

APPENDIX B

Perturbation Theory (a super-fast introduction)

The following is really super condensate (although self-consistent). If you want more details see [RS80, Kat66] in which you probably can find more than you are looking for.

B.1 Bounded operators

In the following we will consider only *separable* Banach spaces, i.e. Banach spaces that have a countable dense set.

Given a Banach space \mathcal{B} we can consider the set $L(\mathcal{B}, \mathcal{B})$ of the linear bounded operators from \mathcal{B} to itself. We can then introduce the norm $\|B\| = \sup_{\|v\| \leq 1} \|Bv\|$.

Problem B.1 *Show that $(L(\mathcal{B}, \mathcal{B}), \|\cdot\|)$ is a Banach space. That is that $\|\cdot\|$ is really a norm and that the space is complete with respect to such a norm.*

To each $A \in L(\mathcal{B}, \mathcal{B})$ are associated two important subspaces: the range $R(A) = \{v \in \mathcal{B} : \exists w \in \mathcal{B} \text{ such that } v = Aw\}$ and the kernel $N(A) = \{v \in \mathcal{B} : Av = 0\}$.

Problem B.2 *Prove, for each $A \in L(\mathcal{B}, \mathcal{B})$, that $N(A)$ is a closed linear subspaces of \mathcal{B} . Show that this is not necessarily the case for $R(A)$.*

An very special, but very important, class of operators are the projectors.

Definition B.1.1 An operator $\Pi \in L(\mathcal{B}, \mathcal{B})$ is called a projector iff $\Pi^2 = \Pi$.

Note that if Π is a projector, so is $\mathbb{1} - \Pi$. We have the following interesting fact.

Lemma B.1.2 If $\Pi \in L(\mathcal{B}, \mathcal{B})$ is a projector, then $N(\Pi) \oplus R(\Pi) = \mathcal{B}$.

PROOF. If $v \in \mathcal{B}$, then $v = \Pi v + (\mathbb{1} - \Pi)v$. Notice that $R(\mathbb{1} - \Pi) = N(\Pi)$ and $R(\Pi) = N(\mathbb{1} - \Pi)$. Finally, if $v \in N(\Pi) \cap R(\Pi)$, then $v = 0$, which concludes the proof. \square

Another, more general, very important class of operators are the compact ones.

Definition B.1.3 An operator $K \in L(\mathcal{B}, \mathcal{B})$ is called compact iff for any bounded set B the closure of $K(B)$ is compact.

Remark B.1.4 Note that not all the linear operator on a Banach space are bounded. For example consider the derivative acting on $\mathcal{C}^1((0, 1), \mathbb{R})$.

B.2 Functional calculus

First of all notice that all the Riemannian theory of integration works verbatim¹ for function $f \in \mathcal{C}^0(\mathbb{R}, \mathcal{B})$, where \mathcal{B} is a Banach space.² We can thus talk of integrals of the type $\int_a^b f(t)dt$, it is special case of the so called Bochner integral. Next, we can talk of analytic functions for functions in $\mathcal{C}^0(\mathbb{C}, \mathcal{B})$: a function is analytic in an open region $U \subset \mathbb{C}$ iff at each point $z_0 \in U$ there exists a neighborhood $B \ni z_0$ and elements $\{a_n\} \subset \mathcal{B}$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \forall z \in B. \quad (\text{B.2.1})$$

¹Of course one cannot talk of sup or inf of the sums, yet one can easily show that any two sufficiently fine partitions yields close values, just take the partition given by the intersection and compare with that.

²Recall that a Banach space is a complete normed vector space (in the following we will consider only vector spaces on the field of complex numbers), that is a normed vector space in which all the Cauchy sequences have a limit in the space.

Problem B.3 Show that if $f \in \mathcal{C}^0(\mathbb{C}, \mathcal{B})$ is analytic in $U \subset \mathbb{C}$, then given any smooth closed curve γ , contained in a sufficiently small disk in U , holds³

$$\int_{\gamma} f(z)dz = 0 \quad (\text{B.2.2})$$

Then show that the same hold for any piecewise smooth close curve in U .

