq \frac{1}{(f^{|n|})'(x_0)} \int_J e^{-\sum_{k=0}^{|n|-1} D|f^k J|} dx \geq L e^{-D} \delta. \end{aligned}$$

Then choosing $L > e^D \delta^{-1}$ leads a length of $|f^{-|n|}J|$ larger than one, which contradicts the fact that f is an homeomorphism. □

The above fact can be used to prove the following result (due to Poincaré).

Theorem 4.5.3 *If $\tau(f) = \omega \notin \mathbb{Q}$, then f is \mathcal{C}^0 -conjugate to $R_\omega(x) = x + \omega \pmod{1}$.*

PROOF. See [HK95] Theorem 11.2.7. □

4.6 Circle maps: smooth theory

We have seen that the qualitative behavior of smooth circle maps with irrational rotation number is similar to the behavior of the rigid rotation in Problem 4.10. What is not clear is if the two dynamics can be smoothly conjugated (i.e. in the spirit of the flow box theorem, but globally). This latter problem turns out to be extremely subtle and to require much finer number theoretical consideration than distinguishing between rational and irrationals.

Since we have seen that more smoothness allows to obtain stronger results, it is natural to start by considering analytic functions.

To make the following easier, we will limit ourselves to the case of a maps close to the identity. That is maps with a covering $F : \mathbb{R} \rightarrow \mathbb{R}$ of the form $F(x) = x + \omega + f(x)$, where $f(x + 1) = f(x)$ is “small”.

4.6.1 Analytic KAM theory

To define the sense in which f is small we assume first that f is an analytic function. That is f is a restriction to the real axes of a function, that abusing notation we will still call f , holomorphic in a strip. Let $D_\alpha = \{z \in \mathbb{C} : |\Im(z)| \leq \frac{\alpha}{2\pi}\}$ and consider the function space $\mathbb{B}_\alpha = \{g \in \mathcal{C}^0(D_\alpha, \mathbb{C}) : g(z + 1) = g(z) \forall z \in D_\alpha, g \text{ holomorphic in } \overset{\circ}{D}_\alpha\}$. This is a Banach space when equipped with the norm $\|g\|_\alpha = \sup_{z \in D_\alpha} |g(z)|$.

Theorem 4.6.1 *If $\int_0^1 f = 0$ and there exist $\alpha_0 \in (0, 1)$, $C_0 > 0$ such that $\|f\|_{\alpha_0} \leq 5C_0\alpha_0^3 10^{-10}$, and $\omega > 0$ satisfies*

$$\left| \omega - \frac{p}{q} \right| \geq \frac{C_0}{q^2}$$

for each $p, q \in \mathbb{N}$, then there exists $h \in \mathbb{B}_{\alpha_0/2}$ such that, setting $H(x) = x + h(x)$, $\|h\|_{\alpha_0/2} \leq 3C_0^{-\frac{1}{3}} \|f\|_{\alpha_0}^{\frac{1}{3}}$ and, for all $x \in \mathbb{R}$,

$$H^{-1} \circ F \circ H(x) = x + \omega. \tag{4.6.1}$$

A natural question is: do irrational numbers with the above properties exists? The answer is yes (for example all the quadratic irrational

satisfy such inequalities), but a bit of theory is needed to see it. For a quick introduction to these problems solve the Problems 4.27, 4.28, 4.29, 4.30, 4.31, 4.32, 4.33.

Remark 4.6.2 *Note that we can always reduce to the case $\int f = 0$ by subtracting the average to f and adding it to ω . This simply means that $\tau(F) = \omega$.*

Remark 4.6.3 *The unaware reader can be horrified by the 10^{-10} in the statement of the above theorem. Such a ridiculous number is in part due to the fact that I have privileged readability over optimality, but in part it comes with the method. Indeed, it is well known among specialist that to obtain optimal estimates for KAM-type theorems is a very hard problem and a currently still active field of research.*

PROOF OF THEOREM 4.6.1. Note that if H is invertible, then equation (4.6.1) is equivalent to, for each $z \in D_{\alpha_0/2}$,

$$h(z + \omega) - h(z) = f(z + h(z)). \quad (4.6.2)$$

In fact, we are interested to solving the above equation only for real z . In the following to avoid confusion I will use z for a complex variable and x for a real one.

It is natural to introduce the linear operator $L_\omega g(x) = g(x + \omega) - g(x)$. If such an operator were invertible, then we could write

$$h = L_\omega^{-1} f \circ H, \quad (4.6.3)$$

that looks like a fixed point problem and hopefully can be studied with known techniques.

We have thus to study the operator L_ω . The best is to compute it in Fourier series:

$$L_\omega g(x) = \sum_{k \in \mathbb{Z}} e^{2\pi i k x} (e^{2\pi i \omega k} - 1) \hat{g}_k$$

where $g(x) = \sum_{k \in \mathbb{Z}} e^{2\pi i k x} \hat{g}_k$. Thus, provided $\hat{g}_0 = 0$,

$$L_\omega^{-1} g(x) = \sum_{k \in \mathbb{Z} \setminus \{0\}} e^{2\pi i k x} \frac{\hat{g}_k}{e^{2\pi i \omega k} - 1}.$$

Thanks to the fact that $\omega \notin \mathbb{Q}$, the coefficients in the above formula are well defined. Yet, it remains the issue of the convergence of the series. Indeed, the coefficients can be very large since,⁶

$$\left| e^{2\pi i \omega k} - 1 \right| \geq 2 \inf_{p \in \mathbb{N}} |\omega k - p| \geq 2C_0 |k|^{-1}.$$

This is the main difficulty of the present problem: the infamous *small divisors*. Clearly, due to the small divisors L_ω^{-1} is not a bounded operator. This makes it very hard to study directly (4.6.3). To bypass this problem we need an idea.

