CHAPTER 4

How to establish hyperbolicity–the magic of cones

In chapter 2 we have taken advantage of the hyperbolic structure of the map in a very explicit manner, yet non linear maps may be hyperbolic but this cannot be seen by naively analyzing their derivative. Hence the necessity to have a tool to establish the hyperbolicity of a given system.

Our next task is to understand a general approach which allows to establish when the Lyapunov exponents are different from zero almost everywhere.

4.1 The two dimensional case

We start by dealing with the area preserving two dimensional case in order to explain the basic idea.

Theorem 4.1.1 (Wojtkowski [69]) *Let (X, μ, T) be a dynamical system where X is a compact two dimensional Riemannian manifold, μ is the Riemannian volume,*[1] *T a diffeomorphism of X and*

$$\int_X \log \|DT\| d\mu < \infty.$$

If there exists a measurable family of convex two sided cones $\mathcal{C}(x) \subset \mathcal{T}_x X$ such that, for almost all $x \in X$, there exists $n \in \mathbb{N}$ with the property[2]

$$D_x T^n \mathcal{C}(x) \subset int(\mathcal{C}(T^n x)) \cup \{0\},$$

then the Lyapunov exponents are different from zero almost surely.

A system with a family of cones satisfying the hypotheses of the Theorem 4.1.1 is called *eventually strictly monotone*.[3]

Proof. Let $x \in X$ and $n \in \mathbb{N}$ such that

$$D_x T^n \mathcal{C}(x) \subset \mathrm{int}(\mathcal{C}(T^n x)) \cup \{0\}.$$

[1] In fact, it is the symplectic not the Riemannian structure that matters; yet, in two dimension, there is not much difference. On the contrary, in section 4.2 this difference will be made explicit.

[2] A "(two sided) cone," in a linear space, is a set such that if ξ belongs to the set then $\lambda\xi$ belongs to the set for each $\lambda \in \mathbb{R}$. In addition, we require that the set is closed, has open interior and its complement is non void. Actually, this last conditions could be relaxed for x in a zero measure set without changing the following proof (see footnote 6 at page 84).By "two sided convex cone" we mean that each half cone is a convex set. Notice that in $\mathbb{R}^2$ a such a cone is defined uniquely by the two edges. By measurable we mean that the functions from X to the unit vectors, in the direction of the edges, are measurable.

[3] In fact, in the case of billiards a similar situation is called *sufficiency*sufficiency but I find the above terminology more appropriate.

The first thing to notice is that it is possible to make an orientation preserving change of coordinates (i.e., a change of coordinates via a matrix with positive determinant) both in $\mathcal{T}_xX$ and in $\mathcal{T}_{T^nx}X$ such that, in the new coordinates, $\mathcal{C}(x)$ and $\mathcal{C}(T^nx)$ become the standard cone $\mathcal{C}_+ = \{v \in \mathbb{R}^2 \mid v_1v_2 \geq 0\}$ and the Riemannian structures–the scalar product and the volume–are the standard ones (see Problem 4.5, Problem 4.6). Viewed in this coordinates D_xT^n becomes a two by two matrix with determinant equal to one, that maps $\mathcal{C}_+$ strictly into itself. Note that, since the cone family is measurable, the change of coordinates depends measurably by x.

To continue it is necessary to study a bit the general properties of the matrices enjoying the above mentioned properties. Notice that if we define a quadratic form $Q : \mathbb{R}^2 \to \mathbb{R}$ by $Q(v) = v_1v_2$, then $\mathcal{C}_+ = \{v \in \mathbb{R}^2 \mid Q(v) \geq 0\}$, so our task is to study the two by two matrices L with $\det(L) = 1$ and such that $Q(v) \geq 0$, $v \neq 0$, implies $Q(Lv) > 0$.[4]

Algebraic considerations

Let $v = (1, u)$ with $u \in \mathbb{R}^+$, which implies $v \in \mathcal{C}_+$, and

$$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $\det(L) = 1$ (note that this is equivalent to L symplectic, see section 4.2). Then, for each $u \geq 0$, we must have

$$0 < Q(Lv) = ac + (ad + bc)u + bdu^2, \tag{4.1.1}$$

Setting $u = 0$, it must be $ac > 0$. On the other hand, since $(0, 1) \in \mathcal{C}_+$ and $L(0, 1) = (b, d)$, it must be $bd > 0$. Finally, if we compute the quadratic polynomial in its minimum $u_0 = -\frac{ad+bc}{2bd}$ we get, calling $v_0 = (1, u_0)$,

$$Q(Lv_0) = -\frac{1}{4bd} < 0.$$

The above relation is possible only if $v_0 \notin \mathcal{C}_+$, which implies $u_0 < 0$ or $ad + bc > 0$, that is $ad > \frac{1}{2}$. Collecting the above results it follows that all the elements of the matrix L must be different from zero and, in addition, they must have the same sign. Since $Q(v) = Q(-v)$, without loss of generality we can assume them to be all positive.

The next step is to define some measure of expansion for a strictly monotone matrix. A natural quantity to consider is:

$$\sigma(L) = \inf_{v \in \text{int}(\mathcal{C}_+)} \sqrt{\frac{Q(Lv)}{Q(v)}}.$$

Choosing again $v = (1, u)$, it follows

$$\frac{(a + bu)(c + du)}{u} \geq ad + bc = 1 + 2bc > 1, \tag{4.1.2}$$

thus $\sigma(L) \geq 1$.

[4]We will call such matrices *Strictly monotone*.strictly monotone

Moreover, given two monotone matrices L_1, L_2, we have

$$\sigma(L_1L_2) = \inf_{v\in\mathrm{int}(\mathcal{C}_+)} \sqrt{\frac{Q(L_1L_2v)}{Q(L_2v)}}\sqrt{\frac{Q(L_2v)}{Q(v)}} \geq \sigma(L_1)\sigma(L_2). \tag{4.1.3}$$

An interesting fact, that follows immediately from (4.1.2), is that $\sigma(L) > 1$ if and only if L is strictly monotone.

Measure theoretical considerations

The point of measuring the expansion via the Q-form is due to the following Lemma.[5]

Lemma 4.1.2 *If the Dynamical System (X, T, μ), X a two dimensional Riemannian manifold, T differentiable a.e. and μ the Riemannian volume, is eventually strictly monotone, then*

$$\liminf_{n\to\infty} \frac{1}{n} \ln \sigma(D_xT^n) > 0 \quad \mu\text{-}a.e..$$

PROOF. Let $\nu : X \to \mathbb{R}^+ \cap \{\infty\}$ be defined by

$$\nu(x) := \liminf_{n\to\infty} \frac{1}{n} \ln \sigma(D_xT^n).$$

Then

$$\begin{aligned}
\nu(T^{-1}x) &= \liminf_{n\to\infty} \frac{1}{n} \ln \sigma(D_{T^{-1}x}T^n) \\
&\geq \liminf_{n\to\infty} \frac{1}{n}\{\ln \sigma(D_xT^{n-1}) + \ln \sigma(D_{T^{-1}x}T)\} \\
&\geq \liminf_{n\to\infty} \frac{1}{n} \ln \sigma(D_xT^n) = \nu(x),
\end{aligned} \tag{4.1.4}$$

where we have used (4.1.3) and the fact that $\sigma(D_\xi T) \geq 1$ by monotonicity.