Problem B.4 Show that if $f \in \mathcal{C}^0(\mathbb{C}, \mathcal{B})$ is analytic in $U \subset \mathbb{C}$, then given any smooth closed curve γ , contained in U and having in its interior a point z , hods the formula

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} (\xi - z)^{-1} f(\xi) d\xi. \quad (\text{B.2.3})$$

Problem B.5 Show that if $f \in \mathcal{C}^0(\mathbb{C}, \mathcal{B})$ satisfies (B.2.3) for each smooth closed curve in an open set U , then f is analytic in U .

Problem B.6 Show that the $n \times n$ matrices form a Banach Algebra.⁴

Problem B.7 Show that the the set of bounded linear operators from a Banach space \mathcal{B} to itself (usually called $L(\mathcal{B}, \mathcal{B})$) form a Banach algebra.

B.3 Spectrum and resolvent

Given $A \in L(\mathcal{B}, \mathcal{B})$ we define the resolvent, called $\rho(A)$, as the set of the $z \in \mathbb{C}$ such that $(z\mathbf{1} - A)$ is invertible and the inverse belongs to $L(\mathcal{B}, \mathcal{B})$. The spectrum of A , called $\sigma(A)$ is the complement of $\rho(A)$ in \mathbb{C} .

Problem B.8 Prove that, for each Banach space \mathcal{B} and operator $A \in L(\mathcal{B}, \mathcal{B})$, if $z \in \rho(A)$, then there exists a neighborhood U of z such that $(z\mathbf{1} - A)^{-1}$ is analytic in U .

³Of course, by $\int_{\gamma} f(z)dz$ we mean that we have to consider any smooth parametrization $g : [a, b] \rightarrow \mathbb{C}$ of γ , $g(a) = g(b)$, and then $\int_{\gamma} f(z)dz := \int_a^b f \circ g(t)g'(t)dt$. Show that the definition does not depend on the parametrization and that one can use piecewise smooth parametrizations as well.

⁴A Banach Algebra \mathcal{A} is a Banach space where it is defined the multiplications between element with the usual properties of an algebra and, in addition, for each $a, b \in \mathcal{A}$ holds $\|ab\| \leq \|a\| \cdot \|b\|$.

From the above exercise follows that $\rho(A)$ is open, hence $\sigma(A)$ is closed.

Problem B.9 Show that, for each $A \in L(\mathcal{B}, \mathcal{B})$, $\sigma(A) \neq \emptyset$.

Problem B.10 Show that if $\Pi \in L(\mathcal{B}, \mathcal{B})$ is a projector, then $\sigma(\Pi) = \{0, 1\}$.

Next, consider a Banach space \mathcal{B} and $A \in L(\mathcal{B}, \mathcal{B})$, given a smooth curve γ containing in its interior $\sigma(A) \cup \{0\}$, for each $n \in \mathbb{N}$ holds

$$\begin{aligned} \frac{1}{2\pi i} \int_{\gamma} (z\mathbf{1} - A)^{-1} dz &= \frac{1}{2\pi i} \int_{\gamma} z^{-1} (\mathbf{1} - z^{-1}A)^{-1} dz \\ &= \lim_{r \rightarrow \infty} \frac{1}{2\pi} \int_0^{2\pi} (\mathbf{1} - r^{-1}e^{-i\theta}A)^{-1} d\theta = \mathbf{1}. \end{aligned}$$

Problem B.11 Show that, if $A \in L(\mathcal{B}, \mathcal{B})$ and p is any polynomial, then for each $n \in \mathbb{N}$ and smooth curve $\gamma \subset \mathbb{C}$, with $\sigma(A)$ in its interior,

$$p(A) = \frac{1}{2\pi i} \int_{\gamma} p(z)(z\mathbf{1} - A)^{-1} dz.$$

Note that if $f(z) = \sum_{n=0}^{\infty} f_n z^n$ is an analytic function in all \mathbb{C} (entire), then formula (B.2.1) allows to define

$$f(A) = \sum_{n=0}^{\infty} f_n A^n$$

Problem B.12 Show that, if $A \in L(\mathcal{B}, \mathcal{B})$ and f is an entire function, then for each smooth curve $\gamma \subset \mathbb{C}$, with $\sigma(A)$ in its interior,

$$f(A) = \frac{1}{2\pi i} \int_{\gamma} f(z)(z\mathbf{1} - A)^{-1} dz.$$

In view of the above fact, the following definition is natural:

Definition B.3.1 For each $A \in L(\mathcal{B}, \mathcal{B})$, f analytic in a region U containing $\sigma(A)$, then for each smooth curve $\gamma \subset U$, with $\sigma(A)$ in its interior, define

$$f(A) = \frac{1}{2\pi i} \int_{\gamma} f(z)(z\mathbf{1} - A)^{-1} dz.$$

Problem B.13 Show that the above definition does not depend on the curve γ .

