Chapter 1

The origins: Differential equations

As this book is about Dynamical Systems, I feel obliged to start by defining the objects of study. The concept of Dynamical System is a very general one and it appears in many branches of mathematics from discrete mathematics, number theory and probability to geometry and analysis and has wide applications in physics, chemistry and biology.

Probably the most general formulation of such a concept is the action of a group over an algebra. Given a group $\mathbb{G}$ and an algebra $\mathcal{A}$ the (left)-action of $\mathbb{G}$ on $\mathcal{A}$ is simply a map $f : \mathbb{G} \times \mathcal{A} \to \mathcal{A}$ such that

1. $f(gh, a) = f(g, f(h, a))$ for each $g, h \in \mathbb{G}$ and $a \in \mathcal{A}$;
2. $f(e, a) = a$ for every $a \in \mathcal{A}$, where e is the identity element of $\mathbb{G}$;
3. $f(g, a + b) = f(g, a) + f(g, b)$ for each $g \in \mathbb{G}$ and $a, b \in \mathcal{A}$;
4. $f(g, ab) = f(g, a)f(g, b)$ for each $g \in \mathbb{G}$ and $a, b \in \mathcal{A}$;

In our discussion we will be mainly motivated by physics. We will consider only the cases in which $\mathbb{G} \in \{\mathbb{N}, \mathbb{Z}, \mathbb{R}_+, \mathbb{R}\}$[1] is interpreted as *time* and $\mathcal{A}$ is given by an algebra of functions over some set X, interpreted ad the *observable* of the system.[2] In addition, we will restrict ourselves

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[1] Although even in physics other possibilities are very relevant, e.g. in the case of Statistical Mechanics it is natural to consider the action of the space translations, i.e. the groups $\{\mathbb{Z}^d, \mathbb{R}^d\}$ for some $d \in \mathbb{N}$, $d > 1$.

[2] Again other possibilities are relevant, e.g. the case of Quantum Mechanics (in the so called Heisenberg picture) where the algebra of the observable is non commutative and consists of the bounded operators over some Hilbert space.

to situations where the action over the algebra is induced by an action over the set X (this is a map $f : \mathbb{G} \times X \to X$ that satisfies condition 1,2 above).[3] Indeed, given an action f of $\mathbb{G}$ on X and an algebra $\mathcal{A}$ of function on X one can define $\tilde{f}(g,a)(x) := a(gx)$ for all $g \in \mathbb{G}$, $a \in \mathcal{A}$ and $x \in X$. It is then easy to verify that $\tilde{f}$ satisfies conditions 1-4 above.

We will call discrete time Dynamical System the ones in which $\mathbb{G} \in \{\mathbb{N}, \mathbb{Z}\}$ and continuos time Dynamical Systems the ones in which $\mathbb{G} \in \{\mathbb{R}_+, \mathbb{R}\}$. Note that, in the first case, $f(n,x) = f(n-1+1,x) = f(1, f(n-1,x))$, thus defining $T : X \to X$ as $T(x) = f(1,x)$, holds $f(n,x) = T^n(x)$.[4] Thus in such a case we can (and will) specify the Dynamical System by writing only (X,T). In the case of continuos Dynamical Systems we will write $\phi_t(x) := f(t,x)$ and call ϕ_t a flow (if the group is $\mathbb{R}$) or a semi-flow (if the group is $\mathbb{R}_+$) and will specify the Dynamical System by writing (X, ϕ_t). In fact, in this notes we will be interested only in Dynamical Systems with more structure i.e. *topological*, *measurable*, *smooth* Dynamical Systems. By topological Dynamical Systems we mean a triplet $(X, \mathcal{T}, T)$, where $\mathcal{B}$ is a topology and T is continuos (if $B \in \mathcal{T}$, then $T^{-1}B \in \mathcal{T}$). By smooth we consider the case in which X has a differentiable structure and T is r-times differentiable for some $n \in \mathbb{N}$. Finally, a measurable Dynamical Systems is a quadruple (X, Σ, T, μ) where Σ is a σ-algebra, T is measurable (if $B \in \Sigma$, then $T^{-1}B \in \Sigma$) and μ is an invariant measure (for all $B \in \Sigma$, $\mu(T^{-1}B) = \mu(B)$).

So far for general definitions that, to be honest, are not so inspiring. Indeed, what characterizes the modern Dynamical Systems is not so much the setting but rather the type of questions that one asks, first and foremost:

- **What happens for very long times?** (asymptotic theory)
- **Is a given behavior visible in nature?** (stability theory).

The rest of this book will deal in various ways with such questions.

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[3] Again relevant cases are not included, for example all Markov Process where the evolution is given by the action of some semigroup.

[4] Obviously $T^2(x) = T \circ T(x) = T(T(x))$, $T^3(x) = T \circ T \circ T(x) = T(T(T(x)))$ and so on.

The original motivation for this type of setting comes from the study of the motion of some body which, after Newton, typically appears as solution of an ordinary differential equation (ODE). It is then natural to start with a brief reminder of some basic facts concerning ODE.

1.1 Few basic facts about ODE: a reminder

We will be mainly interested in the initial Cauchy problem for ODE, that is

$$\begin{aligned} \dot{x}(t) &= V(x(t), t) \\ x(0) &= x_0 \end{aligned} \tag{1.1.1}$$

where $x : I \to B$, for some open interval $I \subset \mathbb{R}$, and a separable Banach space $\mathcal{B}$,[5] and $V \in \mathcal{C}^0(\mathcal{B} \times \mathbb{R}, \mathcal{B})$.[6] Clearly it must be $x \in \mathcal{C}^1(I, \mathcal{B})$.

In the following applications it will be almost always $\mathcal{B} = \mathbb{R}^d$, for some $d \in \mathbb{N}$, and the reader can chose to substitute $\mathbb{R}^d$ to $\mathcal{B}$ in all the subsequent arguments, yet it is interesting that the theory can be developed for general Banach spaces at no extra cost. In any case, for simplicity, in the following we will always assume that all the Banach spaces are separable even if not explicitly mentioned. In essence, this is just a fancy way of saying that much of the following depends only on the Banach structure of $\mathbb{R}^d$, that is, for example, nowhere is used the fact that $\mathbb{R}^d$ has a finite basis.[7]

The first issue concerning (1.1.1) is the existence of solutions. To

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[5] A Banach spaces is a complete normed vector spaces. This means that a Banach space is a vector space V over the $\mathbb{R}$ or $\mathbb{C}$ equipped with a norm $\|\cdot\|$ such that every Cauchy sequence in V has a limit in V. By *separable* we mean that there exists a countable dense set. Check [RS80, Kat66] for more details or [DS88] for a lot more details.