The idea that we will use if due to Kolomogorov and goes as follows: instead of solving (4.6.3) consider the change of variables $H_0(x) = x + h_0(x)$ where $h_0 = L_\omega^{-1} f$. Of course such a change of variable it is not the right one, yet one can try to write

$$H_0^{-1} \circ F \circ H_0(x) = x + \omega + f_1(x)$$

and hope that f_1 is much smaller than f . If this is the case one can iterate the procedure and hope that it converges to a limiting change of variables that is the one we are looking for.

To implement the above idea the first thing we need is to connect the analysis via Fourier series to the analytic properties of the functions.

Consider the norm

$$|g|_\alpha := \sum_{k \in \mathbb{Z}} e^{\alpha|k|} |\hat{g}_k|.$$

Let us call \mathcal{B}_α the Banach space of the periodic functions (of period one) on \mathbb{R} equipped with the above norm.

Note that, for $\beta < \alpha$,⁷

⁶Note that $|e^{ix} - 1| \geq |\sin x| \geq \frac{2x}{\pi}$, provided $x \in [0, \pi/2]$. On the other hand if $x \in [\pi/2, \pi]$, then $|e^{ix} - 1| \geq |1 - \cos x| \geq 1$. Hence we can use the simple, but not very sharp, estimate $|e^{2\pi i x} - 1| \geq \inf_{p \in \mathbb{Z}} 2|x - p|$.

⁷Here we use that, for each $n \in \mathbb{N}$ and $\sigma > 0$,

$$\sup_{k \in \mathbb{N}} k^n e^{-\sigma k} \leq \sup_{x \in \mathbb{R}_+} x^n e^{-\sigma x} = \left(\frac{n}{\sigma}\right)^n e^{-n} \leq e^{-1} \sigma^{-n} n!.$$

The last inequality is an application of Stirling formula. If you do not remember it, here is the baby version used above,

$$n! = e^{\sum_{k=1}^n \ln k} \geq e^{\int_1^n \ln x dx} = e^{n \ln n - n + 1} = n^n e^{-n+1}.$$

$$\begin{aligned}
|L_\omega^{-1}g|_\beta &\leq \sum_{k \in \mathbb{Z}} \frac{|k|}{2C_0} e^{\beta|k|} |\hat{g}_k| \leq \frac{|g|_\alpha}{2C_0} \sup_{k \in \mathbb{Z}} |k| e^{-(\alpha-\beta)|k|} \\
&\leq \frac{|g|_\alpha}{2eC_0(\alpha-\beta)}
\end{aligned} \tag{4.6.4}$$

Thus $L_\omega^{-1} : \mathcal{B}_\alpha \rightarrow \mathcal{B}_\beta$ is a bounded operator for each $\alpha > \beta$.

The point is that there is a connection between the above Banach spaces, namely we can define $\Xi : \mathbb{B}_\beta \rightarrow \mathcal{B}_\alpha$, by $\Xi g(x) = g(x)$, for all $x \in \mathbb{R}$.⁸ To see the relation between the norms, let us compute the Fourier coefficients

$$[\widehat{\Xi g}]_k = \frac{1}{i} \int_0^1 e^{2\pi i k x} g(x) dx$$

Problem 4.23 Show that $|[\Xi g]_k| \leq e^{-\alpha|k|} \|g\|_\alpha$.

Hence, for $\alpha > \beta$, $\|\Xi\|_{\mathbb{B}_\alpha \rightarrow \mathcal{B}_\beta} \leq 2(1 - e^{\beta-\alpha})^{-1}$. Note also that we can easily define the inverse: if $g \in \mathcal{B}_\alpha$, then define

$$\Xi^{-1}g(z) = \sum_{k \in \mathbb{Z}} e^{2\pi i k z} \hat{g}_k$$

Problem 4.24 Verify that the above is really the inverse of Ξ .

If $g \in \mathcal{B}_\alpha$, then

$$\|\Xi^{-1}g\|_\alpha \leq \sum_{k \in \mathbb{Z}} e^{|k|\alpha} |\hat{g}_k| = |g|_\alpha.$$

Thus $\|\Xi^{-1}\|_{\mathcal{B}_\alpha \rightarrow \mathbb{B}_\alpha} \leq 1$.

Problem 4.25 Show that, for each $\alpha > \beta$, $\alpha - \beta < 2$, setting $h_0 = \Xi^{-1}L_\omega^{-1}\Xi f$, holds

$$\begin{aligned}
\|h_0\|_\beta &\leq \frac{4\|f\|_\alpha}{C_0(\alpha-\beta)^2} \\
\|h'_0\|_\beta &\leq \frac{64\pi}{C_0(\alpha-\beta)^3} \|f\|_\alpha.
\end{aligned}$$

⁸In other work we simply take the restriction of the function to the real axis.