Let $A_0 = \{x \in X \mid \nu(x) = 0\}$, to prove the Lemma it suffices to show that $\mu(A_0) = 0$. To this end note that $TA_0 \subset A_0$, since (4.1.4) implies $\nu(Tx) \leq \nu(x)$. Then, consider $\Lambda = \cup_{n\in\mathbb{N}}T^{-n}A_0$, clearly the $\Lambda \supset A_0$ is an invariant set and

$$\mu(\Lambda\backslash A_0) = \mu(\cup_{n\in\mathbb{N}}T^{-1}A_0\backslash A_0) \leq \sum_{n=0}^{\infty}[\mu(T^{-n}A_0) - \mu(A_0)] = 0. \tag{4.1.5}$$

Consequently, if we suppose that $\mu(A_0) > 0$, then $\mu(\Lambda) > 0$. Therefore, for each $m \in \mathbb{N}$,

$$0 = \int_{A_0} \nu(x)\mu(dx) \geq \int_{A_0} \liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{[\frac{n}{m}]-1} \ln \sigma(D_{T^{im}x}T^m),$$

[5]It is interesting to remark that the next Lemma, together with the results of section 4.3, imply the existence of the stable and unstable distribution (see Examples 4.3.1-Cones and Q-forms). Thus, since X is two dimensional, the existence a.e. of the L.E. follows as in Problem 3.6 without invoking Oseledec Theorem. The use of Oseledec Theorem is instead necessary in higher dimensions.

where we have used (4.1.3) again. At this point we would like to use BET, yet we face a technical problem: we do not know if $\ln \sigma(D_x T^m)$ is integrable. Nevertheless, we are not interested in large values of $\ln \sigma(D_x T^m)$. It is then natural to define

$$\varphi_m(x) = \min\{\ln \sigma(D_x T^m),\, 1\}.$$

Now $\varphi_m \in L^\infty(X, \mu)$, thus the ergodic average $\varphi_m^+ \in L^\infty(X, \mu)$; hence, remembering (4.1.5),

$$\begin{aligned} 0 \geq \int_{A_0} \liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{[\frac{n}{m}]-1} \varphi_m(T^{im}x) &= \frac{1}{m} \int_{A_0} \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi_m(T^{im}x) \\ &= \frac{1}{m} \int_{A_0} \varphi_m^+(x) = \frac{1}{m} \int_\Lambda \varphi_m^+ = \frac{1}{m} \int_\Lambda \varphi_m. \end{aligned} \tag{4.1.6}$$

That is $\varphi_m = 0$ a.e. in Λ. But this is a contradiction since, calling $B_m = \{x \in X \mid \sigma(D_x T^m) > 1\}$, the definition of eventually strictly monotone is equivalent to $\mu(\cup_{m\in\mathbb{N}} B_m) = \mu(X)$. Therefore there must exists an $m \in \mathbb{N}$ such that $\mu(\Lambda \cap B_m) > 0$, which implies

$$\int_\Lambda \varphi_m \geq \int_{\Lambda\cap B_m} \varphi_m > 0,$$

whereby contradicting (7.1.1). □

The relevance of what we have seen up to now for the estimation of the Lyapunov exponents depends on the trivial inequality

$$\|v\|^2 \geq 2Q(v). \tag{4.1.7}$$

The only real problem left is that, due to our change of variable to put the cones into their standard form, the euclidean norm $\|\cdot\|$ in the new variables no longer correspond to the original norm in X (let us call such original norm, at the point x, $\|\cdot\|_x$). Nevertheless, the two norms must be equivalent by construction,[6] hence there must exists an everywhere strictly positive measurable function $a(x) \leq 1$ such that, for each $v \in \mathcal{T}_x X$

$$a(x)^{-1}\|v\| \geq \|v\|_x \geq a(x)\|v\|.$$

Let us introduce the set $A(\varepsilon) = \{x \in X \mid a(x) > \varepsilon\}$, clearly $\cup_{\varepsilon>0} A(\varepsilon)$ has full measure. Now Poincaré theorem implies, if $\mu(A(\varepsilon)) \neq 0$, that almost all points in $A(\varepsilon)$ return to $A(\varepsilon)$ infinitely often. Let $x \in A(\varepsilon)$ be one of such points, then there exists a sequence n_k such that $T^{n_k}x \in A(\varepsilon)$ for each $k \in \mathbb{N}$.

Accordingly, for each $v \in \operatorname{int}(\mathcal{C}(x))$, $\|v\|_x = 1$, and $m \in \mathbb{N}$ holds

$$\begin{aligned} \lambda(x,\, v) = \lim_{n\to\infty} \frac{1}{n} \log \|D_x T^n v\|_{T^n x} &\geq \liminf_{k\to\infty} \frac{1}{n_k} \log \|D_x T^{n_k} v\|_{T^{n_k}x} \\ &\geq \liminf_{k\to\infty} \frac{1}{n_k} \log \|D_x T^{n_k} v\| + \liminf_{k\to\infty} \frac{1}{n_k} \log a(T^{n_k}x) \\ &\geq \liminf_{n\to\infty} \frac{1}{n} \log \|D_x T^n v\| - \lim_{k\to\infty} \frac{1}{n_k} \log \varepsilon^{-1} \end{aligned}$$

[6] If we admit that $\mathcal{C}(x)$ can have an empty interior on a set of zero measure in X–see footnote 2 at page 81–then the two norms would be equivalent only almost everywhere. Nevertheless, this does not change the proof: call X_1 the incriminated set, then $X_2 := \cup_{n\in\mathbb{Z}} T^n X_1$ has also zero measure and it is an invariant set. We can then discard such a set and work on its complement without any other change in the following.

and, since $Q(v) \neq 0$ and by (4.1.7),

$$\begin{aligned}\lambda(x,\, v) &\geq \liminf_{n\to\infty} \frac{1}{n} \log \sqrt{\frac{Q(D_x T^n v)}{Q(v)}} \\ &\geq \liminf_{n\to\infty} \frac{1}{n} \log \sigma\left(D_x T^n\right) > 0,\end{aligned}$$

due to Lemma 4.1.2.