[6] Given two Banach spaces $\mathcal{B}_1, \mathcal{B}_2$, an open set $U \subset \mathcal{B}_1$, and $q \in \mathbb{N}$ by $\mathcal{C}^q(U, \mathcal{B}_2)$ we mean the functions from U to $\mathcal{B}_2$ that are q time differentiable with continuity. Such a vector space is itself a Banach space when equipped with the norm given by the sup of all its derivatives till the order q included.

[7] At a little extra cost one could consider ODE on manifolds. I will avoid it for simplicity and treat explicitly the few case of needed in the sequel.

address such an issue it is convenient to consider the equation[8]

$$x(t) = x_0 + \int_0^t V(x(s), s)ds \tag{1.1.2}$$

Problem 1.1 *Show that for each interval* $I \subset \mathbb{R}$*, if* $x \in \mathcal{C}^1(I, \mathcal{B})$ *is a solution of* (1.1.1)*, then it is a solution of* (1.1.2)*. Show that if* $x \in \mathcal{C}^0(I, \mathcal{B})$ *is a solution of* (1.1.2) *then is a solution of* (1.1.1)*.*

1.1.1 Existence and uniqueness

The issue of existence and uniqueness is then solved by applying the following fundamental theorem.

Theorem 1.1.1 (Fixed point contraction) *Given a Banach space* $\mathcal{B}$*, a bounded closed set* $A \subset \mathcal{B}$ *and a map* $K : A \to \mathcal{B}$ *if*

i) $K(A) \subset A$*,*

ii) there exists $\sigma \in (0, 1)$ *such that* $\|K(v) - K(w)\| \leq \sigma\|v - w\|$ *for each* $v, w \in A$*,*

then there exists a unique $v_* \in A$ *such that* $Kv_* = v_*$*.*

Proof. Since A is bounded $\sup_{x,y\in A} \|x - y\| = L < \infty$, i.e. it has a finite diameter. Let $a_0 \in A$ and consider the sequence of points defined recursively by $a_{n+1} = K(a_n)$ and the sequence of sets $A_0 = A$ and $A_{n+1} = K(A_n) \subset A$. Let $d_n := \sup_{x,y\in A_n} \|x - y\|$, the diameter of A_n. Then if $x, y \in A_n$, we have

$$\|K(y) - K(x)\| \leq \sigma\|x - y\| \leq \sigma d_n.$$

That is $d_{n+1} \leq \sigma d_n \leq \sigma^n L$. This means that, for each $n, m \in \mathbb{N}$, $a_n, a_0 \in A$ and $a_m, a_{n+m} \in A_m$, hence $\|a_{n+m} - a_m\| \leq \sigma^m L$. That is

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[8] The most convenient meaning of the integral of a function with values in a Banach space is in Bochner sense, which reduces to the usual Lebesgue integral in the case $\mathcal{B} = \mathbb{R}^d$, see [Yos95] for definition and properties. Yet, for our purposes the equivalent of the Riemannian integral suffices and that is defined in the obvious manner, see 1.7 for details.

$\{a_n\} \subset A$ is a Cauchy sequence and, being $\mathcal{B}$ a Banach space, it must have an accumulation point $v_* \in \mathcal{B}$. Moreover since A is closed it must be $v_* \in A$. Clearly

$$\begin{aligned}\|Kv_* - v_*\| &= \lim_{n\to\infty} \|Kv_* - a_n\| = \lim_{n\to\infty} \|Kv_* - Ka_{n-1}\| \\ &\leq \lim_{n\to\infty} \sigma\|v_* - a_{n-1}\| = 0.\end{aligned}$$

Hence, v_* is a fixed point. Next, suppose there exist $u \in A$, $u \neq v_*$, such that $Ku = u$. Then $v_*, u \in A_n$ for each $n \in \mathbb{N}$,[9] thus $\|u - v_*\| \leq \sigma^n L$ for each n, that is $u = v_*$. □

Corollary 1.1.2 *Given a Banach space $\mathcal{B}$ and a map $K : \mathcal{B} \to \mathcal{B}$ with the property that there exists $\sigma \in (0,1)$ such that $\|K(v) - K(w)\| \leq \sigma\|v - w\|$ for each $v, w \in A$, then there exists a unique $v_* \in \mathcal{B}$ such that $Kv_* = v_*$.*

PROOF. To prove the theorem, for each $L \in \mathbb{R}_+$ consider the sets $B_L := \{v \in \mathcal{B} : \|v\| \leq L\}$. Then $\|K(v)\| \leq \|K(v) - K(0)\| + \|K(0)\| \leq \sigma\|v\| + \|K(0)\| \leq \sigma L + \|K(0)\|$. Thus, for each $L \geq (1-\sigma)^{-1}\|K(0)\|$ we have that $K(B_L) \subset B_L$. The statement follows then by applying Theorem 1.1.1. □

Theorem 1.1.3 (Existence and Uniqueness theorem for ODE) *For each $V \in \mathcal{C}^1(\mathcal{B} \times \mathbb{R}, \mathcal{B})$ there exists $\delta \in \mathbb{R}_+$ such that there exists a unique solution of (1.1.2) in $\mathcal{C}^0((-\delta, \delta), \mathcal{B})$.*

PROOF. Let $\delta \in (0,1)$. The reader can verify that the vector space $\mathcal{C}^0([-\delta,\delta], \mathcal{B})$, equipped with the norm $\|u\|_\infty := \sup_{t\in[-\delta,\delta]} \|u(t)\|$ is a Banach space.[10] For each $u \in \mathcal{C}^0([-\delta,\delta], \mathcal{B})$ let us define the operator $K : \mathcal{C}^0([-\delta,\delta], \mathcal{B}) \to \mathcal{C}^0([-\delta,\delta], \mathcal{B})$ by

$$K(u)(t) := x_0 + \int_0^t V(u(s), s)ds.$$

For $L = \max\{2\|x_0\|, 1\}$ let $M = \sup_{|t|\leq 1, \|u\|\leq L}\{\|V(u,t)\| + \|\partial_u V(u,t)\|\}$, then for each $u \in B_L := \{u \in \mathcal{C}^0([-\delta,\delta], \mathcal{B}) \;:\; \|u\|_\infty \leq L\}$ holds

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[9] Since $u = K^n u$ and $v_* = K^n v_*$.

[10] It suffices to remember that the uniform limit of continuous functions is a continuos function.

