Appendix C

Perturbation Theory (a super-fast introduction)

The following is really super condensate (although self-consistent). If you want more details see [RS80, Kat66] in which you probably can find more than you are looking for.

C.1 Bounded operators

In the following we will consider only *separable* Banach spaces, i.e. Banach spaces that have a countable dense set.[1]

Given a Banch space $\mathcal{B}$ we can consider the set $L(\mathcal{B}, \mathcal{B})$ of the linear bounded operators from $\mathcal{B}$ to itself. We can then introduce the norm $\|B\| = \sup_{\|v\| \leq 1} \|Bv\|$.

Problem C.1 *Show that $(L(\mathcal{B}, \mathcal{B}), \|\cdot\|)$ is a Banach space. That is that $\|\cdot\|$ is really a norm and that the space is complete with respect to such a norm.*

Problem C.2 *Show that the $n \times n$ matrices form a Banach Algebra.*[2]

[1] Recall that a Banach space is a complete normed vector space (in the following we will consider vector spaces on the field of complex numbers), that is a normed vector space in which all the Cauchy sequences have a limit in the space. Again, if you are uncomfortable with Banach spaces, in the following read $\mathbb{R}^d$ instead of $\mathcal{B}$ and matrices instead of operators, but be aware that we have to develop the theory without the use of the determinant that, in general, is not defined for operators on Banach spaces.

[2] A Banach Algebra $\mathcal{A}$ is a Banach space where it is defined the multiplications between element with the usual properties of an algebra and, in addition, for each $a, b \in \mathcal{A}$ holds $\|ab\| \leq \|a\| \cdot \|b\|$.

Problem C.3 *Show that $L(\mathcal{B},\mathcal{B})$ form a Banach algebra.*[3]

To each $A \in L(\mathcal{B},\mathcal{B})$ are associated two important subspaces: the range $R(A) = \{v \in \mathcal{B} \;:\; \exists\, w \in \mathcal{B}$ such that $v = Aw\}$ and the kernel $N(A) = \{v \in \mathcal{B} \;:\; Av = 0\}$.

Problem C.4 *Prove, for each $A \in L(\mathcal{B},\mathcal{B})$, that $N(A)$ is a closed linear subspaces of $\mathcal{B}$. Show that this is not necessarily the case for $R(A)$ if $\mathcal{B}$ is not finite dimensional.*

An very special, but very important, class of operators are the projectors.

Definition C.1.1 *An operator $\Pi \in L(\mathcal{B},\mathcal{B})$ is called a* projector *iff $\Pi^2 = \Pi$.*

Note that if Π is a projector, so is $\mathbb{1} - \Pi$. We have the following interesting fact.

Lemma C.1.2 *If $\Pi \in L(\mathcal{B},\mathcal{B})$ is a projector, then $N(\Pi) \oplus R(\Pi) = \mathcal{B}$.*

PROOF. If $v \in \mathcal{B}$, then $v = \Pi v + (\mathbb{1} - \Pi)v$. Notice that $R(\mathbb{1} - \Pi) = N(\Pi)$ and $R(\Pi) = N(\mathbb{1} - \Pi)$. Finally, if $v \in N(\Pi) \cap R(\Pi)$, then $v = 0$, which concludes the proof. $\square$

Another, more general, very important class of operators are the compact ones.

Definition C.1.3 *An operator $K \in L(\mathcal{B},\mathcal{B})$ is called* compact *iff for any bounded set B the closure of $K(B)$ is compact.*

Remark C.1.4 *Note that not all the linear operator on a Banach space are bounded. For example consider the derivative acting on $\mathcal{C}^1((0,1),\mathbb{R})$.*

C.2 Functional calculus

First of all recall that all the Riemannian theory of integration works verbatim for function $f \in \mathcal{C}^0(\mathbb{R},\mathcal{B})$, where $\mathcal{B}$ is a Banach space. We

[3] The multiplication is given by the composition.

can thus talk of integrals of the type $\int_a^b f(t)dt$,[4] Next, we can talk of *analytic functions* for functions in $\mathcal{C}^0(\mathbb{C}, \mathcal{B})$: a function is analytic in an open region $U \subset \mathbb{C}$ iff at each point $z_0 \in U$ there exists a neighborhood $B \ni z_0$ and elements $\{a_n\} \subset \mathcal{B}$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \forall z \in B. \tag{C.2.1}$$

Problem C.5 *Show that if $f \in \mathcal{C}^0(\mathbb{C}, \mathcal{B})$ is analytic in $U \subset \mathbb{C}$, then given any smooth closed curve γ, contained in a sufficiently small disk in U, holds*[5]

$$\int_\gamma f(z)dz = 0 \tag{C.2.2}$$

Then show that the same hold for any piecewise smooth closed curve with interior contained in U, provided U is simply connected.