Problem B.14 For each $A \in L(\mathcal{B}, \mathbb{B})$ and functions f, g analytic on a domain $D \supset \sigma(A)$, show that $f(A) + g(A) = (f + g)(A)$ and $f(A)g(A) = (f \cdot g)(A)$.

Problem B.15 In the hypotheses of the Definition B.3.1 show that $f(\sigma(A)) = \sigma(f(A))$.

Problem B.16 Consider $f : \mathbb{C} \rightarrow \mathbb{C}$ entire and $A \in L(\mathcal{B}, \mathcal{B})$. Suppose that $\{z \in \mathbb{C} : f(z) = 0\} \cap \sigma(A) = \emptyset$. Show that $f(A)$ is invertible and $f(A)^{-1} = f^{-1}(A)$.

Problem B.17 Let $A \in L(\mathcal{B}, \mathcal{B})$. Suppose there exists a semi-line ℓ , starting from the origin, such that $\ell \cap \sigma(A) = \emptyset$. Prove that it is possible to define an operator $\ln A$ such that $e^{\ln A} = A$.

Remark B.3.2 Note that not all the interesting functions can be constructed in such a way. In fact, $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is such that $A^2 = -\mathbf{1}$, thus it can be interpreted as a square root of $-\mathbf{1}$ but it cannot be obtained directly by a formula of the type (B.2.3).

Problem B.18 Suppose that $A \in L(\mathcal{B}, \mathcal{B})$ and $\sigma(A) = B \cup C$, $B \cap C = \emptyset$, suppose that the smooth closed curve $\gamma \subset \rho(A)$ contains B , but not C , in its interior, prove that

$$P_B := \frac{1}{2\pi i} \int_{\gamma} (z\mathbf{1} - A)^{-1} dz$$

is a projector that does not depend on γ .

Note that from the definition in Problem B.18 easily follows that $P_B A = A P_B$. Hence, $AR(P_B) \subset R(P_B)$ and $AN(P_B) \subset N(P_B)$. Thus $\mathcal{B} = R(P_B) \oplus N(P_B)$ provides an invariant decomposition for A .

Problem B.19 In the hypotheses of Problem B.18, prove that $A = P_B A P_B + (\mathbf{1} - P_B) A (\mathbf{1} - P_B)$.

Problem B.20 In the hypotheses of Problem B.18, prove that $\sigma(P_B A P_B) = B \cup \{0\}$. Moreover, if $\dim(R(P_B)) = D < \infty$, then the cardinality of B is $\leq D$.

B.4 Perturbations

Let us consider $A, B \in L(\mathcal{B}, \mathcal{B})$ and the family of operators $A_\nu := A + \nu B$.

Lemma B.4.1 *For each $\delta > 0$ there exists $\nu_\delta \in \mathbb{R}$ such that, for all $|\nu| \leq \nu_\delta$, $\rho(A_\nu) \supset \{z \in \mathbb{C} : d(z, \sigma(A)) > \delta\}$.*

PROOF. Let $d(z, \sigma(A)) > \delta$, then

$$(z\mathbf{1} - A_\nu) = (z\mathbf{1} - A) [\mathbf{1} - \nu(z\mathbf{1} - A)^{-1}B] \quad (\text{B.4.4})$$

Now $\|(z\mathbf{1} - A)^{-1}B\|$ is a continuous function in z outside $\sigma(A)$, moreover it is bounded outside a ball of large enough radius, hence there exists $M_\delta > 0$ such that $\sum_{d(z, \sigma(A)) > \delta} \|(z\mathbf{1} - A)^{-1}B\| \leq M_\delta$. Choosing $\nu_\delta = (2M_\delta)^{-1}$ yields the result. \square

Suppose that $\bar{z} \in \mathbb{C}$ is an isolated point of $\sigma(A)$, that is there exists $\delta > 0$ such that $\{z \in \mathbb{C} : |z - \bar{z}| \leq \delta\} \cap (\sigma(A) \setminus \{\bar{z}\}) = \emptyset$, then the above Lemma shows that, for ν small enough, $\{z \in \mathbb{C} : |z - \bar{z}| \leq \delta\}$ still contains an isolated part of the spectrum of $\sigma(A_\nu)$, let us call it B_ν , clearly $B_0 = \{\bar{z}\}$.