$\|K(u(t))\| \leq \|x_0\| + \delta M \leq L$ provided we chose $\delta \leq (L - \|x_0\|)M^{-1}$. In addition, for each $u, v \in B_L$,

$$\|K(u) - K(v)\|_\infty \leq \delta M \|u - v\|_\infty \leq \frac{1}{2}\|u - v\|_\infty,$$

provided we chose $\delta \leq (2M)^{-1}$. We can then apply Theorem 1.1.1 and obtain a unique solution of the equation $Ku = u$ in B_L. The Theorem follows then by remembering Problem 1.1. □

Remark 1.1.4 *Note that in the proof of Theorem 1.1.1 one can chose the same δ for an open set of initial condition.*

Remark 1.1.5 *The hypotheses of the above Theorem can be easily weakened to V locally Lipschitz, yet only continuity does not suffice for uniqueness as shown by the example*

$$\begin{aligned} &\dot{x} = \sqrt{x} \\ &x(0) = 0. \end{aligned}$$

which has the solutions $x(t) = 0$ and $x(t) = \frac{1}{4}t^2$.[11]

Remark 1.1.6 *The introduction of δ in Theorem 1.1.1 cannot be avoided as shown by the example*

$$\begin{aligned} &\dot{x} = x^2 \\ &x(0) = 1. \end{aligned}$$

which has the solution $x(t) = (1-t)^{-1}$ which is not continuous, nor bounded, for $t = 1$.

We have seen a mechanism whereby the solution cannot be defined for all times, the next Lemma shows that, for $\mathcal{C}^1$ vector fields is the *only* mechanism.[12]

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[11] If $\mathcal{B}$ is finite dimensional, then for the existence $V \in \mathcal{C}^0$ suffices. This follows by a direct application of Schauder fixed point Theorem, for informations on such a fixed point theorem and fixed point theorem in general see [Zei86].

[12] I state the result for positive times, for negative times is the same.

Lemma 1.1.7 *In the hypotheses of Theorem 1.1.3, if* $x \in \mathcal{C}^1([0,\delta),\mathcal{B})$ *is a solution of* (1.1.1) *and is there exists* $M > 0$ *such that* $\sup_{t\in[0,\delta)} \|x(t)\| \leq M$*, then there exists* $\bar{\delta} > \delta$ *and* $\bar{x} \in \mathcal{C}^1([0,\bar{\delta}),\mathcal{B})$ *that solves* (1.1.1) *(i.e. the solution can be extended for longer times).*

Proof. Let $\{t_n\}$ any sequence that converges to δ, then

$$\|x(t_n)-x(t_m)\| \leq \int_{t_n}^{t_m} \|V(x(s),s)\|ds \leq |t_n-t_m| \sup_{\|z\|\leq M} \sup_{s\in[0,\delta)} \|V(z,s)\|$$

Thus $\{x(t_n)\}$ is a Cauchy sequence and admits a limit $x_* \in \mathcal{B}$. By the same argument it follows $x_* = \lim_{t\to\delta} x(t)$. We can then consider the equation

$$y(t) = x_* + \int_\delta^t V(y(s),s)ds.$$

By Theorem 1.1.3 there exists δ_1 and $y \in \mathcal{C}^1((\delta-\delta_1,\delta+\delta_1),\mathcal{B})$ which satisfy the above equation. Let then $\bar{\delta} = \delta + \delta_1$ and define then

$$\bar{x}(t) := \begin{cases} x(t) & \text{fot all } t \in [0,\delta) \\ y(t) & \text{fot all } t \in [\delta,\bar{\delta}). \end{cases}$$

Clearly $\bar{x} \in \mathcal{C}^0([0,\bar{\delta}),\mathcal{B})$ and, for $t \in [0,\delta)$ holds true

$$\begin{aligned} \bar{x}(t) &= x_* + \int_\delta^t V(\bar{x}(s),s)ds = x_0 + \int_0^\delta V(\bar{x}(s),s)ds + \int_\delta^t V(\bar{x}(s),s)ds \\ &= x_0 + \int_0^t V(\bar{x}(s),s)ds. \end{aligned}$$

Thus, again by Theorem 1.1.3 and remembering 1.1 the Lemma follows.

□

1.1.2 Growald inequality

To understand a bit better how the solution of an ODE can grow in time we will consider a simple case. Before doing so we need a technical but extremely useful Lemma.

Lemma 1.1.8 (Integral Gronwald inequality) *Let* $L, T \in \mathbb{R}_+$ *and* $\xi, f \in \mathcal{C}^0([0,T], \mathbb{R})$. *If, for all* $t \in [0,T]$,

$$\xi(t) \leq L \int_0^t \xi(s)\, ds + f(t),$$

then

$$\xi(t) \leq f(t) + L \int_0^t e^{L(t-s)} f(s)\, ds.$$

Proof. Let us first consider the case in which $f \equiv 0$. In this case the Lemma asserts $\xi(t) \leq 0$. Indeed, since ξ is a continuos function there exists $t_* \in [0, (2L)^{-1}] \cap [0,T] =: I_1$ such that $\xi(t_*) = \sup_{t \in I_1} \xi(t)$. But then,

$$\xi(t_*) \leq L \int_0^{t_*} \xi(s)\, ds \leq \xi(t_*) L t_* \leq \frac{1}{2} \xi(t_*)$$

which implies $\xi(t_*) \leq 0$ and hence $\xi(t) \leq 0$ for each $t \in I_1$. If $I_1 = [0,T]$, then we are done, otherwise letting $t_1 := (2L)^{-1}$ we have

$$\xi(t) \leq L \int_{t_1}^t \xi(s)\, ds$$

and we can make the same argument as before in the interval $[t_1, 2t_1]$. Iterating we have $\xi(t) \leq 0$ for all $t \in [0,T]$.