We have then seen that, for almost every $x \in X$ and $v \in Int\, \mathcal{C}(x)$, $\lambda(x,\, v) \neq 0$. Noticing that the Lyapunov exponents of T are given by minus the Lyapunov exponents of T^{-1} (seeProblem 3.**??** and vicinity). Thus, by Oseledets Theorem [**?**] (or see section 4.4), almost every point must have a vector $v_-(x)$ such that $\lambda(x, v_-(x)) < 0$. Obviously, given any vector $v \in Int\, \mathcal{C}(x)$ it follows $\lambda(x, \alpha v + \beta v_-(x)) = \lambda(x, v)$, provided $\alpha \neq 0$, and this concludes the story. □

Remark 4.1.3 *The measurability assumption is a very weak hypothesis but cannot be eliminated. Indeed, if one constructs a cone family along the trajectories it can easily be made strictly monotone. Hence, if the system has zero Lyapunov exponents such a cone family cannot be measurable (see Problem 4.2).*

The above theorem provides us with a very powerful instrument to establish hyperbolicity for a given dynamical system.

To see how it works let us consider some simple examples.

4.1.1 Examples

Linear automorphysms of the Torus

Consider the matrix

$$L = \begin{pmatrix} 1 & a \\ a & 1+a^2 \end{pmatrix}$$

with $a \in \mathbb{N}$ and the standard cone $\mathcal{C}_+ = \{(v, v) \in \mathbb{R}^2 \mid uv \geq 0\}$. Then

$$L \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u + av \\ au + (1+a^2)v \end{pmatrix}$$

shows that $\mathcal{C}_+$ is strictly monotone for L. Of course, this is a rather silly example since it is completely obvious that the map is hyperbolic, the next example is a little less trivial.

Perturbations of linear automorphysms of the Torus

Consider a diffeomorphism $\Phi : \mathbb{T}^2 \to \mathbb{T}^2$ such that

$$\|(D\Phi - \mathbb{1})L\| < 1; \tag{4.1.8}$$

where L is defined as in the previous example, then the map T defined by $Tx := \Phi(Lx)$ is hyperbolic. To see this write

$$DT \begin{pmatrix} u \\ v \end{pmatrix} = L \begin{pmatrix} u \\ v \end{pmatrix} + (D\Phi - \mathbb{1})L \begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} u + av \\ au + (1+a^2)v \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

but $\max\{|\alpha|, |\beta|\} \leq \|(D\phi - \mathbb{1})L\|\|(u,v)\| \leq u + v$. Thus $\mathcal{C}_+$ is strictly monotone for DT.

It is interesting to notice that, already for this simple example, it would be not immediately clear how to establish hyperbolicity without using a cone language.[7] In addition, remark that the full strength of Wojtkowski theorem it is not used here–since the cone family is strictly monotone.

Levowich map

Levowichmap, Levowich Let us consider the map $T : \mathbb{T}^2 \to \mathbb{T}^2$ defined by[8]

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x - \sin x + y \\ x - \sin x + y \end{pmatrix},$$

It is immediate to verify that

$$D_{(x,y)}T = \begin{pmatrix} 2 - \cos x & 1 \\ 1 - \cos x & 1 \end{pmatrix}$$

Since $\det(DT) = 1$, it follows that $(\mathbb{T}^2, T, m)$ is a Dynamical Systems. In addition, $D_xT\mathcal{C}_+ \subset \mathcal{C}_+$ strictly, apart from the zero measure set $\{x = 0\}$, so Theorem 4.1.1 applies.

4.2 Higher dimension–an overview

The difficulties in extending to higher dimensions the previous results stem mainly from the large variety of possible cone shapes in higher dimension. It is far from obvious how to relate monotone properties of a given cone to the behavior of the Lyapunov exponents. One possible way is to generalize the approach based on quadratic forms. We will comment on this possibility in section 4.6. Yet, in the special case of *Symplectic Systems*symplectic systems, it is possible to develop a very rich theory which is astonishingly similar to the two dimensional one. Here we give a brief insight into this theory, but see [49], [50] and [69] for a much more detailed account.

Definition 4.2.1 *By symplectic Systems we mean a Dynamical System (X, T, μ) where X is a symplectic manifoldsymplectic manifold,*[9] *μ is the symplectic volume*[10] *and T is a symplectic mapSymplectic Maps.*[11]

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[7]Of course, the above result would follow from the Structural stability as well, yet the structural stability holds for small perturbations and the question would remain: how small is small? In our case the answer is provided by (4.1.8) (**maybe 4.1.8 is the condition for structural stability as well (toral stuff is quite rigid , check it!)**).

[8]Note that here our torus has the periodicity of 2π instead than one as in the previous examples, this is just to have simpler formulae; the reader can easily reformulate the problem on the torus $\mathbb{R}^2 \mod 1$.

[9]A symplectic manifold is a smooth manifold of even dimensions together with a symplectic formsymplectic form. By *symplectic form* we mean an antisymmetric differential two form ω which is close, see [3] for more details.

[10]Given a symplectic form ω on a manifold of dimension $2d$ the $2d$ form $\wedge^d\omega$ is a volume form: the symplectic volume.

[11]A map is symplectic if is conserves the symplectic two form ω, that is, for each $x \in X$ and vectors $v, w \in \mathcal{T}_xX$, holds $\omega(D_xTv, D_xTw) = \omega(v, w)$.

Clearly one can also define a Symplectic Systems in continuous time, we have already seem the typical example: Hamiltonian systems (see Examples 1.1.1).

For the convenience of the reader we will present here some of the material from [71] and [49].

Let $\mathcal{W}$ be a linear symplectic space of dimension $2d$ with the symplectic form ω. For instance we call $\mathcal{W} = \mathbb{R}^d \times \mathbb{R}^d$ the standard linear symplectic spacesymplect space if

$$\omega(w_1, w_2) = \langle \xi^1, \eta^2 \rangle - \langle \xi^2, \eta^1 \rangle, \tag{4.2.9}$$

where $w_i = (\xi^i, \eta^i)$, $i = 1, 2$, and $\langle \xi, \eta \rangle = \xi_1\eta_1 + \cdots + \xi_d\eta_d$.

The symplectic groupsymplect group $Sp\,(d, \mathbb{R})$ is the group of linear maps of $\mathcal{W}$ ($2d \times 2d$ matricessymplect matrices if $\mathcal{W} = \mathbb{R}^d \times \mathbb{R}^d$) preserving the symplectic form i.e., $L \in Sp\,(d, \mathbb{R})$ if

$$\omega(Lw_1, Lw_2) = \omega(w_1, w_2) \tag{4.2.10}$$

for every $w_1, w_2 \in \mathcal{W}$.

By definition a Lagrangian subspace of a linear symplectic space $\mathcal{W}$ is a d-dimensional subspace on which the restriction of ω is zero (equivalently it is a maximal subspace on which ω vanishes).