Problem C.6 *Show that if $f \in \mathcal{C}^0(\mathbb{C}, \mathcal{B})$ is analytic in a simply connected $U \subset \mathbb{C}$, then given any smooth closed curve γ, with interior contained contained in U and having in its interior a point z, hods the formula*

$$f(z) = \frac{1}{2\pi i} \int_\gamma (\xi - z)^{-1} f(\xi) d\xi. \tag{C.2.3}$$

Problem C.7 *Show that if $f \in \mathcal{C}^0(\mathbb{C}, \mathcal{B})$ satisfies (C.2.3) for each smooth closed curve in a simply connected open set U, then f is analytic in U.*

C.3 Spectrum and resolvent

Given $A \in L(\mathcal{B}, \mathcal{B})$ we define the *resolvent*, called $\rho(A)$, as the set of the $z \in \mathbb{C}$ such that $(z\mathbb{1} - A)$ is invertible and the inverse belongs to $L(\mathcal{B}, \mathcal{B})$. The *spectrum* of A, called $\sigma(A)$ is the complement of $\rho(A)$ in $\mathbb{C}$.

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[4] This is special case of the so called Bochner integral [Yos95].

[5] Of course, by $\int_\gamma f(z)dz$ we mean that we have to consider any smooth parametrization $g : [a, b] \to \mathbb{C}$ of γ, $g(a) = g(b)$, and then $\int_\gamma f(z)dz := \int_a^b f \circ g(t) g'(t) dt$. Show that the definition does not depend on the parametrization and that one can use piecewise smooth parametrizations as well.

Problem C.8 *Prove that, for each Banach space $\mathcal{B}$ and operator $A \in L(\mathcal{B}, \mathcal{B})$, if $z \in \rho(A)$, then there exists a neighborhood U of z such that $(z\mathbb{1} - A)^{-1}$ is analytic in U.*

From the above exercise follows that $\rho(A)$ is open, hence $\sigma(A)$ is closed.

Problem C.9 *Show that, for each $A \in L(\mathcal{B}, \mathcal{B})$, $\sigma(A) \neq \emptyset$.*

Problem C.10 *Show that if $\Pi \in L(\mathcal{B}, \mathcal{B})$ is a projector, then $\sigma(\Pi) = \{0, 1\}$.*

Up to now the theory for operators seems very similar to the one for matrices. Yet, the spectrum for matrices is always given by a finite number of points while the situation for operators can be very dfferenct.

Problem C.11 *Consider the operator $\mathcal{L} : \mathcal{C}^0([0, 1], \mathbb{C}) \to \mathcal{C}^0([0, 1], \mathbb{C})$ defined by*

$$(\mathcal{L}f)(x) = \frac{1}{2}f(x/2) + \frac{1}{2}f(x/2 + 1/2).$$

Show that $\sigma(\mathcal{L}) = \{z \in \mathbb{C} \, : \, |z| \leq 1\}$.

Problem C.12 *Show that, if $A \in L(\mathcal{B}, \mathcal{B})$ and p is any polynomial, then for each $n \in \mathbb{N}$ and smooth curve $\gamma \subset \mathbb{C}$, with $\sigma(A)$ in its interior,*

$$p(A) = \frac{1}{2\pi i} \int_\gamma p(z)(z\mathbb{1} - A)^{-1} dz.$$

Problem C.13 *Show that, for each $A \in L(\mathcal{B}, \mathcal{B})$ the limit*

$$r(A) = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}$$

exists.

The above limit is called the *spectral radius* of A.