The point of introducing the spaces \mathbb{B}_α is that the equation for f_1 reads

$$f_1(x) = h_0(x) - h_0(x + \omega + f_1(x)) + f(x + h_0(x)). \quad (4.6.5)$$

To study such equation in \mathcal{B}_α is highly non trivial, while \mathbb{B}_α is much better suited to estimate the norms of composition of functions.

To study (4.6.5) in \mathbb{B}_α the first step is to verify that it makes sense. Obviously one can see it as the restriction to the real axes of an equation involving functions defined on the complex plane, yet it is necessary to check that the composition is well defined, that is we have to carefully analyze domains and ranges of the various functions. For later use we carry out the needed estimates in the following Lemma.

Lemma 4.6.4 *Given functions $f \in \mathbb{B}_\alpha$ and $h \in \mathbb{B}_\beta$, $\alpha > \beta > \alpha/2$ such that $\|f\|_\alpha \leq \frac{\alpha-\beta}{2\pi}$, h satisfies (4.6.2) with $\|h\|_\beta \leq \frac{\alpha-\beta}{8\pi}$, $\|h'\|_\beta \leq \frac{1}{2}$, there exists a function $f_1 \in \mathbb{B}_\gamma$ such that, setting $F(z) = z + \omega + f(z)$, $H(z) = z + h(z)$, H is invertible, $H^{-1} \in \mathbb{B}_\gamma$, $\gamma = 2\beta - \alpha$, $\|f_1\|_\gamma \leq \frac{1}{2}\|f\|_\alpha$ and*

$$H^{-1} \circ F \circ H(z) = z + \omega + f_1(z).$$

PROOF. First of all H is invertible when restricted to the real axis since $H'_0 \geq \frac{1}{2}$. Let $H_0^{-1}(z) = z + \psi(z)$, clearly

$$\psi(z) = -h(z + \psi(z)).$$

So the inverse is the fixed point of the operator $K(\psi)(z) = -h(z + \psi(z))$ which is well defined on the set $A = \{\psi \in \mathbb{B}_\gamma : \|\psi\|_\gamma \leq \frac{\alpha-\beta}{2\pi}\}$. It is easy to verify that such a fixed point exists and is unique.

Note that the function f_1 must satisfy equation (4.6.5). To solve (4.6.5) we must look for a fixed point for the operator $K(\varphi)(z) = h(z) - h(z + \omega + \varphi(z)) + f(z + h(z))$ on the set $A = \{\varphi \in \mathbb{B}_\gamma : \|\varphi\|_\gamma \leq \frac{1}{2}\|f\|_\alpha\}$.

Let us check that $K(A) \subset A$.

$$\begin{aligned} K(\varphi)(z) &= h(z) - h(z + \omega) + h(z + \omega) - h(z + \omega + \varphi(z)) + f(z + h(z)) \\ &= f(z + h(z)) - f(z) + h(z + \omega) - h(z + \omega + \varphi(z)) \end{aligned}$$

Thus, using the estimate in Problem 4.34,

$$\|K(\varphi)\|_\gamma \leq \|f'\|_\beta \|h\|_\gamma + \|h'\|_\beta \|\varphi\|_\gamma \leq \frac{1}{4}\|f\|_\alpha + \frac{1}{2}\|\varphi\|_\gamma \leq \frac{1}{2}\|f\|_\alpha.$$

Thus, by the usual contraction argument, exists $f_1 \in A$ such that $K(f_1) = f_1$. \square

Since we need to restrict the domain several time it is convenient to do it in a systematic fashion. Let $\rho_n := e^{-n\tau_0}\alpha$, and apply Lemma 4.6.4 with $\beta = \rho_2$ and $\gamma = \rho_3$. Setting $\varepsilon_0 = \|f\|_\alpha$ the Lemma applies provided $\varepsilon_0 \leq \min\{\frac{\tau_0\alpha}{\pi e}, \frac{C_0\tau_0^3\alpha^3}{16e\pi}\}$.⁹ We then choose $\tau_0 = \alpha^{-1}e[16\pi C_0^{-1}\varepsilon]^{1/3} =: C_1\alpha^{-1}\varepsilon^{1/3}$, hence $\min\{\frac{\tau_0\alpha}{\pi e}, \frac{C_0\tau_0^3\alpha^3}{16e\pi}\} = \frac{C_0\tau_0^3\alpha^3}{16e\pi}$ provided $\varepsilon \leq \frac{4}{e^3\pi\sqrt{C_0}}$.

We can thus implement an iterative procedure by setting: $f_0 = f$,
 $h_n(z + \omega) - h_n(z) = f_n(z)$, $H_n(z) = z + h_n(z)$, $F_n(z) = z + f_n(z)$,
 $H_n^{-1} \circ F_n \circ H_n(z) = z + \omega + f_{n+1}(z)$.