Definition 4.2.2 *Given two transversal Lagrangian subspaces V_1 and V_2 we define the sector between V_1 and V_2 by*

$$\mathcal{C} = \mathcal{C}\,(V_1, V_2) = \{w \in \mathcal{W} \mid \omega(v_1, v_2) \geq 0 \textit{ for } w = v_1 + v_2, v_i \in V_i, i = 1,\ 2\}$$

Equivalently, if we define the quadratic form associated with an ordered pair of transversal Lagrangian subspaces,

$$\mathcal{Q}(w) = \omega(v_1, v_2)$$

where $w = v_1 + v_2$, is the unique decomposition of w with the property $v_i \in V_i, i = 1, 2$, then we have

$$\mathcal{C} = \{w \in \mathcal{W} \mid \mathcal{Q}(w) \geq 0\}.$$

In the case of the standard symplectic space, $V_1 = \mathbb{R}^d \times \{0\}$ and $V_2 = \{0\} \times \mathbb{R}^d$ we get

$$\mathcal{Q}\,((\xi, \eta)) = \langle \xi, \eta \rangle$$

and

$$\mathcal{C}_+ = \{(\xi, \eta) \in \mathbb{R}^d \times \mathbb{R}^d \mid \langle \xi, \eta \rangle \geq 0\}.$$

We will refer to this $\mathcal{C}_+$ as the standard sector. Since any two pairs of transversal Lagrangian subspaces are symplectically equivalent (see Problem Problem 4.**??**) we may consider only this case without any loss of generality.

It is natural to ask if a sector determines uniquely its sides. It is not a vacuous question since, for $d > 1$, there are many Lagrangian subspaces in the boundary of a sector. The answer is positive.

Proposition 4.2.3 *For two pairs of transversal Lagrangian subspaces V_1, V_2 and V_1', V_2' if*

$$\mathcal{C}\,(V_1, V_2) = \mathcal{C}\left(V_1', V_2'\right)$$

then

$$V_1 = V_1' \quad and \quad V_2 = V_2'.$$

Moreover V_1 and V_2 are the only isolated Lagrangian subspaces contained in the boundary of the sector $\mathcal{C}(V_1, V_2)$.

Based on the notion of the sector between two transversal Lagrangian subspaces (or the quadratic form $\mathcal{Q}$) we define two monotonicity properties of a linear symplectic map. By $int\,\mathcal{C}$ we denote the interior of the sector, i.e.,

$$int\,\mathcal{C} = \{w \in \mathcal{W} | \mathcal{Q}(w) > 0\}.$$

Definition 4.2.4 *Given the sector $\mathcal{C}$ between two transversal Lagrangian subspaces we call a linear symplectic map L monotone if*

$$L\mathcal{C} \subset \mathcal{C}$$

and strictly monotone if

$$L\mathcal{C} \subset int\,\mathcal{C} \cup \{0\}.$$

A very useful characterization of monotonicity is given in the following

Proposition 4.2.5 *L is (strictly) monotone if and only if $\mathcal{Q}(Lw) \geq \mathcal{Q}(w)$ for every $w \in \mathcal{W}$ ($\mathcal{Q}(Lw) > \mathcal{Q}(w)$ for every $w \in \mathcal{W}$, $w \neq 0$). In particular, $Q(Lw) = Q(w)$, that is, L is a Q-isometry iff $L\mathcal{C} = \mathcal{C}$.*

The fact that monotonicity implies the increase of the quadratic form defining the cone is a manifestation of a very special geometric structure of a sector and does not hold for cones defined by general quadratic forms.

Proposition 4.2.6 *A monotone map L is strictly monotone if and only if*

$$LV_i \subset int\,\mathcal{C} \cup \{0\}, \; i = 1, 2.$$

For the proofs of the above facts see [71] and [**?**] or look at Problems Problem 4.**??**...

Remark 4.2.7 *Proposition 4.2.3 and 4.2.6 are trivial in the two dimensional case. As already notice, proposition 4.2.5 follows, in the two dimensional case, by 4.1.2.*

The relevance of the above discussion is the possibility to extend Theorem 4.1.1 to the present setting.

Theorem 4.2.8 (Wojtkowski [69]) *Let (X, μ, T) be a dynamical system where X is the finite union of Symplectic Manifolds, μ is the symplectic volume, T an invertible almost everywhere differentiable symplectic map of X and*

$$\int_X \log \|DT\| d\mu < \infty.$$

If there exists a measurable, a.e. non degenerate, eventually strictly invariant family of sectors then the Lyapunov exponents are different from zero almost surely.

Proof. The proof follows the one of Theorem 4.1.1 where the algebraic considerations are replaced by Propositions 4.2.3, 4.2.5, 4.2.6, while the measure theoretical part is exactly the same. □

4.2.1 Examples

Linear symplectic maps

We will consider the following generalization of the Arnold cat. Let us consider the Dynamical Systems $(\mathbb{T}^{2d}, T, m)$, where m is the Lebesgue measure and $Tx = Lx \mod 1$, with the following matrix L

$$L = \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ M & \mathbb{1} + M \end{pmatrix}$$

where $M > 0$ and $M_{ij} \in \mathbb{Z}$ (see Problem 4.8 for a more concrete examples and Problem 4.15 to realize how general the example is). Then the system is symplectic and strictly monotone with respect to the standard sector, thus this Dynamical Systems is hyperbolic.

4.3 Metrics and cones

This section is devoted to a little digression on semi–metrics that can be associated to one sided cones convex. This is a vast field, here we will consider only few basic facts.

There is a very geometric approach to this: consider the projectivization of the cone (that is the set of equivalence classes with respect to the equivalence relation $\sim$ defined by $v \sim w$ iff there exists $\lambda \in \mathbb{R}^+$ such that $v = \lambda w$) whereby obtaining a convex set in the projective space and then use the associated projective metric [23]. We will use a more direct, yet equivalent, approach (see Problem 4.20 and Problem 4.**??** for further informations on the above point of view and the connection with the following). Hopefully, the reader will excuse the setting which is a bit abstract in order to be applicable to some unexpected situations.

We start by illustrating some results in lattice theory originally due to Garrett Birkhoff. For more details see [**?**], and [**?**] for a recent overview of the field. Consider a topological vector space $\mathbb{V}$ with a partial ordering "$\preceq$," that is a vector lattice.[12] We require the partial order to be *continuous*, i.e. given $\{v_n\} \in \mathbb{V}$ $\lim_{n\to\infty} v_n = v$, if $v_n \succeq w$ for each n, then $v \succeq w$. We call such vector lattices *integrally closed*.[13]

We define the closed convex cone[14] $\mathcal{C} = \{v \in \mathbb{V} \mid v \neq 0, v \succeq 0\}$ (hereafter, the term "closed cone" $\mathcal{C}$ will mean that $\mathcal{C} \cup \{0\}$ is closed). Conversely, given a closed convex cone $\mathcal{C} \subset \mathbb{V}$, enjoying the property $\mathcal{C} \cap -\mathcal{C} = \emptyset$, we can define an order relation by (see Problem 4.19)

$$v \preceq w \iff w - v \in \mathcal{C} \cup \{0\}.$$

Henceforth, each time that we specify a convex cone we will assume the corresponding order relation and vice versa. The reader must therefore be advised that "$\preceq$" will mean different

[12]We are assuming the partial order to be well behaved with respect to the algebraic structure: for each $v, w \in \mathbb{V}$ $v \succeq w \iff v - w \succeq 0$; for each $v \in \mathbb{V}$, $\lambda \in \mathbb{R}^+\backslash\{0\}$ $v \succeq 0 \Longrightarrow \lambda f \succeq 0$; for each $v \in \mathbb{V}$ $v \succeq 0$ and $v \preceq 0$ imply $v = 0$ (antisymmetry of the order relation).