Lemma C.3.1 *For each $A \in L(\mathcal{B}, \mathcal{B})$ holds true $\sup_{z\in\sigma(A)} |z| = r(A)$.*

PROOF. Since we can write

$$(z\mathbb{1} - A)^{-1} = z^{-1}(\mathbb{1} - z^{-1}A)^{-1} = z^{-1}\sum_{n=0}^{\infty} z^{-n}A^n,$$

and since the series converges if it converges in norm, from the usual criteria for the convergence of a series follows $\sup_{z\in\sigma(A)}|z| \leq r(A)$. Suppose now that the inequality is strict, then there exists $0 < \eta < r(A)$ and a curve $\gamma \subset \{z \in \mathcal{C} \,:\, |z| \leq \eta\}$ which contains $\sigma(A)$ in its interior. Then applying Problem C.12 yields $\|A^n\| \leq C\eta^n$, which contradicts $\eta < r(A)$. □

Note that if $f(z) = \sum_{n=0}^{\infty} f_n z^n$ is an analytic function in all $\mathbb{C}$ (entire), then we can define

$$f(A) = \sum_{n=0}^{\infty} f_n A^n.$$

Problem C.14 *Show that, if $A \in L(\mathcal{B}, \mathcal{B})$ and f is an entire function, then for each smooth curve $\gamma \subset \mathbb{C}$, with $\sigma(A)$ in its interior,*

$$f(A) = \frac{1}{2\pi i}\int_\gamma f(z)(z\mathbb{1} - A)^{-1}dz.$$

In view of the above fact, the following definition is natural:

Definition C.3.2 *For each $A \in L(\mathcal{B}, \mathcal{B})$, f analytic in a region U containing $\sigma(A)$, then for each smooth curve $\gamma \subset U$, with $\sigma(A)$ in its interior, define*

$$f(A) = \frac{1}{2\pi i}\int_\gamma f(z)(z\mathbb{1} - A)^{-1}dz. \tag{C.3.4}$$

Problem C.15 *Show that the above definition does not depend on the curve γ.*

Problem C.16 *For each $A \in L(\mathcal{B}, \mathcal{B})$ and functions f, g analytic on a domain $D \supset \sigma(A)$, show that $f(A)+g(A) = (f+g)(A)$ and $f(A)g(A) = (f \cdot g)(A)$.*

Problem C.17 *In the hypotheses of the Definition C.3.2 show that* $f(\sigma(A)) = \sigma(f(A))$ *and* $[f(A), A] = 0$.

Problem C.18 *Consider* $f : \mathbb{C} \to \mathbb{C}$ *entire and* $A \in L(\mathcal{B}, \mathcal{B})$. *Suppose that* $\{z \in \mathbb{C} \,:\, f(z) = 0\} \cap \sigma(A) = \emptyset$. *Show that* $f(A)$ *is invertible and* $f(A)^{-1} = f^{-1}(A)$.

Problem C.19 *Let* $A \in L(\mathcal{B}, \mathcal{B})$. *Suppose there exists a semi-line* ℓ, *starting from the origin, such that* $\ell \cap \sigma(A) = \emptyset$. *Prove that it is possible to define an operator* $\ln A$ *such that* $e^{\ln A} = A$.

Remark C.3.3 *Note that not all the interesting functions can be constructed in such a way. In fact,* $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ *is such that* $A^2 = -\mathbb{1}$, *thus it can be interpreted as a square rooth of* $-\mathbb{1}$ *but it cannot be obtained directly by a formula of the type* (C.3.4).

Problem C.20 *Suppose that* $A \in L(\mathcal{B}, \mathcal{B})$ *and* $\sigma(A) = B \cup C$, $B \cap C = \emptyset$, *suppose that the smooth closed curve* $\gamma \subset \rho(A)$ *contains* B, *but not* C, *in its interior, prove that*

$$P_B := \frac{1}{2\pi i} \int_\gamma (z\mathbb{1} - A)^{-1} dz$$

is a projector that does not depend on γ.

Note that by Problem C.17 easily follows that $P_B A = A P_B$. Hence, $AR(P_B) \subset R(P_B)$ and $AN(P_B) \subset N(P_B)$. Thus $\mathcal{B} = R(P_B) \oplus N(P_B)$ provides an invariant decomposition for A.

Problem C.21 *In the hypotheses of Problem C.20, prove that* $A = P_B A P_B + (\mathbb{1} - P_B) A (\mathbb{1} - P_B)$.