In addition, we set $\alpha_0 = \alpha$, $\alpha_{n+1} = e^{-3\tau_n}\alpha_n$, $\varepsilon_{n+1} = \frac{\varepsilon_n}{2}$ and $\tau_n = C_1\alpha_n^{-1}\varepsilon_n^{1/3}$. Note that this choices imply that Lemma 4.6.4 can be applied to each stage of the iteration. Now, if $\alpha_n \geq \frac{1}{2}\alpha_0$, holds $\varepsilon_n = 2^{-n}\varepsilon$, $\tau_n \leq 2C_1\alpha_0^{-1}2^{-n/3}\varepsilon^{1/3}$. This implies $\alpha_n = \alpha_0 e^{-3\sum_{k=0}^{n-1}\tau_k} \geq e^{-30C_1\alpha_0^{-1}\varepsilon^{1/3}}\alpha_0$ which is larger than $\alpha_0/2$ provided $\varepsilon \leq \left[\frac{\alpha_0 \ln 2}{30C_1}\right]^3$ which is implied by $\varepsilon \leq 5C_0\alpha_0^3 10^{-10}$.

We have thus a sequence of changes of variables $H_n(z) = z + h_n(z)$, the next question is if it exists $H(z) = \lim_{n \rightarrow \infty} H_0 \circ H_1 \circ \dots \circ H_n(z)$. It suffices to prove that the sequence is uniformly bounded on $D_{\alpha_0/2}$

$$\begin{aligned} |H_0 \circ H_1 \circ \dots \circ H_n(z) - z| &\leq \sum_{k=0}^n \|h_k\|_{\alpha_k} \leq \sum_{k=0}^n \frac{\tau_k}{4\pi} \alpha_k \\ &\leq \sum_{k=0}^{\infty} 2^{k/3} \varepsilon^{1/3} \frac{C_1}{4\pi} \leq \varepsilon^{1/3} \frac{5C_1}{4\pi} \leq 4C_0^{-1/3} \|f\|_{\alpha_0}^{1/3} \end{aligned}$$

Similarly it follows that the H_n form a Chauchy sequence, hence they have a limit $H \in \mathbb{B}_{\alpha_0/2}$ with $\|\text{id} - H\|_{\alpha_0/2} \leq 4C_0^{-1/3} \|f\|_{\alpha_0}^{1/3}$. From this it follows also (see Problem 4.34)

$$\|1 - H'\|_{\alpha_0/4} \leq \frac{32\pi \|f\|_{\alpha_0}^{1/3}}{C_0^{1/3} \alpha_0} \leq \frac{1}{2}, \quad (4.6.6)$$

⁹Just use Problem 4.25 and the fact that $1 - e^{-x} = \int_0^x e^{-y} dy \geq e^{-1}x$, for $x \in (0, 1)$, to check the hypotheses of the Lemma.

Hence H is invertible. \square

4.6.2 Smooth KAM theory

The final question is: do similar results hold assuming less smoothness? The answer is yes, yet to explore the optimal results it is not an easy task. Here we content ourselves with a partial result.

Theorem 4.6.5 *For each $r > 3$, if $\|f\|_{C^r} \leq 10^{-18}C_0e^{-(r-3)^{-1}}$ and $\omega > 0$ satisfies*

$$\left| \omega - \frac{p}{q} \right| \geq \frac{C_0}{q^2}$$

for each $p, q \in \mathbb{N}$, then there exists $\mathfrak{h} \in C^1$ such that, setting $\mathcal{H}(x) = x + \mathfrak{h}(x)$, \mathcal{H} is invertible and

$$\mathcal{H}^{-1} \circ F \circ \mathcal{H}(x) = x + \omega.$$

PROOF. The basic idea is to write $f = \sum_{m=0}^{\infty} \tilde{f}_m$ where $\tilde{f}_m(x) = \sum_{e^{am} \leq |k| < e^{a(m+1)}} f_k e^{2\pi i k x}$ where $a > 1$ is a parameter to be chosen later.¹⁰ Then one can apply Theorem 4.6.1 to each \tilde{f}_m . Indeed, let $\alpha_m = (m+1)e^{-a(m+1)}$ where then

$$\begin{aligned} \|\tilde{f}_m\|_{\alpha_m} &\leq \sum_{e^{am} \leq |k| < e^{a(m+1)}} |f_k| e^{\alpha_m |k|} \leq \sum_{e^{am} \leq |k| < e^{a(m+1)}} |f|_{C^r} (2\pi)^{-r} |k|^{-r} e^{\alpha_m |k|} \\ &\leq \sum_{e^{am} \leq |k| < e^{a(m+1)}} |f|_{C^r} (2\pi)^{-r} e^{-arm} e^{m+1} \\ &\leq |f|_{C^r} e^{-(ar-2)m}. \end{aligned}$$

If $|f|_{C^r}$ is small enough, we can apply Theorem 4.6.1 to \tilde{f}_0 . Hence there exists \tilde{h}_0 such that, setting $\tilde{H}_0(z) = z + \tilde{h}_0(z)$ and $\tilde{F}_0(z) = z + \tilde{f}_0(z)$,

$$\tilde{H}_0^{-1} \circ \tilde{F}_0 \circ \tilde{H}_0(z) = z + \omega =: R_\omega(z).$$

The obvious next step is to compute $\tilde{f}_{1,k}$ such that, for each $n \in \mathbb{N}$,

$$\tilde{H}_0^{-1} \circ \left(R_\omega + \sum_{k=0}^n \tilde{f}_k \right) \circ \tilde{H}_0(z) = z + \omega + \sum_{k=1}^n \tilde{f}_{1,k}(z).$$

¹⁰This choice it is not optimal, yet it makes the latter computations simpler.