[13]To be precise, in the literature "integrally closed" is used in a weaker sense. First, $\mathbb{V}$ does not need a topology. Second, it suffices that for $\{\alpha_n\} \in \mathbb{R}$, $\alpha_n \to \alpha$; $v, w \in \mathbb{V}$, if $\alpha_n v \succeq w$, then $\alpha v \succeq w$. Here we will ignore these and other subtleties: our task is limited to a brief account of the results relevant to the present context.

[14]Attention!!: here, by "cone," we mean any set such that, if v belongs to the set, then λv belongs to it as well, for each $\lambda > 0$. The reason for this change in the definition of cone is that two sided cones, viewed as sets, are never convex, while convexity plays a central rôle in the following. As we will see in Examples 4.3.1–Cones and Q-forms this change in definition does not limit the applicability of the present theory to the cones introduced in the previous section.

things in different contexts.

It is then possible to define a projective metric Θ (Hilbert metric),[15] in $\mathcal{C}$, by the construction:

$$
\begin{aligned}
\alpha(v,\, w) &= \sup\{\lambda \in \mathbb{R}^+ \mid \lambda v \preceq w\} \\
\beta(v,\, w) &= \inf\{\mu \in \mathbb{R}^+ \mid w \preceq \mu v\} \\
\Theta(v,\, w) &= \log\left[\frac{\beta(v,\, w)}{\alpha(v,\, w)}\right]
\end{aligned}
\tag{4.3.11}
$$

where we take $\alpha = 0$ and $\beta = \infty$ if the corresponding sets are empty.

Lemma 4.3.1 *The function Θ is a semi-metric in $\mathcal{C}$.*

Proof. Clearly $\Theta(v, w) =$ implies $v = \lambda w$ for some $\lambda \in \mathbb{R}^+$, also $\Theta(v, w) = \Theta(w, v)$ and the tringle inequlity can be easily checked. □

The importance of the previous constructions is due, in our context, to the following theorem.

Theorem 4.3.2 *Let $\mathbb{V}_1$, and $\mathbb{V}_2$ be two integrally closed vector lattices; $L : \mathbb{V}_1 \to \mathbb{V}_2$ a linear map such that $L(\mathcal{C}_1) \subset \mathcal{C}_2$, for two closed convex cones $\mathcal{C}_1 \subset \mathbb{V}_1$ and $\mathcal{C}_2 \subset \mathbb{V}_2$ with $\mathcal{C}_i \cap -\mathcal{C}_i = \emptyset$. Let Θ_i be the Hilbert metric corresponding to the cone $\mathcal{C}_i$. Setting $\Delta = \sup_{v,\, w \in L(\mathcal{C}_1)} \Theta_2(v,\, w)$ we have*

$$
\Theta_2(Lv,\, Lw) \leq \tanh\left(\frac{\Delta}{4}\right)\Theta_1(v,\, w) \qquad \forall v,\, w \in \mathcal{C}_1
$$

($\tanh(\infty) \equiv 1$).

Proof. Let $v,\, w \in \mathcal{C}_1$. On the one hand if $\alpha := \alpha(v, w) = 0$ or $\beta := \beta(v, w) = \infty$, then the inequality is obviously satisfied. On the other hand, if $\alpha \neq 0$ and $\beta \neq \infty$, then

$$
\Theta_1(v,\, w) = \ln\frac{\beta}{\alpha}
$$

where $\alpha v \preceq w$ and $\beta v \succeq w$, since $\mathbb{V}_1$ is integrally closed. Notice that $\alpha \geq 0$, and $\beta \geq 0$ since $v \succeq 0$, $w \succeq 0$. If $\Delta = \infty$, then the result follows from $\alpha Lv \preceq Lw$ and $\beta Lv \succeq Lw$. If $\Delta < \infty$, then, by hypothesis,

$$
\Theta_2\left(L(w - \alpha v),\, L(\beta v - w)\right) \leq \Delta
$$

which means that there exist $\lambda,\, \mu \geq 0$ such that

$$
\begin{aligned}
\lambda L(w - \alpha v) &\preceq L(\beta v - w) \\
\mu L(w - \alpha v) &\succeq L(\beta v - w)
\end{aligned}
$$

with $\ln\frac{\mu}{\lambda} \leq \Delta$. The previous inequalities imply

$$
\begin{aligned}
&\frac{\beta + \lambda\alpha}{1 + \lambda} Lv \succeq Lw \\
&\frac{\mu\alpha + \beta}{1 + \mu} Lv \preceq Lw.
\end{aligned}
$$

[15] In fact, we define a semi–metric, since $v \sim w \Rightarrow \Theta(v,\, w) = 0$. The metric that we describe corresponds to the conventional Hilbert metric on $\widetilde{\mathcal{C}}$, the quotient of $\mathcal{C}$ with respect to the relation "$\sim$".

Accordingly,

$$
\begin{aligned}
\Theta_2(Lv,\, Lw) &\leq \ln \frac{(\beta+\lambda\alpha)(1+\mu)}{(1+\lambda)(\mu\alpha+\beta)} = \ln \frac{e^{\Theta_1(v,\,w)}+\lambda}{e^{\Theta_1(v,\,w)}+\mu} - \ln\frac{1+\lambda}{1+\mu} \\
&= \int_0^{\Theta_1(v,\,w)} \frac{(\mu-\lambda)e^{\xi}}{(e^{\xi}+\lambda)(e^{\xi}+\mu)} d\xi \leq \Theta_1(v,\,w)\frac{1-\frac{\lambda}{\mu}}{\left(1+\sqrt{\frac{\lambda}{\mu}}\right)^2} \\
&\leq \tanh\left(\frac{\Delta}{4}\right)\Theta_1(v,\,w).
\end{aligned}
$$

□

Remark 4.3.3 *In general, it suffices to know $L(\mathcal{C}_1) \subset \mathcal{C}_2$ in order to conclude $\Theta_2(Lv,\, Lw) \leq \Theta_1(v,\, w)$. However, a strict contraction depends on the diameter of the image being finite.*[16]

In particular, if an operator maps a convex cone strictly inside itself (in the sense that the diameter of the image is finite), then it is a contraction in the Hilbert metric. This implies the existence of a "positive" eigenfunction (provided the cone is complete with respect to the Hilbert metric), and, with some additional work, the existence of a gap in the spectrum of L (see [**?**] for details or solve Problem 4.**??**).