To treat the general case we reduce to the previous one. Let

$$\zeta(t) := \xi(t) - f(t) - L \int_0^t e^{L(t-s)} f(s)\, ds.$$

Then

$$\begin{aligned}
\zeta(t) &\leq L \int_0^t \xi(s)\, ds - \int_0^t L e^{L(t-s)} f(s)\, ds \\
&= L \int_0^t \zeta(s)\, ds + L \int_0^t \left\{ f(s) ds + L \int_0^s e^{L(s-\tau)} f(\tau) d\tau \right\} \\
&\quad - \int_0^t L e^{L(t-s)} f(s)\, ds.
\end{aligned}$$

Next, notice that

$$\begin{aligned}
\int_0^t ds L \int_0^s e^{L(s-\tau)} f(\tau)\, d\tau &= L \int_0^t d\tau f(\tau) \int_\tau^t ds e^{L(s-\tau)} \\
&= \int_0^t f(s) \{ e^{L(t-s)} - 1 \} ds.
\end{aligned}$$

Thus,

$$\zeta(t) \leq L \int_0^t \zeta(s)\, ds.$$

We have then reduced the problem to the previous case which implies that it must be $\zeta(t) \leq 0$ from which the Lemma follows. □

We can now discuss the announced examples. Let $L(\mathcal{B}, \mathcal{B})$ be the Banach space of the linear bounded operators from $\mathcal{B}$ to $\mathcal{B}$.[13]

Lemma 1.1.9 *For each $A \in \mathcal{C}^1(\mathbb{R}, L(\mathcal{B}, \mathcal{B}))$, consider the Cauchy problem*

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) \\ x(0) &= x_0. \end{aligned}$$

If $\|A(t)\| \leq L$ for all $0 \leq t \leq T \in \mathbb{R}$, then $\|x(t)\| \leq e^{Lt}\|x_0\|$ for all $0 \leq t \leq T$.

Proof. If we write the equation in the equivalent integral form we have

$$\|x(t)\| \leq \|x_0\| + \int_0^t \|A(s)x(s)\|\, ds \leq \|x_0\| + L \int_0^t \|x(s)\|\, ds.$$

Setting $\xi(t) := \|x(t)\|$ and applying Lemma 1.1.8 the Lemma follows. □

Corollary 1.1.10 *In the hypotheses of the previous Lemma the solution is defined on all $\mathbb{R}$.*

1.1.3 Flows

In this section we analize th case in which the vector field is time independent and grows at most linearly.

Lemma 1.1.11 *Given $V \in \mathcal{C}^1(\mathcal{B}, \mathcal{B})$, if there exists $L \geq 0$ such that $\|V(x)\| \leq L\|x\|$, then the solution of (1.1.1) exists for all times and for all initial conditions.*

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[13] The norm of $L \in L(\mathcal{B}, \mathcal{B})$ is given by $\|L\| := \sup_{\substack{v \in \mathcal{B} \\ \|v\|=1}} \|Lv\|$. If $\mathcal{B} = \mathbb{R}^d$, then $L(\mathcal{B}, \mathcal{B})$ is just the vector space of the $d \times d$ matrices.

PROOF. For each $x_0 \in \mathcal{B}$ we have that there exists $\delta > 0$ such that for all $0 \leq t \leq \delta$

$$\|x(t)\| \leq \|x_0\| + L \int_0^t \|x(s)\| ds.$$

Thus Gronwald inequality implies $\|x(t)\| \leq \|x_0\| e^{Lt}$ for $t \in (0, \delta)$. This implies $\|\dot{x}\| = \|V(x(t))\| \leq L\|x(t)\| \leq \|x_0\| e^{L\delta}$. The Lemma follows by applying Lemma 1.1.7. □

For each $x_0 \in \mathcal{B}$ and $t \in \mathbb{R}$ we can then define $x(t, x_0)$ to be the solution of (1.1.1) at time t.

Lemma 1.1.12 *For each V as in Lemma 1.1.11, setting $\phi_t(x_0) := x(t, x_0)$, $\phi_{-t} = \phi_t^{-1}$ for $t \geq 0$, we have that $(\mathcal{B}, \phi_t)$, $t \in \mathbb{R}$, is a Dynamical System.*

PROOF. All we need to prove is that ϕ_t is an action of $\mathbb{R}$ on $\mathcal{B}$. First of all note that ϕ_t is indeed invertible. If not then there would be $x, x' \in \mathcal{B}$ such that $\phi_t(x) = \phi_t(x')$. But then the uniqueness of the solutions of the ODE implies $x = x'$. Moreover it is easy to check that $\phi_{-t}(x_0) = x(-t, x_0)$. Finally, $\phi_t(\phi_s(x)) = \phi_{t+s}(x)$. □

We have thus proved that a large class of vector fields gives rise to *flows*.

1.1.4 Dependence on a parameter

Once one knows the existence and uniqueness of the solution the first natural question is to ask how does the solution depend on the initial condition. To this end it is convenient to first analyze a related problem: the dependence form an external paramenter. A first fact is given by a general results extending Theorem 1.1.1.

Theorem 1.1.13 *Given Banach spaces $\mathcal{B}, \mathcal{B}_1$, a bounded closed set $A \subset \mathcal{B}$, an open set $U \in \mathcal{B}_1$ and a map $K : A \times U \to \mathcal{B}$. If*

- *i)* $K(A, \lambda) \subset A$, *for each* $\lambda \in U$;
- *ii) there exists* $\sigma \in (0, 1)$ *such that* $\|K(v, \lambda) - K(w, \lambda)\| \leq \sigma \|v - w\|$ *for each* $v, w \in A$, $\lambda \in U$;

iii) there exists $L \geq 0$ such that $\|K(v,\lambda) - K(v,\lambda')\| \leq L\|\lambda - \lambda'\|$ for each $v \in A$, $\lambda, \lambda' \in U$;

then for each $\lambda \in U$, there exists a unique $v(\lambda) \in A$ such that $K(v(\lambda),\lambda) = v(\lambda)$ and, for each $\lambda, \lambda' \in U$

$$\|v(\lambda) - v(\lambda')\| \leq (1-\sigma)^{-1} L \|\lambda - \lambda'\|. \tag{1.1.3}$$

Proof. The existence of $v(\lambda)$ follows by Theorem 1.1.1, it thus suffices to prove (1.1.3).

$$\begin{aligned}\|v(\lambda) - v(\lambda')\| &= \|K(v(\lambda),\lambda) - K(v(\lambda'),\lambda')\| \\ &\leq \|K(v(\lambda),\lambda) - K(v(\lambda'),\lambda) + L\|\lambda - \lambda'\| \\ &\leq \sigma\|v(\lambda) - v(\lambda')\| + L\|\lambda - \lambda'\|\end{aligned}$$

which readily implies the result. □

If we take $K(u,\lambda)(t) = \lambda + \int_0^t V(u(s),s)ds$, we can apply the above theorem and obtain a Lipschitz dependence on the initial conditions. In fact, more holds true. As an example of what can be done I provide a theorem which hypotheses are not optimal but suffice for our purposes.