Problem C.22 *In the hypotheses of Problem C.20, prove that* $\sigma(P_B A P_B) = B \cup \{0\}$. *Moreover, if* $\dim(R(P_B)) = D < \infty$, *then the cardinality of* B *is* $\leq D$.

C.4 Perturbations

Let us consider $A, B \in L(\mathcal{B}, \mathcal{B})$ and the family of operators $A_\nu := A + \nu B$.

Lemma C.4.1 *For each $\delta > 0$ there exists $\nu_\delta \in \mathbb{R}$ such that, for all $|\nu| \leq \nu_\delta$, $\rho(A_\nu) \supset \{z \in \mathbb{C} \;:\; d(z, \sigma(A)) > \delta\}$.*

Proof. Let $d(z, \sigma(A)) > \delta$, then

$$(z\mathbb{1} - A_\nu) = (z\mathbb{1} - A)\left[\mathbb{1} - \nu(z\mathbb{1} - A)^{-1}B\right] \tag{C.4.5}$$

Now $\|(z\mathbb{1} - A)^{-1}B\|$ is a continuous function in z outside $\sigma(A)$, moreover it is bounded outside a ball of large enough radius, hence there exists $M_\delta > 0$ such that $\sum_{d(z,\sigma(A))>\delta} \|(z\mathbb{1} - A)^{-1}B\| \leq M_\delta$. Choosing $\nu_\delta = (2M_\delta)^{-1}$ yields the result. □

Suppose that $\bar{z} \in \mathbb{C}$ is an isolated point of $\sigma(A)$, that is there exists $\delta > 0$ such that $\{z \in \mathbb{C} \;:\; |z - \bar{z}| \leq \delta\} \cap (\sigma(A) \setminus \{\bar{z}\}) = \emptyset$, then the above Lemma shows that, for ν small enough, $\{z \in \mathbb{C} \;:\; |z - \bar{z}| \leq \delta\}$ still contains an isolated part of the spectrum of $\sigma(A_\nu)$, let us call it B_ν, clearly $B_0 = \{\bar{z}\}$.

Problem C.23 *Let P_{B_ν} be defined as in Problem C.20. Prove that, for ν small enough, it is an analytic function of ν.*

Problem C.24 *If P, Q are two projectors and $\|P - Q\| < 1$, then* $\dim(R(P)) = \dim(R(Q))$.

The above two exercises imply that the dimension of the eigenspace $R(P_{B_\nu})$ is constant.

Next, we consider the case in which B_0 consist of one point and $\dim(R(P_{B_0})) = 1$, it follows that also B_ν must consist of only one point, let us set $P_\nu := P_{B_\nu}$.

Lemma C.4.2 *If* $\dim(R(P_0)) = 1$*, then A_ν has a unique eigenvalue z_ν in a neighborhood of $\bar{z}$, $z_0 = \bar{z}$. In addition z_ν is an analytic function of ν.*

PROOF. From the previous exercises it follows that P_ν is a rank one operator which depend analytically on ν. In addition, since P_ν is a rank one projector it must have the form $P_\nu w = v_\nu \ell_\nu(w)$, where $\ell_\nu \in \mathcal{B}'$.[6] Then $z_\nu P_\nu = P_\nu A_\nu P_\nu$. Next, setting $a(\nu) := \ell_0(P_\nu v_0) = \ell_\nu(v_0)\ell_0(v_\nu)$, we have that a is analytic and $a(0) = 1$. Thus $a \neq 0$ in a neighborhood of zero and $z_\nu = a(\nu)^{-1}\ell_0(P_\nu A_\nu P_\nu v_0)$ is analytic in such a neighborhood. □

Problem C.25 *If* $\dim(R(P_0)) = 1$, *then there exists* $h_\nu \in \mathcal{B}$ *and* $\ell_\nu \in \mathcal{B}'$ *such that* $P_\nu f = h_\nu \ell_\nu(f)$ *for each* $f \in \mathcal{B}$. *Prove that* h_ν, ℓ_ν *can be chosen to be analytic functions of* ν.