Usually the space $\mathbb{V}$ comes endowed with its own metric, in such a case it is natural to wonder about the strength of the Hilbert metric compared to other metrics. While, in general, the answer depends on the cone, it is nevertheless possible to state an interesting result.

Definition 4.3.4 *A function $\rho : \mathbb{V} \to \mathbb{R}^+$ is called homogeneous of degree one if for all $\lambda \in \mathbb{R}$ and $v \in \mathbb{V}$*

$$\rho(\lambda v) = |\lambda|\rho(v).$$

Remark 4.3.5 *Note that a norm or a linear functional are both homogeneous function of degree one.*

Definition 4.3.6 *A homogeneous function of degree one is called* adapted to a cone $\mathcal{C}$ *if, for each $v, w \in \mathbb{V}$,*

$$-v \preceq w \preceq v \Longrightarrow \rho(v) \geq \rho(w),$$

and $v \in int\ \mathcal{C}$ implies $\rho(v) > 0$.

Lemma 4.3.7 *Let ρ_i be two homogeneous functions of degree one adapted to the cone $\mathcal{C} \subset \mathbb{V}$. Then, given $v,\, w \in int\ \mathcal{C} \subset \mathbb{V}$ for which $\rho_1(v) = \rho_1(w)$,*

$$\rho_2(v-w) \leq \left(e^{\Theta(v,\,w)} - 1\right)\min\{\rho_2(v), \rho_2(w)\}.$$

Proof. We know that $\Theta(v,\, w) = \ln\frac{\beta}{\alpha}$, where $\alpha v \preceq w \preceq \beta v$. This implies that $-w \preceq 0 \preceq \alpha v \preceq w$, i.e. $\rho_1(w) \geq \alpha\rho_1(v)$, or $\alpha \leq 1$. In the same manner it follows that $\beta \geq 1$. Hence,

[16] In the theory of Markov processes this corresponds to the so called *positivity improving* (see also Example 4.3.1).

$$
\begin{aligned}
w - v \preceq &(\beta - 1)v \preceq (\beta - \alpha)v \\
w - v \succeq &(\alpha - 1)v \succeq -(\beta - \alpha)v \\
w - v \preceq &(1 - \beta^{-1})w \preceq (\alpha^{-1} - \beta^{-1})w \\
w - v \succeq &(1 - \alpha^{-1})w \succeq -(\alpha^{-1} - \beta^{-1})w
\end{aligned}
$$

which implies

$$
\begin{aligned}
\|w - v\| &\leq (\beta - \alpha)\|v\| \leq \frac{\beta - \alpha}{\alpha}\|v\| = \left(e^{\Theta(v,w)} - 1\right)\|v\| \\
\|w - v\| &\leq (\alpha^{-1} - \beta^{-1})\|w\| \leq (\frac{\beta}{\alpha} - 1)\beta^{-1}\|w\| \leq \left(e^{\Theta(v,w)} - 1\right)\|w\|.
\end{aligned}
$$

□

Many normed vector lattices satisfy the hypothesis of Lemma 1.3 (e.g. Banach lattices[17]).

4.3.1 Examples

Perron-Frobenius Theorem

Consider a matrix $L : \mathbb{R}^n \to \mathbb{R}^n$ of all strictly positive elements: $L_{ij} \geq \gamma > 0$. The Perron-Frobenius theorem states that there exists a unique eigenvector v^+ such that $v_i^+ > 0$, in addition the corresponding eigenvalue λ is simple, maximal and positive. There quite a few proofs of this theorem a possible one is based on Birkhoff theorem. Consider the cone $\mathcal{C}^+ = \{v \in \mathbb{R}^2 \mid v_i \geq 0\}$, then obviously $L\mathcal{C}^+ \subset \mathcal{C}^+$. Moreover an explicit computation (see Problem 4.**??**) shows that

$$
\Theta(v, w) = \sum_{ij} \ln \frac{v_i w_j}{v_j w_i}.
$$

Then, setting $M = \max_{ij} L_{ij}$, it follows that

$$
\Theta(Lv, Lw) \leq 2 \ln \frac{M}{\gamma} := \Delta < \infty.
$$

We have then a contraction in the Hilbert metric and the result follows from usual fixed points theorems. Note that, since $\Theta(v, \lambda v) = 0$, for all $\lambda \in \mathbb{R}^+$, the fixed point $v_+ \in \mathbb{R}^n$ is only projective, that is $Lv_+ = \lambda v_+$ for some $\lambda \in \mathbb{R}$; in other words, we have an eigenvalue.

Remark that L^* satisfies the same conditions as L, thus there exists $w^+ \in \mathcal{C}^+$, $\mu \in \mathbb{R}^+$, such that $L^* w^+ = \mu w^+$. Next, define $\rho_1(v) = |\langle w^+, v\rangle|$ and $\rho_2(v) = \|v\|$. It is easy to check that they are two homogeneous forms of degree one adapted to the cone.

In addition, if $\rho_1(v) = \rho_2(v)$, then $\rho_1(L^n v) = \rho_1(L^n w)$. Hence, by Lemma 4.3.7

$$
\begin{aligned}
\|L^n v - L^n w\| &\leq \left(e^{\Theta(L^n v, L^n w)} - 1\right) \min\{\|L^n v\|, \|L^n w\|\} &\quad (4.3.12)\\
&\leq K\Lambda^n \min\{\|L^n v\|, \|L^n w\|\}, &\quad (4.3.13)
\end{aligned}
$$

[17] A Banach lattice $\mathbb{V}$ is a vector lattice equipped with a norm satisfying the property $\| \, |v| \, \| = \|v\|$ for each $f \in \mathbb{V}$, where $|v|$ is the least upper bound of v and $-v$. For this definition to make sense it is necessary to require that $\mathbb{V}$ is "directed," i.e. any two elements have an upper bound.

for some constant K depending only on v, w. The estimate 4.3.12 means that all the vectors in the cone grow at the same rate. In fact, for all $v \in \text{int}\mathcal{C}$,

$$\|\lambda^{-n}L^n v - \lambda^{-n}L^n w\| \leq K\Lambda^n.$$

Hence, $\lim_{n\to\infty} \lambda^{-n}L^n v = v_+$.