Theorem 1.1.14 (Smooth dependence on a parameter) *Let $V \in \mathcal{C}^2(\mathcal{B} \times \mathbb{R} \times \mathbb{R}^d, \mathcal{B})$, and let $\delta > 0$, $A \subset \mathcal{B}$ and $U \subset \mathbb{R}^d$ so that Theorem 1.1.1 applies for all $x \in A$, $\lambda \in U \subset \mathcal{B} \times \mathbb{R} \times \mathbb{R}^d$. For each $x_0 \in A$, $\lambda \in U$, let $X(t, x_0, \lambda)$ be the unique solution of (1.1.1). Then, , for each $(x_0, t) \in A \times [-\delta, \delta]$, $X(t, x_0, \cdot) \in \mathcal{C}^1(U, \mathcal{B})$.*

Proof. For each $x_0 \in U$ let us consider the ODE in $L(\mathcal{B}, \mathcal{B})$

$$\begin{aligned}&\dot\xi(t) = \partial_x V(X(t,x_0,\lambda),t,\lambda) \cdot \xi(t) + \partial_\lambda V(X(t,x_0,\lambda),t,\lambda) \\ &\xi(0) = 0.\end{aligned} \tag{1.1.4}$$

We claim that $\xi(t) = \partial_\lambda X(t,x_0,\lambda)$. To verify it suffices to prove that there exists $C > 0$ such that, for $h \in \mathcal{B}$ small enough, if $\zeta(t,h,\lambda) := X(t,x_0,\lambda+h) - X(t,x_0,\lambda) - \xi(t)h$, then $\|\zeta(t,h)\| \leq C\|h\|^2$. By Taylor formula we have

$$\begin{aligned}\dot\zeta(t,h) &= V(X(t,x_0,\lambda+h),t,\lambda+h) - V(X(t,x_0),t,\lambda) \\ &\quad - \partial_x V(X(t,x_0,\lambda),t) \cdot \xi(t)h - \partial_\lambda V(X(t,x_0,\lambda),t,\lambda)h \qquad (1.1.5) \\ &= \partial_x V(X(t,x_0),t) \cdot \zeta(t,h) + R(t)\end{aligned}$$

where

$$\begin{aligned}\|R(t)\| &\leq C\left(\|X(t,x_0+\xi_0)-X(t,x_0)\|^2+|h\|^2\right)\\ &\leq 2C(\|\zeta(t,h)\|^2+\|\xi(t)\|^2\|h\|^2).\end{aligned}$$

with $C=\|V\|_{\mathcal{C}^2}$. Note that $\zeta(0)=0$. We can then conclude by using Lemma 1.1.8. Indeed such a Lemma applied to (1.1.6) implies $\|\xi(t)\|\leq e^{Ct}$. Next, let $T>0$ be the maximal time such that $\|\zeta(t,h)\|\leq 1/2$ and $e^{Ct}\leq 2$. Then, for $t\leq T$, (1.1.5) yields

$$\|\zeta(t,h)\|\leq\int_0^t 2C\|\zeta(s)\|ds+e^{2Ct}\|h\|^2$$

and Lemma 1.1.8 implies the announced estimate. □

Corollary 1.1.15 (Smooth dependence on the initial conditions) *Let $V\in\mathcal{C}^2(\mathcal{B}\times\mathbb{R},\mathcal{B})$, and let δ so that Theorem 1.1.1 applies for all x_0 in some open set U. For each $x_0\in U$ let $X(t,x_0)$ be the unique solution of (1.1.1), then, for each $t\in[-\delta,\delta]$, $X(t,\cdot)\in\mathcal{C}^1(U,\mathcal{B})$ and $\xi=\partial_{x_0}X$ is a soltuon of*

$$\begin{aligned}&\dot{\xi}(t)=\partial_x V(X(t,x_0),t)\cdot\xi(t)\\ &\xi(0)=\mathbb{1}.\end{aligned}\tag{1.1.6}$$

Proof. Set $z=x-x_0$ and consider the resulting equation

$$\begin{aligned}&\dot{z}=V(z+x_0,t)=:\bar{V}(z,t,x_0)\\ &z(0)=0.\end{aligned}$$

One can then consider x_0 as an external parameter, applying Theorem 1.1.14 yields the result. □

1.2 Linear ODE and Floquet theory

Let us briefly discuss the simplest possible differential equation: the affine ones. We restrict ourselves to the case $\mathcal{B}=\mathbb{R}^d$ for some $d\in\mathbb{N}$ since we will use some spectral theory which is substantially more complex in the general case.

1.2.1 Linear equations

Consider

$$\begin{aligned}\dot{x} &= Ax\\ x(0) &= x_0.\end{aligned} \tag{1.2.7}$$

Problem 1.2 *Show, by induction, that for each $n \in \mathbb{N}$ the solution of* (1.2.7) *satisfies*

$$x(t) = \sum_{k=0}^{n} \frac{1}{k!} A^k t^k x_0 + \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n A^{n+1} x(t_n).$$

Taking the limit for $n \to \infty$ in the above expression one readily obtains $x(t) = \sum_{n=0}^{\infty} \frac{1}{n!} A^n t^n x_0$. That this is a solution can be also by verified directly inserting this formula in (1.2.7) (and noticing that the series and the series obtained by deviating term by term are uniformly convergent). By the standard analytic functional calculus for matrices (and operators, see Appendix B) we can thus write $x(t) = e^{At} x_0$.

The above discussion provides a general solution for all equations of the type 1.2.7. In reality life it is not that simple: if one has a concrete matrix A and wants to compute e^{At}, this is may be quite unpleasant. A general strategy, although not necessarily the simplest one, is to perform a linear change of variables $x = Uz$. Then $\dot{z} = U^{-1}AUz$, where U is chosen so that $\Lambda = U^{-1}AU$ is in Jordan normal form. Then

$$x(t) = Uz(t) = Ue^{\Lambda t} z_0 = Ue^{\Lambda t} U^{-1} x_0.$$

It suffices then to know how to take exponentials of Jordan blocks, and this can be computed by using the defining series.