This can be done if $|f|_{\mathcal{C}^r}$ is small enough. If the latter series is convergent in $\mathcal{C}^0(\mathbb{R}, \mathbb{R})$ we obtain

$$\tilde{H}_0^{-1} \circ F \circ \tilde{H}_0(x) = x + \omega + \sum_{k=1}^{\infty} \tilde{f}_{1,k}(x).$$

We can then try to iterate the above procedure by applying Theorem 4.6.1 to $f_{1,1}$ and so on. Setting $\tilde{f}_{0,k} = \tilde{f}_k$, we have then the following iterative scheme: for all $k, m \in \mathbb{N}$

$$\begin{aligned} \tilde{H}_n^{-1} \circ \left(R_\omega + \tilde{f}_{n,n} \right) \circ \tilde{H}_n &= R_\omega \\ \tilde{H}_n^{-1} \circ \left(R_\omega + \sum_{k=n}^m \tilde{f}_{n,k} \right) \circ \tilde{H}_n &= R_\omega + \sum_{k=n+1}^m \tilde{f}_{n+1,k} \end{aligned} \quad (4.6.7)$$

It follows that the $\tilde{f}_{n+1,k}$, $k \geq n+1$, satisfy the equations

$$\tilde{f}_{n+1,k}(z) = \tilde{f}_{n,k}(z + \tilde{h}_n(z)) + \tilde{h}_n(z + \omega) - \tilde{h}_n(z + \omega + \tilde{f}_{n,k}(z)). \quad (4.6.8)$$

While, setting $\mathcal{H}_n(z) := \tilde{H}_0 \circ \tilde{H}_1 \circ \cdots \circ \tilde{H}_{n-1}(z) =: z + \mathfrak{h}_n(z)$, $\mathcal{H}_0(z) = z$,

$$\mathcal{H}_n^{-1} \circ \left(R_\omega + \sum_{k=0}^m \tilde{f}_k \right) \circ \mathcal{H}_n = R_\omega + \sum_{k=n}^m \tilde{f}_{n,k}. \quad (4.6.9)$$

Next, let us assume by induction that there exists $A > 1$ such that, for each $n \in \mathbb{N}$ and $k \geq n$, $\|\tilde{f}_{n,k}\|_{\alpha_k} \leq A\|\tilde{f}_k\|_{\alpha_k}$. Then Theorem 4.6.1 implies that the solution to the first of the (4.6.7) exists provided $\|\tilde{f}_{n,n}\|_{\alpha_n} \leq C^*\alpha_n^3$, with $C^* = 3C_0 10^{-13}$. This is implied by $\|\tilde{f}_n\|_{\alpha_n} \leq C^*A^{-1}\alpha_n^3$ which, in turns is implied by

$$|f|_{\mathcal{C}^r} e^{-(ar-2)n} \leq C^*A^{-1}e^{-3a(n+1)}.$$

This is satisfied provided $|f|_{\mathcal{C}^r} \leq C^*A^{-1}e^{-3a}$ and $r > 3 + 2a^{-1}$. By Theorem 4.6.1 we have then

$$\|\tilde{h}_n\|_{\alpha_n/2} \leq 3C_0^{-\frac{1}{3}} \|\tilde{f}_{n,n}\|_{\alpha_n}^{\frac{1}{3}} \leq 3C_0^{-\frac{1}{3}} A^{\frac{1}{3}} e^{-\frac{1}{3}(ar-2)n} |f|_{\mathcal{C}^r}^{\frac{1}{3}} \leq \frac{\alpha_n}{4}.$$

Note that if $a > \ln 8$, then $\alpha_n/4 > \alpha_{n+1}$. In addition, note (see Problem 4.34) that

$$\begin{aligned} \|\tilde{h}'_n\|_{\frac{3}{8}\alpha_n} &\leq 48\pi(n+1)e^{-a(n+1)}C_0^{-\frac{1}{3}}A^{\frac{1}{3}}e^{-\frac{1}{3}(ar-2)n}|f|_{\mathcal{C}^r}^{\frac{1}{3}} \\ &=: C_1^*A^{\frac{1}{3}}e^{-\frac{a}{3}(r-3-2a^{-1})n}|f|_{\mathcal{C}^r}^{\frac{1}{3}} \leq \frac{1}{2} \end{aligned}$$

provided $|f|_{\mathcal{C}^r} \leq \frac{1}{8}A^{-1}(C_1^*)^{-3}$. To solve equation (4.6.8) it is then necessary that $\|\tilde{f}_{n,k}\|_{\alpha_k} \leq \frac{1}{8}\alpha_n$ for all $k \geq n$ which is implied by $|f|_{\mathcal{C}^r} \leq [8eA]^{-1}$. It follows (by the usual application of the fixed point theorem)

$$\begin{aligned} \|\tilde{f}_{n+1,k}\|_{\alpha_k} &\leq \left[1 - C_*A^{\frac{1}{3}}e^{\frac{a}{3}(3+2a^{-1}-r)k}|f|_{\mathcal{C}^r}^{\frac{1}{3}}\right]^{-1} \|\tilde{f}_{n,k}\|_{\alpha_k} \\ &\leq e^{2C_*A^{\frac{1}{3}}e^{\frac{a}{3}(3+2a^{-1}-r)k}|f|_{\mathcal{C}^r}^{\frac{1}{3}}} \|\tilde{f}_{n,k}\|_{\alpha_k} \\ &\leq e^{2C_*A^{\frac{1}{3}}[1-e^{\frac{a}{3}(3+2a^{-1}-r)}]^{-1}|f|_{\mathcal{C}^r}^{\frac{1}{3}}} \|\tilde{f}_k\|_{\alpha_k} \leq e^{4C_*A^{\frac{1}{3}}|f|_{\mathcal{C}^r}^{\frac{1}{3}}} \|\tilde{f}_k\|_{\alpha_k} \end{aligned}$$