Problem B.21 *Let P_{B_ν} be defined as in Problem B.18. Prove that, for ν small enough, it is an analytic function of ν .*

Problem B.22 *If P, Q are two projectors and $\|P - Q\| < 1$, then $\dim(R(P)) = \dim(R(Q))$.*

The above two exercises imply that the dimension of the eigenspace $R(P_\nu)$ is constant.

Lemma B.4.2 *If $\dim(R(P_0)) = 1$, then A_ν has a unique eigenvalue z_ν in a neighborhood of \bar{z} , $z_0 = \bar{z}$. In addition z_ν is an analytic function of ν .*

PROOF. From the previous exercises it follows that P_ν is a rank one operator which depend analytically on ν . In addition, since P_ν is a rank one projector it must have the form $P_\nu w = v_\nu \ell_\nu(w)$, where

$\ell_\nu \in \mathcal{B}'$.⁵ Then $z_\nu P_\nu = P_\nu A_\nu P_\nu$. Next, setting $a(\nu) := \ell_0(P_\nu v_0) = \ell_\nu(v_0)\ell_0(v_\nu)$, we have that a is analytic and $a(0) = 1$. Thus $a \neq 0$ in a neighborhood of zero and $z_\nu = a(\nu)^{-1}\ell_0(P_\nu A_\nu P_\nu v_0)$ is analytic in such a neighborhood. \square

Hence in the case of $A \in L(\mathcal{B}, \mathcal{B})$ with an isolated simple⁶ eigenvalue \bar{z} we have that the corresponding eigenvalue z_ν of $A_\nu = A + \nu B$, $B \in L(\mathcal{B}, \mathcal{B})$, for ν small enough, depend smoothly from ν . In addition, using the notation of the previous Lemma, we can easily compute the derivative: differentiating $A_\nu v_\nu = z_\nu v_\nu$ with respect to ν and then setting $\nu = 0$, yields

$$Bv + Av'_0 = z'_0 v + \bar{z}v'_0.$$

But, for all $w \in \mathcal{B}$, $Pw = v\ell(w)$, with $\ell(Aw) = \bar{z}\ell(w)$ and $\ell(v) = 1$, thus applying ℓ to both sides of the above equation yields

$$z'_0 = \ell(Bv).$$

Problem B.23 Compute v'_0 .

Problem B.24 What does it happen if the eigenspace associated to \bar{z} is finite dimensional, but with dimension strictly larger than one?

Hints to solving the Problems

B.1 The triangle inequality follows trivially from the triangle inequality of the norm of \mathcal{B} . To verify the completeness suppose that $\{B_n\}$ is a Cauchy sequence in $L(\mathcal{B}, \mathcal{B})$. Then, for each $v \in \mathcal{B}$, $\{B_n v\}$ is a Cauchy sequence in \mathcal{B} , hence it has a limit, call it $B(v)$. We have so defined a function from \mathcal{B} to itself. Show that such a function is linear and bounded, hence it defines an element of $L(\mathcal{B}, \mathcal{B})$, which can easily be verified to be the limit of $\{B_n\}$.

⁵By \mathcal{B}' , the dual space, we mean the set of bounded linear functionals on \mathcal{B} . Verify that is a Banach space with the norm $\|\ell\| = \sum_{w \in \mathcal{B}} \frac{|\ell(w)|}{\|w\|}$.

⁶That is with the associated eigenprojector of rank one.

B.2 The first part is trivial. For the second one can consider the vector space $\ell^2 = \{x \in \mathbb{R}^{\mathbb{N}} : \sum_{i=0}^{\infty} x_i^2 < \infty\}$. Equipped with the norm $\|x\| = \sqrt{\sum_{i=0}^{\infty} x_i^2}$ it is a Banach (actually Hilbert) space. Consider now the vectors $e_i \in \ell^2$ defined by $(e_i)_k = \delta_{ik}$ and the operator $(Ax)_k = \frac{1}{k}x_k$. Then $R(A) = \{x \in \ell^2 : \sum_{k=0}^{\infty} k^2 x_k^2 < \infty\}$, which is dense in ℓ^2 but strictly smaller.