Finally, consider $\mathbb{V}_1 = \{v \in \mathbb{V} \mid \langle w^+, v\rangle = 0\}$. Clearly $L\mathbb{V}_1 \subset \mathbb{V}_1$ and $\mathbb{V}_1 \oplus \text{span}\{v_+\} = \mathbb{V}$. Let $w \in \mathbb{V}_1$, clearly there exists $\alpha \in \mathbb{R}^+$ such that $\alpha v_+ + w \in \mathcal{C}$,[18] thus

$$\|L^n w\| \leq \|L^n(\alpha v_+ + w) - \alpha L^n v_+\| \leq L\Lambda^n\lambda^n.$$

This immediately implies that L restricted to the subspace $\mathbb{V}_1$ has spectral radius less that $\lambda\Lambda$. In other words, λ is the maximal eigenvalue, it is simple and any other eigenvalue must be smaller than $\lambda\Lambda$. We have thus obtained an estimate of the spectral gap between the first and the second eigenvalue.

Cones and Q-forms

Here we would like to consider only half of the cone defined by the Q-form in order to apply the present theory. If $L_{ij} > 0$, then we can choose the first quadrant; on the other hand , if $L_{ij} < 0$, then L maps the first into the third quadrant. In both cases a monotone matrix L maps a one sided cone into a one sided cone cone. Here we will consider only the first case and leave the other–essentially identical–to the reader.[19] Consequently, L is a monotone matrix with respect to the standard sector $\mathcal{C}_+$ and $L_{ij} > 0$, then $L\mathcal{C}^+ \subset \mathcal{C}^+$ where, as in the previous example, $\mathcal{C}^+ = \{v \in \mathbb{R}^2 \mid v_i \leq 0\}$. Thus all the results of the previous example apply.

In particular, we have seen that, if $v = (1, \alpha)$, $w = (1, \beta) \in \mathcal{C}_+$, then

$$\Theta(v, w) = \left|\ln \frac{\alpha}{\beta}\right|.$$

Another interesting formula is (see Problem 4.**??**)

$$2\sinh(\frac{1}{2}\Theta(v, w)) = \frac{|\omega(v, w)|}{\sqrt{Q(v)Q(w)}}. \tag{4.3.14}$$

This means that here exists a relation between the Hilbert metric and the Q-form.

To understand this relation better, let us compute

$$\text{diam}(L\mathcal{C}^+) = \sup_{\alpha,\beta>0} \left|\ln \frac{(a+\alpha b)(c+\beta d)}{(a+\beta b)(c+\alpha d)}\right|.$$

Since

$$\frac{a}{c} = \frac{b}{d}(\frac{1+bc}{bc}) > \frac{b}{d}$$

it follows $\frac{a}{c} \geq \frac{b}{d}$. Thus

$$\text{diam}(L\mathcal{C}_+) = \left|\ln \frac{ad}{cb}\right| = \ln \frac{1+bc}{bc}, \tag{4.3.15}$$

[18] this is a special case of the general fat that any vector can be written as the linear combination of two vectors belonging to the cone.

[19] An easy way out is to consider L^2 instead of L.

which implies that, if L is strictly monotone, then $\text{diam}(L\mathcal{C}^+) < \infty$. Accordingly, the rate of contraction of the Hilbert metric is given by

$$\Lambda = \frac{1 - \sqrt{\frac{bc}{1+bc}}}{1 + \sqrt{\frac{bc}{1+bc}}} = \frac{1}{(\sqrt{1+bc} + \sqrt{bc})^2} = \frac{1}{\sigma(L)^2}, \tag{4.3.16}$$

where the last equality follows by a straightforward computation (see Problem 4.??).

Remark 4.3.8 *It is not immediately clear how to extend the above considerations to the higher dimensional setting discussed in section 4.2. In fact, to do so it is necessary to introduce a different metric [49] of Caratheodory type [67]. We will not do it here but the reader should be aware that such a generalization it is possible.*

Expanding maps–uniqueness of the a.c. measureexpanding maps, a.c.i.m.

A remarkable fact of Birkhoff theorem is that it applies to infinite dimensional vector spaces. In Example 1.4.1 we have studied the properties of $\mathcal{L}$. A computation similar to the one done there shows that, given a twice differentiable expanding map of the torus, the cone

$$\mathcal{C}_\alpha = \{h \in \mathcal{C}^{(0)}(\mathbb{T}) \mid h \geq 0; \frac{h(x)}{h(y)} \leq e^{\alpha d(x,y)}\} \tag{4.3.17}$$

is invariant. In fact, if $h \in \mathcal{C}_\alpha$

$$\begin{aligned} \mathcal{L}h(x) &= \sum_{z \in T^{-1}x} |D_z T|^{-1} h(z) \leq \sum_{w \in T^{-1}y} \frac{|D_w T|}{|D_z T|} |D_w T|^{-1} h(w) e^{\alpha d(z,w)} \\ &\leq \sum_{w \in T^{-1}y} |D_w T|^{-1} h(w) e^{(\lambda^{-1}\alpha + C)d(x,y)} = e^{(\lambda^{-1}\alpha + C)d(x,y)} \mathcal{L}h(y). \end{aligned}$$

By choosing α large enough, there exists $\sigma \in (\lambda^{-1}, 1)$ such that $\mathcal{L}\mathcal{C}_\alpha \subset \mathcal{C}_{\sigma\alpha}$.

A direct computation shows that the diameter is finite (see Problem 4.??). Accordingly, we have a contraction in the Hilbert metric. This implies that there exists only one invariant measure μ which is absolutely continuous with respect to Lebesgue ($d\mu = h_* dm$). Moreover, if $\rho_1(f) = |\int f|$ and $\rho_2(f) = \|f\|_\infty$, we have that Lemma 4.3.7 applies whereby showing that $\mathcal{L}h \to h_*$ in the sup norm for all $h \in \mathcal{C}_\alpha$, $\rho_1(h) = \rho_2(h_*)$. By arguments similar to the one employed in 4.3.1 it is possible to see that $\mathcal{L}$, viewed as an operator in $\mathcal{C}^{(1)}(\mathbb{T})$, has a maximal eigenvalue one while all the rest of the spectrum is separate by a gap (see Problem 4.??), clearly this implies not only the mixing but provides as well an estimate on the mixing rate for $\mathcal{C}^{(1)}(\mathbb{T})$ functions (see Problem 4.??).

4.4 Cones and invariant distributions

Here we use the machinery developed in the previous section to obtain a constructive proof of the existence of the unstable distribution in a special, but very interesting, case.