provided $a > (r - 3)^{-1}$. Next, note that if $|f|_{\mathcal{C}^r} \leq (8C_1^*)^{-3}$, then there exists $A \leq 8$ such that $e^{4C_*A^{\frac{1}{3}}|f|_{\mathcal{C}^r}^{\frac{1}{3}}} \leq A$, hence proving the induction hypotheses. To conclude we must prove that the change of coordinate \mathcal{H}_n is convergent. Note that

$$|\mathcal{H}'_n(x)| \leq \prod_{k=0}^n \|\tilde{H}'_k\|_{\frac{3}{8}\alpha_k} \leq \prod_{k=0}^n e^{C_1^*A^{\frac{1}{3}}e^{-\frac{a}{3}(r-3-2a^{-1})k}|f|_{\mathcal{C}^r}^{\frac{1}{3}}}$$

It is then easy to see that the \mathcal{H}_n form a Cauchy sequence in \mathcal{C}^1 . The theorem follows by collecting all the above inequalities and setting $a = 3 + (r - 3)^{-1}$. \square

Problems

4.26. If M is a \mathcal{C}^r manifold, $f \in \mathcal{C}^r(M, M)$ is a diffeomorphism and $\tau \in \mathcal{C}^r(M, (0, \infty))$, show that the associated suspension flow is defined on a \mathcal{C}^r manifold and is \mathcal{C}^r .

4.27. Consider the Dynamical System $([0, 1], T)$ where

$$T(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor = \frac{1}{x} \pmod{1}$$

($\lfloor a \rfloor$ is the integer part of a). This is called the *Gauss map*. Prove that for each $x \in \mathbb{Q} \cap [0, 1]$ holds $\lim_{n \rightarrow \infty} T^n(x) = 0$.

4.28. Prove that any infinite continuous fraction of the form

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$$

with $a_i \in \mathbb{N}$ defines a real number.

4.29. Prove that, for each $a \in \mathbb{N}$,

$$x = \frac{1}{a + \frac{1}{a + \frac{1}{a + \dots}}} = \frac{-a + \sqrt{a^2 + 4}}{2}.$$

4.30. Prove that, for all $s > 2$, for Lebesgue almost all numbers $x \in [0, 1]$ there exists $C > 0$ such that¹¹

$$\left| x - \frac{p}{q} \right| \geq \frac{C}{q^s}$$

for all $p, q \in \mathbb{N}$.

4.31. Let $f_a(x) = \frac{1}{a+x}$. Given a sequence $[a_0, a_1, \dots, a_n]$ show that

$$f_{a_0} \circ \dots \circ f_{a_n}(x) = \frac{1}{a_0 + \frac{1}{a_1 + \frac{1}{\dots \frac{1}{a_n + x}}}} = \frac{p_n + p_{n-1}x}{q_n + q_{n-1}x},$$

where $p_{n+1} = a_{n+1}p_n + p_{n-1}$ and $q_{n+1} = a_{n+1}q_n + q_{n-1}$, $p_{-1} = 0$, $q_{-1} = 1$, $p_0 = 1$, $q_0 = a_0$. In addition, show that, for all $n \in \mathbb{N}$, $p_n q_{n-1} - q_n p_{n-1} = (-1)^n$ and deduce that p_n, q_n have no common divisor different from one. Finally, verify that

$$f_{a_0} \circ \dots \circ f_{a_n}(x) - f_{a_0} \circ \dots \circ f_{a_{n+1}}(x) = \frac{(-1)^{n+1}[x^2 + a_{n+1}x - 1]}{(q_n + q_{n-1}x)(q_{n+1} + q_n x)}.$$

¹¹The composition below is often called *iterated function system*, it can be naturally viewed as a time dependent dynamical system.

4.32. Let $\omega \in [0, 1)$. Show that there exists infinitely many $p, q \in \mathbb{N}$ such that

$$\left| \omega - \frac{p}{q} \right| \leq \frac{1}{q^2}.$$

4.33. Let $\omega \in [0, 1)$ have the continuous fraction expansion given by $[a_0, a_1 \dots]$. Suppose that $\inf_n a_n > 0$ and $\sup_n a_n < \infty$.¹² Show that there exists a constant $c > 0$ such that for all $p, q \in \mathbb{N}$

$$\left| \omega - \frac{p}{q} \right| \geq \frac{c}{q^2}.$$

4.34. For each $\varphi \in \mathbb{B}_\alpha$ and $\beta < \alpha$ show that $\|\varphi'\|_\beta \leq \frac{2\pi\|\varphi\|_\alpha}{\alpha-\beta}$.

4.35. Let us consider an holomorphic function $f : U \subset \mathcal{C} \rightarrow \mathcal{C}$ where U is an open set containing zero. Assume that $f(0) = 0, f'(0) = e^{2\pi i \omega}$. Prove that, if ω is Diophantine, then it is possible to find an open set $D \subset U$ on which f is conjugated to the map $f_\omega(z) = e^{2\pi i \omega} z$.