Problem 1.3 *Compute $e^{\Lambda t}$ for*

$$\Lambda = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \;;\quad \Lambda = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} \;;\quad \Lambda = \begin{pmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix}.$$

Another, equivalent point of view, is to look for solutions of the type $x(t) = e^{at} v$, substituting in the first of (1.2.7) one obtains $av = Av$. Thus, as we know already, each eigenvalue of A provides a solution of

(1.2.7) (ignoring the initial condition). If there exists real eigenvectors $\{v_i\}_{i=1}^d$ which span all $\mathbb{R}^d$ then one can write the general solution, depending on d parameters α_i, as $x(t) = \sum_{i=1}^d \alpha_i v_i e^{a_i t}$, where a_i is he eigenvalue associated to the eigenvector v_i. One can then satisfy the initial condition by solving $x_0 = \sum_{i=1}^d \alpha_i v_i$. The same can be done is the eigenvectors are complex, by working in $\mathbb{C}^d$ instead then $\mathbb{R}^d$. If Jordan blocks are present one can look for solutions of the form $x(t) = \sum_{k=0}^p \frac{1}{(p-k)!} t^k e^{at} v_k$, compare this with your solution of Exercise 1.3.

Remark 1.2.1 *Note that the matrix A does not have eigenvalues with zero imaginary part then (by spectral decomposition) one can write $\mathbb{R}^d = V_- \oplus V_+$, where $AV_\pm = V_\pm$ and A restricted to V_- has eigenvalues with negative real part while on V_+ has eigenvalues with positive real part. Hence if $x_0 \in V_-$ it will hold $\lim_{n\to\infty} x(t) = 0$, and if $x_0 \in V_+$ it will hold $\lim_{n\to\infty} \|x(t)\| = \infty$. Accordingly if $x_0 \notin V_-$ we can write it as $x_0 = x_- + x_+$, where $x_\pm \in V_\pm$, then it will hold $\lim_{n\to\infty} \|x(t)\| = \infty$ and the trajectory will escape to infinity while getting exponentially close to the subspace V_+. This is our first* long time *result.*

A slightly more complex situation is given by

$$\begin{aligned} \dot{x} &= Ax + b(t) \\ x(0) &= x_0, \end{aligned} \tag{1.2.8}$$

where $b \in \mathcal{C}^0(\mathbb{R}, \mathbb{R}^d)$. The solution of (1.2.8) is given by[14]

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-s)} b(s) ds.$$

1.2.2 Floquet theory

Let us consider the simplest case of a linear time dependent equation: the case when there exists T such that $A(t+T) = A(t)$. More precisely, let $\Phi(x_0, s, t)$ be the solution of the Cauchy problem[15]

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) \\ x(s) &= x_0. \end{aligned} \tag{1.2.9}$$

---

[14] To obtain it just look for a solution of the form $x(t) = e^{At} z(t)$ and deduce the differential equation for z.

[15] The solution is well defined for all times by Lemma 1.1.11.

It is an easy exercise to verify $\Phi(ax_0+by_0,s,t)=a\Phi(x_0,s,t)+b\Phi(y_0,s,t)$. This means that there exists $K\in\mathcal{C}^1(\mathbb{R}^2,L(\mathcal{B},\mathcal{B}))$ such that $\Phi(x_0,s,t)=K(s,t)x_0$. In addition, $\Phi(x_0,s+T,t+T)=\Phi(x_0,s,t)$, thus $K(s+T,t+T)=K(s,t)$. The next step is the first occurrence in this book of a very simply but very powerful idea to analyze dynamical systems: a Poincaré section. Essentially the idea consist in looking at the system only at specially selected moments in time. In this case it is convenient to look at $t\in\{nT\}_{n\in\mathbb{Z}}$. That is we want to investigate $\Phi(x_0,0,nT)=:F(x_0,n)$.

Lemma 1.2.2 *The couple $(\mathbb{R}^d,F)$ is a discrete Dynamical System.*

PROOF. We have to show that F is an action of $\mathbb{Z}$ on $\mathbb{R}^d$. Let $F(x_0,n)=f^n(x_0)$ where $f(x_0):=F(x_0,1)$.

$$\begin{aligned}F(x_0,n)&=\Phi(\Phi(x_0,0,(n-1)T),(n-1)T,T))\\&=\Phi(\Phi(x_0,0,(n-1)T),0,T))=f(\Phi(x_0,0,(n-1)T))=f^n(x_0).\end{aligned}$$

In addition, note that the uniqueness of the solutions of the ODE implies that $f(x_0)=0$ implies $x_0=0$. Now, by construction, $f(x_0)=K(0,T)x_0$, thus $K(0,T)$ is an invertible matrix. Hence $F(x_0,-n)=f^{-n}(x_0)$ for all $n\in\mathbb{N}$. □

By using the functional calculus (see Problem **??**) one can define $B:=T^{-1}\ln K(0,T)$, so $e^{BT}=K(0,T)$.[16] Let us now consider $P(t):=K(0,t)e^{-Bt}$.

$$\begin{aligned}P(t+T)&=K(0,t+T)e^{-B(t+T)}=K(T,t+T)K(0,T)K(0,T)^{-1}e^{-Bt}\\&=K(0,t)e^{-Bt}=P(t),\end{aligned}$$

since, for all $x_0\in\mathbb{R}^d$,

$$\begin{aligned}K(0,t+T)x_0&=\Phi(x_0,0,t+T)=\Phi(\Phi(x_0,0,T),T,t)=\Phi(\Phi(x_0,0,T),0,t)\\&=K(0,t)K(0,T)x_0.\end{aligned}$$

We have just proven Floquet theorem.

---

[16] Mention also Jordan block theory and give example for simple blocks, show that $\ln K(0,2T)$ is a real matrix.

Theorem 1.2.3 *The solutions of the equation* (1.2.9) *can be written as* $x(t) = P(t)e^{Bt}x_0$ *where* $P(t+T) = P(t)$ *is periodic.*

Note that the matrix B can be complex valued. This can be avoided at a little extra cost.

Problem 1.4 *Prove that the solutions of the equation* (1.2.9) *can be written as* $x(t) = P(t)e^{Bt}x_0$ *where* B *is real and* $P(t+2T) = P(t)$ *is periodic of period* $2T$.

Note that Theorem 1.2.3 implies that the long time behavior is completely contained in the eigenvalues of the matrix B often called *floquet exponents.*

Problem 1.5 *Find the solutions of*

$$\dot{x} = a(t)Ax$$

where $a \in \mathcal{C}^0(\mathbb{R}, \mathbb{R})$ *is periodic of period* t *and* A *is a fixed matrix.*

Problem 1.6 *Given a fixed matrix* A *and a function at matrix values* $B(t)$ *of period* T*, consider the equation* $\dot{x} = (A + \varepsilon B(t))x$, $\varepsilon \in \mathbb{R}$. *Show that, for* ε *small enough, calling* ν_i *the Floquet exponents and setting* $\lambda_i = e^{\nu_i}$ *(often called* Floquet multiplier*), holds that the* λ_i *are* ε*-close to the eigenvalues of* A.