B.3 Check that the same argument used in the well known case $\mathcal{B} = \mathbb{C}$ works also here.

B.4 Check that the same argument used in the well known case $\mathcal{B} = \mathbb{C}$ works also here.

B.5 Check that the same argument used in the well known case $\mathcal{B} = \mathbb{C}$ works also here.

B.6 Use the norm $\|A\| = \sup_{v \in \mathbb{R}^n} \frac{\|Av\|}{\|v\|}$.

B.7 Use the same norm as in Problem B.6.

B.8 Note that

$$(\zeta \mathbf{1} - A) = (z \mathbf{1} - A - (z - \zeta) \mathbf{1}) = (z \mathbf{1} - A) [\mathbf{1} - (z - \zeta)(z \mathbf{1} - A)^{-1}]$$

and that if $\|(z - \zeta)(z \mathbf{1} - A)^{-1}\| < 1$ then the inverse of $\mathbf{1} - (z - \zeta)(z \mathbf{1} - A)^{-1}$ is given by $\sum_{n=0}^{\infty} (z - \zeta)^n [(z \mathbf{1} - A)^{-1}]^n$ (the Von Neumann series—which really is just the geometric series).

B.9 If $\sigma(A) = \emptyset$, then $(z \mathbf{1} - A)^{-1}$ is an entire function, then the Von Neumann series shows that $(z \mathbf{1} - A)^{-1} = z^{-1}(\mathbf{1} - z^{-1}A)^{-1}$ goes to zero for large z , and then (B.2.3) shows that $(z \mathbf{1} - A)^{-1} = 0$ which is impossible.

B.10 Verify that $(z \mathbf{1} - \Pi)^{-1} = z^{-1} [\mathbf{1} - (z - 1)^{-1} \Pi]$.

B.11 From the above equality follows

$$\begin{aligned} \frac{1}{2\pi i} \int_{\gamma} z^n (z \mathbf{1} - A)^{-1} dz &= A^n + \frac{1}{2\pi i} \int_{\gamma} (z^n - A^n) (z \mathbf{1} - A)^{-1} dz \\ &= A^n + \sum_{k=0}^{n-1} \frac{1}{2\pi i} \int_{\gamma} z^k A^{n-k} dz = A^n. \end{aligned}$$

The statement for polynomial follows trivially.

B.12 Approximate by polynomials.

B.15 For $z \notin f(\sigma(A))$ it is well defined

$$K(z) := \frac{1}{2\pi i} \int_{\gamma} (z - f(\zeta))^{-1} (\zeta \mathbf{1} - A)^{-1} d\zeta,$$

with γ containing $\sigma(A)$ in its interior. By direct computation, using definition B.3.1, one can verify that $(z\mathbf{1} - f(A))K(z) = \mathbf{1}$, thus $\sigma(f(A)) \subset f(\sigma(A))$. On the other hand if, if f is not constant, then for each $z \in \mathcal{C}$ $f(z) - f(\xi) = (z - \xi)g(\xi)$. Hence, applying Definition B.3.1 and Problem B.14 it follows $f(z)\mathbf{1} - f(A) = (z - A)g(A)$ which shows that if $z \in \sigma(A)$, then $f(z) \in \sigma(A)$ (otherwise $(z - A) [g(A)(f(z)\mathbf{1} - f(A))^{-1}] = \mathbf{1}$).

B.17 Since one can define the logarithm on $\mathbb{C} \setminus \ell$, one can use Definition B.3.1 to define $\ln A$. It suffices to prove that if $f : U \rightarrow \mathcal{C}$ and $g : V \rightarrow \mathcal{C}$, with $\sigma(A) \subset U$, $f(U) \subset V$, then $g(f(A)) = g \circ f(A)$. Whereby showing that the definition B.3.1 is a reasonable one. Indeed, remembering Problems B.15, B.16,

$$\begin{aligned} g(f(A)) &= \frac{1}{2\pi i} \int_{\gamma} g(z)(z\mathbf{1} - f(A))^{-1} dz \\ &= \frac{1}{(2\pi i)^2} \int_{\gamma_1} \int_{\gamma} \frac{g(z)}{z - f(\xi)} (\xi \mathbf{1} - A)^{-1} dz d\xi \\ &= \frac{1}{2\pi i} \int_{\gamma_1} g(f(\xi)) (\xi \mathbf{1} - A)^{-1} d\xi = f \circ g(A). \end{aligned}$$

From this immediately follows $e^{\ln A} = A$.