Hints to solving the Problems

4.2 Consider a system $([0, 1], T)$ such that T is piecewise linear, it has an unstable fixed point at x_0 and an attracting fixed point at $z \in (0, x_0)$ so that the set $[z, x_0]$ is forward invariant. Finally arrange so that $T(0) = x_0$ and $T(x) \leq x_0$ for x near zero.

4.10 The equation $\dot{x} = \omega = (\omega_1, \omega_2)$ on \mathbb{T}^2 has the solution $x(t) = (x_1(t), x_2(t)) = x_0 + \omega t \pmod{1}$. If one looks at the flow only at the times $\tau_n = n\omega_1^{-1}$, then $x(n\tau) = x_0 + (0, \alpha n) \pmod{1}$ where $\alpha := \frac{\omega_2}{\omega_1}$. One can then consider the circle map $f : S^1 \rightarrow S^1$ defined by $f(z) = z + \alpha \pmod{1}$. Clearly, if the orbits of such a map are dense in S^1 the original flow will be dense in \mathbb{T}^2 . The density follows in the case $\alpha \notin \mathbb{Q}$. In fact this implies that f has no periodic orbits. Then $\{f^n(0)\}$ is made of distinct points and contains a converging subsequence (by compactness) hence for each $\varepsilon > 0$ exists $\bar{n} \in \mathbb{N}$ such that $|z - f^{\bar{n}}(z)| \leq \varepsilon$, that is $f^{\bar{n}}$

¹²Such numbers ω are called of *constant type*.

is a rotation by less than ε . Hence the orbit $\{f^{k\bar{n}}(z)\}$ enters in the ε -neighborhood of each point of S^1 .

4.13 Suppose that there exists $\varphi(r, s)$, $\varphi \in \mathcal{C}^0([0, 1] \times \mathbb{T}^1, \mathbb{R})$, such that $\varphi(1, \cdot)$ is a parametrization of Γ and $\varphi(0, s) = y$ for some fixed $y \in \mathbb{T}^2$ (i.e. Γ is homotopic to y).

4.12 First of all notice that if $\xi(t)$ is the derivative with respect to the initial condition and $\xi(0) = \lambda V(x(0))$, for some λ , then $\xi(t) = \lambda V(x(t))$ for all t . Define then $\omega(x, y) = x_1y_2 - x_2y_1$ and verify that $x, y \neq 0$ and $\omega(x, y) = 0$ imply that there exists $\lambda \in \mathbb{R}$ such that $x = \lambda y$.¹³ This means that $\omega(\xi(t), V(x(t)))$ cannot change sign. Hence the result.

4.17 Let $\liminf_{n \rightarrow \infty} \frac{a_n}{n} = a > -\infty$, then for each $\varepsilon, m > 0$ exists $\bar{n} \in \mathbb{N}$, $\bar{n} > m$, such that $|a_{\bar{n}} - a\bar{n}| \leq \varepsilon\bar{n}$. Let $l \in \mathbb{N}$, $l > \bar{n}$, and write $l = k\bar{n} + r$, $r < \bar{n}$, then

$$\begin{aligned} a - \varepsilon &\leq \frac{a_l}{l} \leq \frac{ka_{\bar{n}} + kL + a_r}{l} \leq \frac{k\bar{n}(a + \varepsilon) + kL + a_r}{l} \\ &= a + \varepsilon + \frac{L}{m} + \frac{a_r}{l}. \end{aligned}$$

From which the claim follows.

4.18 Setting $I = [a, b]$ note that $g(x) = f(x) - x$ has a zero in I .

4.19 This is the same than saying $\bigcup_{k \in \mathbb{N}} f^{-k}[x, f^n(x)] = S^1$. If not consider $f^{-kn}[x, f^n(x)]$, these are contiguous intervals. If they do not cover all S^1 , then their length must go to zero and $f^{-kn}x$ must have an accumulation point, call it z . Then

$$z = \lim_{k \rightarrow \infty} f^{-kn}(x) = \lim_{k \rightarrow \infty} f^{-kn}(f^n(x)) = f^n(z).$$

Hence f must have a fixed point contradicting $\tau(f) \notin \mathbb{Q}$.

¹³By the way, ω is a symplectic form and its existence implies that the manifold is orientable.