Hence in the case of $A \in L(\mathcal{B}, \mathcal{B})$ with an isolated simple[7] eigenvalue $\bar{z}$ we have that the corresponding eigenvalue z_ν of $A_\nu = A + \nu B$, $B \in L(\mathcal{B}, \mathcal{B})$, for ν small enough, depend smoothly from ν. In addition, using the notation of the previous Lemma, we can easily compute the derivative: differentiating $A_\nu v_\nu = z_\nu v_\nu$ with respect to ν and then setting $\nu = 0$, yields

$$Bv + Av_0' = z_0' v + \bar{z} v_0'.$$

But, for all $w \in \mathcal{B}$, $Pw = v\ell(w)$, with $\ell(Aw) = \bar{z}\ell(w)$ and $\ell(v) = 1$, thus applying ℓ to both sides of the above equation yields

$$z_0' = \ell(Bv).$$

Problem C.26 *Compute* v_0'.

Problem C.27 *What does it happen if the eigenspace associated to* $\bar{z}$ *is finite dimensional, but with dimension strictly larger than one?*

[6] By $\mathcal{B}'$, the dual space, we mean the set of bounded linear functionals on $\mathcal{B}$. Verify that is a Banach space with the norm $\|\ell\| = \sum_{w \in \mathcal{B}} \frac{|\ell(w)|}{\|w\|}$.

[7] That is with the associated eigenprojector of rank one.

Hints to solving the Problems

C.1. The triangle inequality follows trivially from the triangle inequality of the norm of $\mathcal{B}$. To verify the completeness suppose that $\{B_n\}$ is a Cauchy sequence in $L(\mathcal{B}, \mathcal{B})$. Then, for each $v \in \mathcal{B}$, $\{B_n v\}$ is a Cauchy sequence in $\mathcal{B}$, hence it has a limit, call it $B(v)$. We have so defined a function from $\mathcal{B}$ to teself. Show that such a function is linear and bounded, hence it defines an element of $L(\mathcal{B}, \mathcal{B})$, which can easily be verified to be the limit of $\{B_n\}$.

C.2. Use the norm $\|A\| = \sup_{v \in \mathbb{R}^n} \frac{\|Av\|}{\|v\|}$.

C.3. Use the same norm as in Problem C.2.

C.4. The first part is trivial. For the second one can consider the vector space $\ell^2 = \{x \in \mathbb{R}^{\mathbb{N}} : \sum_{i=0}^{\infty} x_i^2 < \infty\}$. Equipped with the norm $\|x\| = \sqrt{\sum_{i=0}^{\infty} x_i^2}$ it is a Banach (actually Hilbert) space. Consider now the vectors $e_i \in \ell^2$ defined by $(e_i) = \delta_{ik}$ and the operator $(Ax)_k = \frac{1}{k} x_k$. Then $R(A) = \{x \in \ell^2 : \sum_{k=0}^{\infty} k^2 x_k^2 < \infty\}$, which is dense in ℓ^2 but strictly smaller.

C.5. Check that the same argument used in the well known case $\mathcal{B} = \mathbb{C}$ works also here.

C.6. Check that the same argument used in the well known case $\mathcal{B} = \mathbb{C}$ works also here.

C.7. Check that the same argument used in the well known case $\mathcal{B} = \mathbb{C}$ works also here.

C.8. Note that

$$(\zeta \mathbb{1} - A) = (z\mathbb{1} - A - (z - \zeta)\mathbb{1}) = (z\mathbb{1} - A)\left[\mathbb{1} - (z - \zeta)(z\mathbb{1} - A)^{-1}\right]$$

and that if $\|(z - \zeta)(z\mathbb{1} - A)^{-1}\| < 1$ then the inverse of $\mathbb{1} - (z - \zeta)(z\mathbb{1} - A)^{-1}$ is given by $\sum_{n=0}^{\infty} (z - \zeta)^n [(z\mathbb{1} - A)^{-1}]^n$ (the Von Neumann series–which really is just the geometric series).

C.9. If $\sigma(A) = \emptyset$, then $(z\mathbb{1} - A)^{-1}$ is an entire function, then the Von Neumann series shows that $(z\mathbb{1} - A)^{-1} = z^{-1}(\mathbb{1} - z^{-1}A)^{-1}$ goes to zero for large z, and then (C.2.3) shows that $(z\mathbb{1} - A)^{-1} = 0$ which is impossible.

C.10. Verify that $(z\mathbb{1} - \Pi)^{-1} = z^{-1}\left[\mathbb{1} - (z-1)^{-1}\Pi\right]$.