B.18 The non dependence on γ is obvious. A projector is characterized by the property $P^2 = P$. Thus

$$\begin{aligned} P_B^2 &:= \frac{1}{(2\pi i)^2} \int_{\gamma_1} \int_{\gamma_2} (z\mathbf{1} - A)^{-1} (\zeta \mathbf{1} - A)^{-1} dz d\zeta \\ &= \frac{1}{(2\pi i)^2} \int_{\gamma_1} dz \int_{\gamma_2} d\zeta (z - \zeta)^{-1} [(z\mathbf{1} - A)^{-1} - (\zeta \mathbf{1} - A)^{-1}]. \end{aligned}$$

If we have chosen γ_1 in the interior of γ_2 , then $(z - \zeta)^{-1} (\zeta \mathbf{1} - A)^{-1}$ is analytic in the interior of γ_1 , hence the corresponding integral gives zero. The other integral gives P_B , as announced.

B.19 Use the above decomposition and the fact that $(\mathbb{1} - P_B)$ is a projector.

B.20 The first part follows from the previous decomposition. Indeed, for z large (by Neumann series)

$$(z\mathbb{1} - A)^{-1} = (z\mathbb{1} - P_B A P_B)^{-1} + (z\mathbb{1} - (\mathbb{1} - P_B)A(\mathbb{1} - P_B))^{-1}.$$

Since the above functions are analytic in the respective resolvent sets it follows that $\sigma(A) \subset \sigma(P_B A P_B) \cup \sigma((\mathbb{1} - P_B)A(\mathbb{1} - P_B))$. Next, for $z \notin B$, define the operator

$$K(z) := \frac{1}{2\pi i} \int_{\gamma} (z - \xi)^{-1} (\xi \mathbb{1} - A)^{-1} d\xi,$$

where γ contains B , but no other part of the spectrum, in its interior. By direct computation (using Fubini and the standard facts about residues and integration of analytic functions) verify that

$$(z\mathbb{1} - P_B A P_B)K(z) = P_B.$$

This implies that, for $z \neq 0$, $(z\mathbb{1} - P_B A P_B)(K(z) + z^{-1}(\mathbb{1} - P_B)) = \mathbb{1}$, that is $(z\mathbb{1} - P_B A P_B)^{-1} = K(z) + z^{-1}(\mathbb{1} - P_B)$. Hence $\sigma(P_B A P_B) \subset B \cup \{0\}$. Since P_B has a kernel, zero must be in the spectrum. On the other hand the same argument applied to $\mathbb{1} - P_B$ yields $\sigma((\mathbb{1} - P_B)A(\mathbb{1} - P_B)) \subset C \cup \{0\}$, hence $\sigma(P_B A P_B) = B \cup \{0\}$.

The second property follows from the fact that $P_B A P_B$, when restricted to the space $R(P_B)$ is described by a $D \times D$ matrix A_B and the equation $\det(z\mathbb{1} - A_B) = 0$ is a polynomial of degree D in z and hence has exactly D solutions (counted with multiplicity).⁷

⁷This is the real reason why spectral theory is done over the complex rather than the real. You should be well aquatinted with the fact that a polynomial p of degree D has D root over \mathbb{C} but, in case you have forgotten, consider the following: first a polynomial of degree larger than zero must have at least a root, otherwise $\frac{1}{p(z)}$ would be an entire function and hence

$$\frac{1}{p(z)} = \lim_{r \rightarrow \infty} \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{1}{p(z + r e^{i\theta})} = 0.$$

Let z_1 be a root. By the Taylor expansion in z_1 follows the decomposition $p(z) = (z - z_1)p_1(z)$ where p_1 has degree $D - 1$. The result follows by induction.

- B.21** Use the representation in Problem B.18 and formula (B.4.4).
- B.22** Note that $Q(\mathbb{1} + P - Q) = QP$, then $Q = (\mathbb{1} - (Q - P))^{-1}QP$, hence $\dim(R(P)) \geq \dim(R(Q))$, exchanging the role of P and Q the result follows.
- B.24** Think hard.⁸

⁸ A good idea is to start by considering concrete examples, for instance

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mu \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + \mu \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$