- 4.27** If $x = \frac{p_0}{q_0}$, $p_0 \leq q_0$, then $q_0 = k_1 p_0 + p_1$, with $p_1 < p_0$, and $T(x) = \frac{p_1}{p_0}$. Let $q_1 = p_0$ and go on noticing that $p_{i+1} < p_i$.¹⁴
- 4.28** Note that if you fix the first n $\{a_i\}$, this corresponds to specifying which elements of the partition $\{[\frac{1}{i+1}, \frac{1}{i}]\}$ are visited by the trajectory of $\{T^i x\}$, T being the Gauss map. By the expansivity of the map readily follows that x must belong to an interval of size λ^{-n} for some $\lambda > 1$.
- 4.29** Note that $T(x) = x$, where T is the Gauss map. Study periodic continuous fractions of period two.
- 4.30** To see it consider the sets $I_{p,q} := [\frac{p}{q} - Cq^{-s}, \frac{p}{q} + Cq^{-s}]$. If $p \leq q$, then $I_{p,q} \subset [0, 1]$. Clearly if $\alpha \notin I_{p,p}$ for all $q \geq p \in \mathbb{N}$, then α satisfies the Diophantine condition. But $\sum_{q \geq p} |I_{p,q}| \leq C \sum_{q=1}^{\infty} q^{-s+1}$ which converges provided $s > 2$ and can be made arbitrarily small by choosing C small. Accordingly, almost all numbers are Diophantine for any $s > 2$.
- 4.31** By induction.
- 4.32** The result is trivial for rational numbers. By Problem 4.28, $\omega = \lim_{n \rightarrow \infty} f_{a_0} \circ \dots \circ f_{a_n}(0)$. Moreover, $f_a([0, \infty)) \subset [0, a^{-1}]$. Thus for each $n \in \mathbb{N}$ there exists $x_n \in [0, a_{n+1}^{-1}]$ such that $\omega = f_{a_0} \circ \dots \circ f_{a_n}(x_n)$. Thus, by the monotonicity of the f_a it follows that either $\omega \in [f_{a_0} \circ \dots \circ f_{a_n}(0), f_{a_0} \circ \dots \circ f_{a_{n+1}}(0)]$ or $\omega \in [f_{a_0} \circ \dots \circ f_{a_{n+1}}(0), f_{a_0} \circ \dots \circ f_{a_n}(0)]$. One can then use the equalities of Problem 4.31 to conclude all the rationals $f_{a_0} \circ \dots \circ f_{a_n}(0)$ satisfy

$$|\omega - f_{a_0} \circ \dots \circ f_{a_n}(0)| \leq \frac{1}{a_{n+1} q_n^2}.$$

¹⁴This is nothing else than the *Euclidean algorithm* to find the greatest common divisor of two integers [Euc78, Elements, Book VII, Proposition 1 and 2]. The greatest common divisor is clearly the last non-zero p_i . This provides also a remarkable way of writing rational numbers: *continued fractions*

$$\frac{p_0}{q_0} = \frac{1}{k_1 + \frac{1}{k_2 + \dots + \frac{1}{k_n}}}$$

You did not like this argument? Here is an interesting alternative. Problem 4.31 implies that

$$f_{a_0} \circ \cdots \circ f_{a_n}(0) = \sum_{k=0}^n \frac{(-1)^k}{q_k q_{k-1}}.$$

Since the odd and even partial sum of an alternating series form monotone sequences that converge to the limit from opposite sides, it follows that

$$\begin{aligned} |\omega - f_{a_0} \circ \cdots \circ f_{a_n}(0)| &\leq |f_{a_0} \circ \cdots \circ f_{a_n}(0) - f_{a_0} \circ \cdots \circ f_{a_{n+1}}(0)| \\ &\leq \frac{1}{a_{n+1} q_n^2}. \end{aligned}$$

4.33 As we have argued at the end of the hint of Problem 4.33, $\omega \in [f_{a_0} \circ \cdots \circ f_{a_n}(0), f_{a_0} \circ \cdots \circ f_{a_{n+1}}(0)] =: I_n$. Note that if $q < q_n$ then

$$\left| \frac{p}{q} - \frac{p_n}{q_n} \right| \geq \frac{1}{q_n q}; \quad \left| \frac{p}{q} - \frac{p_n}{q_n} \right| \geq \frac{1}{q_{n+1} q}.$$

But $|I_n| = \frac{1}{q_n q_{n+1}}$ so it cannot contain any rational number with denominator strictly less than q_n . Accordingly, $\frac{p}{q} \notin I_n$ and thus $|\omega - \frac{p}{q}| \geq \frac{1}{q_{n+1} q} > \frac{1}{q_{n+1} q_n}$. In other words the fraction determined by $[a_0, \dots, a_n]$ are the best approximation of ω among all the numbers with denominator smaller than q_n . Since,

$$\begin{aligned} |\omega - f_{a_0} \circ \cdots \circ f_{a_n}(0)| &\geq |f_{a_0} \circ \cdots \circ f_{a_n}(0) - f_{a_0} \circ \cdots \circ f_{a_{n+2}}(0)| \\ &\geq \frac{1}{(a_{n+1} + 2) q_n^2}. \end{aligned}$$

the result follows by simple computations.

4.34 Since φ is holomorphic by Riemann formula we have

$$\varphi'(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{\varphi(\zeta)}{(z - \zeta)^2} d\zeta$$

where γ is a simple closed curve in D_α surrounding $z \in D_\beta$. For γ we chose the curve $\{z + \frac{\alpha - \beta}{2\pi} e^{i\theta}\}_{\theta \in [0, 2\pi]}$. Hence

$$\|\varphi'\|_\beta \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{2\pi |\varphi|_\alpha}{\alpha - \beta} d\theta = \frac{2\pi |\varphi|_\alpha}{\alpha - \beta}.$$

4.35 Mimic Theorem 4.6.1.

Notes

Lemma 4.3.2 is due to Siegel [Sie45], see [NZ99] for a detailed treatment of flows on surfaces. A detailed treatment of circle rotations can be found in [Her83, Her86]. A general treatment of KAM theory for Hamiltonian Systems, with an emphasis on concrete applications, can be found in [CC95].