C.11. The idea is to look for eigenvalues by using Fourier series. Let $f = \sum_{k\in\mathbb{Z}} f_k e^{2\pi ikx}$ and consider the equation $\mathcal{L}f = zf$,

$$\sum_{k\in\mathbb{Z}} f_k \frac{1}{2}\left\{e^{\pi ikx} + e^{\pi ikx+\pi ik}\right\} = z\sum_{k\in\mathbb{Z}} f_k e^{2\pi ikx}.$$

Let us then restrict to the case in which $f_{2k+1} = 0$, then

$$\sum_{k\in\mathbb{Z}} f_{2k}e^{2\pi ikx} = z\sum_{k\in\mathbb{Z}} f_k e^{2\pi ikx}.$$

Thus we have a solution provided $f_{2k} = zf_k$, such conditions are satisfied by any sequence of the type

$$f_k = \begin{cases} z^j & \text{if } k = 2^j m, j \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

for $m \in \mathbb{N}$. It remains to verify that $\sum_{j=0}^{\infty} z^j e^{2\pi i 2^j x}$ belong to $\mathcal{C}^0$. This is the case if the series is uniformly convergent, which happens for $z| < 1$. Thus all the points in $\{z \in \mathbb{C} \ : \ |z| < 1\}$ are point spectrum of infinite multiplicity. Since the spectrum is closed the statement of the Problem follows.

C.12. Let $p(z) = z^n$, then

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma z^n(z\mathbb{1} - A)^{-1}dz &= A^n + \frac{1}{2\pi i}\int_\gamma (z^n - A^n)(z\mathbb{1} - A)^{-1}dz \\ &= A^n + \sum_{k=0}^{n-1}\frac{1}{2\pi i}\int_\gamma z^k A^{n-k}dz = A^n.\end{aligned}$$

The statement for general polynomial follows trivially.

C.14. Approximate by polynomials.

C.15. For $z \notin f(\sigma(A))$ it is well defined

$$K(z) := \frac{1}{2\pi i}\int_\gamma (z - f(\zeta))^{-1}(\zeta\mathbb{1} - A)^{-1}\, d\zeta,$$

with γ containing $\sigma(A)$ in its interior. By direct computation, using definition C.3.2, one can verify that $(z\mathbb{1} - f(A))K(z) = \mathbb{1}$, thus $\sigma(f(A)) \subset f(\sigma(A))$. On the other hand if, if f is not constant, then for each $z \in \mathcal{C}$ $f(z) - f(\xi) = (z - \xi)g(\xi)$. Hence, applying Definition C.3.2 and Problem C.16 it follows $f(z)\mathbb{1} - f(A) = (z - A)g(A)$ which shows that if $z \in \sigma(A)$, then $f(z) \in \sigma(A)$ (otherwise $(z - A)\left[g(A)(f(z)\mathbb{1} - f(A))^{-1}\right] = \mathbb{1}$).

C.17. Since one can define the logarithm on $\mathbb{C}\backslash\ell$, one can use Definition C.3.2 to define $\ln A$. It suffices to prove that if $f : U \to \mathcal{C}$ and $g : V \to \mathcal{C}$, with $\sigma(A) \subset U$, $f(U) \subset V$, then $g(f(A)) = g \circ f(A)$. Whereby showing that the definition C.3.2 is a reasonable one. Indeed, rememebring Problems C.17, C.18,

$$\begin{aligned} g(f(A)) &= \frac{1}{2\pi i}\int_\gamma g(z)(z\mathbb{1} - f(A))^{-1}dz \\ &= \frac{1}{(2\pi i)^2}\int_{\gamma_1}\int_\gamma \frac{g(z)}{z - f(\xi)}(\xi\mathbb{1} - A)^{-1}dzd\xi \\ &= \frac{1}{2\pi i}\int_{\gamma_1} g(f(\xi))(\xi\mathbb{1} - A)^{-1}d\xi = f \circ g(A). \end{aligned}$$

From this imediately follows $e^{\ln A} = A$.