1.3 Qualitative study of ODE

The previous discussion has shed some light on the behavior of linear ODE, unfortunately the interesting ODE are typically non linear. Although some nonlinear ODE can be solved explicitly (see any ODE book for examples) typically this is not possible, hence the need of a qualitative theory. As for the qualitative study of functions this can be quite naively in one an two dimensions, while higher dimensions requires some non trivial theory. Let us see such a naive qualitative theory for ODE via few examples.

1.3.1 The one dimensional case

This situation is very similar to the study of the graph of a function of one variable. Indeed to draw the graph one studies the first derivative and here the first derivative is specified by the equation. Let us consider a couple of simple examples. Consider

$$\begin{aligned}&\dot{x} = e^{-x^2} + x - 2 = V(x)\\ &x_0 = 0.\end{aligned}$$

One cannot integrate the function $V(x)^{-1}$ (which would yield an explicit solution of the ODE), yet from the equation follows that there exists a close to 2 such that $\dot{x}$ is negative if $x \leq a$ and positive otherwise. This implies that the solution starts to be decreasing and keeps decreasing forever. Next, consider

$$\begin{aligned}&\dot{x} = 1 - 2tx\\ &x_0 = a.\end{aligned}$$

Such an equation cannot be solved by separation of variables, yet the above arguments still apply. In particular, for $t \geq 0$, we have $\dot{x}(t) < 0$ iff $x(t) > \frac{1}{2t}$. On the other hand if $x(t) > \frac{1}{2t}$ it will be so forever. In fact, consider $g(t) = x(t) - \frac{1}{2t}$, then $g'(t) = \dot{x}(t) + \frac{1}{2t^2}$. So if $g(t) = 0$, then $g'(t) > 0$ hence for $t' < t$ one has $g(t') < 0$. Thus the solution will increase until it will intersect the curve $\frac{1}{2t}$ and then it will start decreasing but always staying above such a curve.

1.3.2 Autonomous equations in two dimensions

In this case the basic idea is to consider one component as a function of the other and in this way reduce to the previous case. Let us see some examples.

Van Der Pol equation

Consider the equation

$$\begin{aligned}&\dot{x} = y - f(x)\\ &\dot{y} = -x\end{aligned}$$

with $f(x) = x^3 - x$. The idea is to forget the time dependence and concentrate only on the shape of the trajectories. Thus we can represent trajectories on the xy plane. Indeed, apart from the point $(0,0)$, either $\dot{x}$ or $\dot{y}$ are different from zero. In the first case one can locally invert $x(t)$ and write $y(x) = y(t(x))$. When this is possible one obtains

$$\frac{dy}{dx} = \frac{x}{f(x) - y},$$

which can be studied as in the previous examples.

Volterra equation

$$\begin{aligned}\dot{x} &= ax - Ax^2 - \lambda xy\\ \dot{y} &= -dy + \lambda xy.\end{aligned}$$

Higer order

Consider the equation

$$\begin{aligned}&\ddot{x} = -\gamma\dot{x} + \frac{x^2}{1+x^4}\\ &x(0) = 0; \quad \dot{x}(0) = v.\end{aligned}$$

Setting $(z, w) = (x, \dot{x})$, we can write it as

$$\begin{aligned}\dot{z} &= w\\ \dot{w} &= -\gamma w + \frac{z^2}{1+z^4}\end{aligned}$$

which is the type discussed above.

Clearly if we consider higher still higher dimensional case the above naive approach cannot help us very much, hence the need of a more sophisticated theory.

Problems

1.7. Given a compact interval $I \subset \mathbb{R}$, a Banach space $\mathcal{B}$, and a continuous function $f \in \mathcal{C}^0(I, \mathcal{B})$, shows that one can define the equivalent of the Riemannian integral.

1.8. A more general strategy to define the integral (simpler, but weaker, than the Bochner one) is as follows. Consider an interval $I \subset \mathbb{R}$, a Banach space $\mathcal{B}$, and a bounded function $f : I \to \mathcal{B}$), such that there exist a sequence of simple functions[17] $\{s_n\}$ with the property

$$\lim_{n\to\infty}\int_I \|f(t) - s_n(t)\| dt = 0.$$

Shows that there exists $A \in \mathcal{B}$ such that for each $\ell \in \mathcal{B}'$ holds[18]

$$\ell(A) = \int_I \ell(f(t))dt.$$

1.9. Study the solutions of the following equations for all possible initial conditions and $p \in \mathbb{N}$

$$\begin{aligned}\dot{x} &= |x|^p \\ \dot{x} &= x(\ln|x|)^p\end{aligned}$$

1.10. Let $K \in \mathcal{C}^1(\mathbb{R} \times [0,1])$. Show that the equation

$$\begin{aligned}&\partial_t u(t,s) = \int_0^1 K(t+s,\tau)u(t,\tau)d\tau \\ &u(0,s) = s^2.\end{aligned}$$

has a unique continuos solution for t small enough.

1.11. Under the same hypotheses of Problem 1.6 show that if $\int_0^T B(s)ds = 0$ and the eigenvalues of A have all multiplicity one, then the Floquet multiplier differ from the eigenvalues of e^{AT} only of order ε^2.

1.12. Study the equation

$$(1+x)y\dot{y} + (x+y^2) = 0.$$

---

[17] That is s_n is a piecewise constant function.

[18] Remember that $\mathcal{B}'$ is the vector space of all the linear *bounded* function from $\mathcal{B}$ to $\mathbb{R}$ (or $\mathcal{C}$ if the Banach space is over the field of complex numbers). When equipped with the norm $\|\ell\| := \sup_{\|v\|\leq 1}|\ell(v)|$ it is itself a Banach Space. See [RS80] for details.

1.13. Study the equation (Bernoulli)

$$\dot{y} + p(x)y = q(x)y^n.$$

1.14. Study the equation

$$\ddot{x} = -\gamma\dot{x} - x^3.$$

Hints to solving the Problems

In this section I give hints for the solution of some of the Problems. It is a very very good idea to try very hard to solve the problems *before* looking at the hints. Also I suggest one way to proceed, often other ways are possible and, maybe, even better.

1.1. Start by showing that $x \in \mathcal{C}^1(I, \mathcal{B})$.

1.2. For $n = 0$ it is just (1.1.2). To verify it for any n it suffices to show that

$$\int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n 1 = \frac{t^n}{(n+1)!}.$$

This follows since the domain of integration is $D = \{x \in [0,t]^{n+1} : t_{n+1} \leq t_n \leq \cdots \leq t\}$. On the other hand, for each permutation σ of the set $\{1, \dots, n+1\}$ the sets $D_\sigma = \{x \in [0,t]^{n+1} : t_{\sigma_{n+1}} \leq t_{\sigma_n} \leq \cdots \leq t\}$ have the same measure, all the D_σ are disjoint and the union of all of them gives $[0,t]^{n+1}$.