C.18. The non dependence on γ is obvious. A projector is characterized by the property $P^2 = P$. Thus

$$\begin{aligned} P_B^2 :=& \frac{1}{(2\pi i)^2}\int_{\gamma_1}\int_{\gamma_2}(z\mathbb{1} - A)^{-1}(\zeta\mathbb{1} - A)^{-1}dzd\zeta \\ =& \frac{1}{(2\pi i)^2}\int_{\gamma_1} dz \int_{\gamma_2} d\zeta(z - \zeta)^{-1}\left[(z\mathbb{1} - A)^{-1} - (\zeta\mathbb{1} - A)^{-1}\right]. \end{aligned}$$

If we have chosen γ_1 in the interior of γ_2, then $(z-\zeta)^{-1}(\zeta\mathbb{1}-A)^{-1}$ is analytic in the interior of γ_1, hence the corresponding integral gives zero. The other integral gives P_B, as announced.

C.19. Use the above decomposition and the fact that $(\mathbb{1} - P_B)$ is a projector.

C.20. The first part follows from the previous decomposition. Indeed, for z large (by Neumann series)

$$(z\mathbb{1} - A)^{-1} = (z\mathbb{1} - P_BAP_B)^{-1} + (z\mathbb{1} - (\mathbb{1} - P_B)A(\mathbb{1} - P_B))^{-1}.$$

Since the above functions are analytic in the respective resolvent sets it follows that $\sigma(A) \subset \sigma(P_B A P_B) \cup \sigma((\mathbb{1} - P_B)A(\mathbb{1} - P_B))$. Next, for $z \notin B$, define the operator

$$K(z) := \frac{1}{2\pi i} \int_\gamma (z - \xi)^{-1} (\xi \mathbb{1} - A)^{-1} \, d\xi,$$

where γ contains B, but no other part of the spectrum, in its interior. By direct computation (using Fubini and the standard facts about residues and integration of analytic functions) verify that

$$(z\mathbb{1} - P_B A P_B) K(z) = P_B.$$

This implies that, for $z \neq 0$, $(z\mathbb{1} - P_B A P_B)(K(z) + z^{-1}(\mathbb{1} - P_B)) = \mathbb{1}$, that is $(z\mathbb{1} - P_B A P_B)^{-1} = K(z) + z^{-1}(\mathbb{1} - P_B)$. Hence $\sigma(P_B A P_B) \subset B \cup \{0\}$. Since P_B has a kernel, zero must be in the spectrum. On the other hand the same argument applied to $\mathbb{1} - P_B$ yields $\sigma((\mathbb{1} - P_B)A)\mathbb{1} - P_B)) \subset C \cup \{0\}$, hence $\sigma(P_B A P_B) = B \cup \{0\}$.

The second property follows from the fact that $P_B A P_B$, when restricted to the space $R(P_B)$ is described by a $D \times D$ matrix A_B and the equation $\det(z\mathbb{1} - A_B) = 0$ is a polynomial of degree D in z and hence has exactly D solutions (counted with multiplicity).[8]

C.21. Use the representation in Problem C.20 and formula (C.4.5).

C.22. Note that $Q(\mathbb{1} + P - Q) = QP$, then $Q = (\mathbb{1} - (Q - P))^{-1} QP$, hence $\dim(R(P)) \geq \dim(R(Q))$, exchanging the role of P and Q the result follows.

[8] This is the real reason why spectral theory is done over the complex rather than the real. You should be well aquatinted with the fact that a polynomial p of degree D has D root over $\mathbb{C}$ but, in case you have forgotten, consider the following: first a polynomial of degree larger than zero must have at least a root, otherwise $\frac{1}{p(z)}$ would be an entire function and hence

$$\frac{1}{p(z)} = \lim_{r \to \infty} \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{1}{p(z + re^{i\theta})} = 0.$$

Let z_1 be a root. By the Taylor expansion in z_1 follows the decomposition $p(z) = (z - z_1)p_1(z)$ where p_1 has degree $D - 1$. The result follows by induction.

C.25. Note that $\ell_\nu(h_\nu) = 1$ since P_ν is a projector, hence they are unique apart from a noralization factor. Then we can chose the normalization $\ell_\nu(h_0) = 1$ for all ν small enough. Thus $P_\nu f = h_\nu$, that is h_ν is analytic. Hence, for each $g \in \mathcal{B}$ and ν small, $\ell_\nu(g)\ell_0(h_\nu) = \ell_0(P_\nu g)$, which implies ℓ_ν analytic for ν small.

C.25. Think hard.[9]

[9] A good idea is to start by considering concrete examples, for instance

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mu \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; ; \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + \mu \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$