1.3. First notice that if a matrix has no eigenvalues on the negative axis then the contour γ in B.3.1 can be taken symmetric around the real axis and, by using B.3.1 with the standard definition of ln with a cut on the negative real axis, this defines $\ln K(0,T)$ with real entries (since the formula for his complex conjugate is the same). In general use the spectral decomposition to write $K(0,T) = C + D$ where $\sigma(C) \cap \mathbb{R}_- = \emptyset$ and $\sigma(D) \subset \mathbb{R}_-$. Then $\sigma(D^2) \subset \mathbb{R}_+$, hence $B = \frac{1}{T}\ln C + \frac{1}{2T}\ln D^2$ is real and $e^{2BT} = C^2 + D^2 = K(0,T)^2$. The rest of the argument is as before.

1.4. Show that the solution satisfies

$$x(t) = e^{At}x_0 + \varepsilon \int_0^t e^{A(t-s)}B(s)x(s)ds.$$

and apply the perturbation theory in Appendix B.

1.5. Let $I = [a, b]$. Since the function is continuos, it is uniformly continuous, hence for $\varepsilon > 0$ there exists $\delta > 0$ such that, for each partition $\xi = \{[x_0, x_1], \dots, [x_{n-1}, x_n]\}$, $x_0 = a, x_n = b$, $x_{n+1} - x_n \leq \delta$, holds $\sup_{z,y\in[x_{n+1},x_n]} \|f(z) - f(y)\| \leq \varepsilon$. Accordingly, for each choice of $z_n, y_n \in [x_{n+1}, x_n]$ we have

$$\left\| \sum_{k=0}^{n-1} f(z_k)(x_{k+1} - x_k) - \sum_{k=0}^{n-1} f(y_k)(x_{k+1} - x_k) \right\| \leq \varepsilon.$$

By similar arguments one can compare the sum defined on one partition with the sum defined on a finer partition. Finally sum on different partitions can be compared with the sum on the coarser partition finer of both. This shows that all sufficiently fine partitions yield the same approximate value, hence one can consider the partitions $\xi_n = \{[a + i\frac{b-a}{n}, a + (i+1)\frac{b-a}{n}]\}_{i=0}^{n-1}$ and define

$$\int_I f(t)dt := \lim_{n\to\infty} \sum_{i=0}^{n-1} f(a + i\frac{b-a}{n})\frac{b-a}{n}.$$

By the above discussion this is equivalent to the same limit taken along any other partition the diameter of which elements tend uniformly to zero.

1.6. Since f is bounded, i.e. exists $M > 0$ such that $\sup_{t\in I} \|f(t)\| \leq M$, for each $\ell \in \mathcal{B}'$ holds $\left|\int_I \ell(f(t))dt\right| \leq |I|\, M \,\|\ell\|$. This means that the integral defines and element $\tilde{A}$ of the double dual $(\mathcal{B}')'$, which normally is larger that $\mathcal{B}$. On the other hand for $s_n = \sum_i \alpha_i \mathbb{1}_{I_i}$, where I_i are the interval of constancy of s_n, we can define $A_n = \sum_i \alpha_i |I_i|$. Clearly $\ell(A_n) = \int_I \ell(s_n(t))dt$. By hypothesis A_n is a Cauchy sequence, hence will have a limit $A \in \mathcal{B}$, since for each $\ell \in \mathcal{B}'$ holds

$$\ell(A) = \lim_{n\to\infty} \ell(A_n) = \lim_{n\to\infty} \int_I \ell(s_n(t))dt \int_I \ell(f(t))dt = \ell(\tilde{A})$$

it follows $\tilde{A} = A \in \mathcal{B}$.

1.7. Consider the Banach space $\mathcal{B} = \mathcal{C}^0([0,1],\mathbb{R})$. Then $u(t,\cdot) \in \mathcal{B}$ and one can apply Theorem 1.1.3.

1.8. By Problem 1.6 we know that the solution at time T is given by the matrix $D(\varepsilon) := e^{AT}\left[\mathbb{1} + \varepsilon \int_0^T e^{-As}B(s)e^{As}ds\right]$. By the results in Appendix B it follows that, for ε small enough, the eigenvalues of $D(\varepsilon)$ are still simple and analytic on ε. Thus, let $\lambda(\varepsilon)$ one of such eigenvalues and $\Pi(\varepsilon)$ the associated eigenprojector. We have $D(\varepsilon)\Pi(\varepsilon) = \lambda(\varepsilon)\Pi(\varepsilon)$. Differentiating yields $\dot{D}(\varepsilon)\Pi(\varepsilon) + D(\varepsilon)\dot{\Pi}(\varepsilon) = \dot{\lambda}(\varepsilon)\Pi(\varepsilon) + \lambda(\varepsilon)\dot{\Pi}(\varepsilon)$. Multiplying on the right by $\Pi(\varepsilon)$, since $\Pi(\varepsilon)D(\varepsilon) = D(\varepsilon)\Pi(\varepsilon)$, we have

$$\Pi(\varepsilon)\dot{D}(\varepsilon)\Pi(\varepsilon) = \dot{\lambda}(\varepsilon)\Pi(\varepsilon).$$

Since $\Pi(\varepsilon)v = \langle a(\varepsilon), v\rangle b(\varepsilon)$ for some vectors a, b analytic in ε, $\dot{\lambda}(\varepsilon) = \langle a(\varepsilon), \dot{D}(\varepsilon)b(\varepsilon)\rangle$. We can now apply such a general formula to our specific case:

$$\begin{aligned}\langle a(0), \dot{D}(0)b(0)\rangle &= \langle a(0), e^{AT}\int_0^T e^{-As}B(s)e^{As}b(0)ds\rangle \\ &= \langle a(0), e^{AT}\int_0^T e^{-As}B(s)e^{As}b(0)ds\rangle \\ &= \lambda(0)\int_0^T \langle a(0), B(s)b(0)\rangle ds = 0.\end{aligned}$$

Notes

I presented the results in a really condensed way and with no pretension to exhaust the theory of ODE. If one wants to have a better understanding of the field and some ideas of how an ODE can be solved in special cases better consult [HS74, Arn92, CL55].