Lemma 4.4.1 *Given a smooth Symplectic Dynamical Systems with syngularities (X, T, μ), X a symplectic two dimensional manifold, μ the symplectic volume, if the systems is eventually strictly monotone, then $\{E^u(x)\}$ is almost everywhere well defined. Moreover, if $\mathcal{C}(x)$ is continuous, then $\{E^u(x)\}$ is continuous (where it is defined). In addition, if the cone family is strictly monotone, then $\{E^u(x)\}$ is everywhere defined.*

Proof. Let $\mathcal{C}_n(x) := D_{T^{-n}x}T^n\mathcal{C}(T^{-n}x)$ and $\Delta_n(x) := \operatorname{diam}(\mathcal{C}_n(x))$, then Δ_n is decreasing, thus we can define

$$\Delta_\infty(x) := \lim_{n\to\infty} \Delta_n(x).$$

The key consequence of the results of section 4.3 (in particular Examples 4.3.1–Cones and Q-forms) is

$$\begin{aligned}\Delta_\infty(T^m x) &= \lim_{n\to\infty} \operatorname{diam}(D_{T^{m-n}x}T^n\mathcal{C}(T^{-n+m}x))\\ &= \lim_{n\to\infty} \operatorname{diam}(D_xT^m D_{T^{-n}x}T^n\mathcal{C}(T^{-n}x))\\ &\le \frac{1}{\sigma(D_xT^m)^2}\Delta_\infty(x).\end{aligned}$$

Next, let $\Omega = \{x \in X \mid \Delta_\infty(x) = \infty\}$, we claim that $\mu(\Omega) = 0$. In fact, let $B_m = \{x \in X \mid \sigma(D_xT^m) \ge 2\}$, by eventual strict monotonicity of the cone field and Lemma 4.1.2 follows $\mu(\cup_{m\in\mathbb{N}}B_m) = \mu(X)$. In addition, $B_m \supset B_{m_0}$ for all $m > m_0$. Moreover, if $x \in B_m$, then $\Delta_\infty(T^m x) < \infty$ (see 4.3.16). Thus $T^{-n}\Omega \cap B_m = \emptyset$ for all $n \ge m$, and

$$\mu(\Omega) = \lim_{n\to\infty} \mu(T^{-n}\Omega) \le \lim_{n\to\infty} \mu(X \setminus \cup_{m\le n} B_m) = 0.$$

Finally, let $\Omega_L = \{x \in X \mid \frac{L}{2} \le \Delta_\infty(x) \le L\}$ and suppose $\mu(\Omega_L) > 0$. Then, there exists $n \in \mathbb{N}$ such that $\mu(\Omega_L \cap B_m) > 0$. Consequently, for almost all $x \in \Omega_L \cap B_m$ there exists a return time $\bar{n}m \in \mathbb{N}$ in the past (that is $T^{-m\bar{n}}x \in \Omega_L \cap B_m$). Accordingly,

$$\frac{L}{2} \le \Delta_\infty(x) \le \frac{1}{\sigma(D_xT^m)^2}\Delta_\infty(T^{-m\bar{n}}x) \le \frac{L}{4},$$

which is a contradiction unless $L = 0$. We have so proven that $\mu(\Omega_0) = \mu(X)$. In other words the cones $\mathcal{C}_\infty = \cap_{n\ge 0}\mathcal{C}_n(x)$ is almost everywhere degenerate since, having zero diameter, it consists of a single direction, such a direction is precisely the unstable direction.

To prove the continuity of the above distribution note that the cone family $\mathcal{C}_n(x)$ is continuous. Let x be such that $\Delta_\infty(x) = 0$, then, for each $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that $\Delta_m(x) < \frac{\varepsilon}{2}$. Then one can chose δ such that the edges of $\mathcal{C}_m(y)$ vary by an amount less than $\frac{\varepsilon}{2}$ if $d(x, y) < \delta$. The result follows then taking into account that the Hilbert metric bounds the angle (see Problem ???) and that the unstable distribution is contained in $\mathcal{C}_n$ for each $n \in \mathbb{N}$.

The proof of the last fact is obvious: just a simplification of the above arguments. □

With similar techniques it is also possible to construct the stable and unstable foliations, as we will see in chapter 7.

Let us conclude with an interesting simple fact.

Lemma 4.4.2 *A smooth two-dimensional Symplectic Dynamical System* (X, T, μ) *is AnosovAnosov systems iff it admits a strictly monotone continuous cone family.*

Proof. By Lemma 4.4.1 it follows that the stable and unstable distribution are continuous. But then, by continuity, there exists $\alpha > 0$ and $\sigma > 1$ such that

$$\begin{aligned}&\alpha\sqrt{Q(v)} \leq \|v\| \leq \alpha^{-1}\sqrt{Q(v)} \quad \forall x, \in X \text{ and } v \in E^u(x)\\ &\sigma(D_xT) \geq \sigma \quad \forall x \in X.\end{aligned}$$

Thus,

$$\|D_xT^nv\| \geq \alpha\sqrt{Q(D_xT^nv)} \geq \alpha\sigma^n\sqrt{Q(v)} \geq \alpha^2\sigma^n\|v\|.$$

Analogously one can obtain the statement for the stable direction by using the cone family given by the complementary cones (see Problem 4.4).

The proof that an Anosov systems admit a continuous strictly invariant cone family is obvious and it is left to the reader.[20] □

4.5 The case of Hamiltonian flows

talk about cocycles and derivative cocycles, forms in ambient space

. . . .

4.5.1 Examples

Geodesic flows

use Jacobi fields

Smooth flows with collisions

Billiards with potential

maybe in next chapter?

. . . .

[20] See Problem 7.4.

4.6 General quadratic forms–the non-symplectic case

Potapov stuff, kobayashy and caratheodory metric

. . . .

Before continuing in the development of the theory it can be helpful to develop and study some more interesting and totally non-trivial examples. To this end are dedicated the next two chapters.

Problems

4.1 Show that the hypothesis of Theorem 4.1.1 can be relaxed, in particular it holds for smooth systems with singularities (see section 3.5.1). (Hint: Just follow the proof step by step and notice that nothing substantial need to be changed.)

4.2 Construct a strictly invariant cone family for the irrational translation on $\mathbb{T}^2$ (see Examples 1.1.1) and show that it is not measurable. (Hint: For each trajectory choose a point x. At such a point choose the standard cone $\mathcal{C}_+$, let $\mathcal{C}_n^- = \{(v_1, v_2) \in \mathbb{R}^2 \mid 1 + \frac{1}{n} \leq \frac{v_2}{v_1} \leq 2 + \frac{1}{n}\}$ and $\mathcal{C}_n^+ = \{(v_1, v_2) \in \mathbb{R}^2 \mid -2 - \frac{1}{n} \leq \frac{v_2}{v_1} \leq -1 - \frac{1}{n}\}^c$. Then set $\mathcal{C}(T^n x) = \mathcal{C}_n^+$ and $\mathcal{C}(T^{-n}x) = \mathcal{C}_n^-$. Such a cone family is strictly monotone by construction (since $D_x T = 1$), yet the system has obviously zero Lyapunov exponents. Since all the other hypothesis of Theorem 4.1.1 are satisfied, it follows that the above cone family cannot be measurable.)

4.3 Show that for two dimensional symplectic maps the sum of the Lyapunov exponent is zero (*pairing of the Lyapunov exponents*pairing of the exponents). (Hint:If $\omega(v, w) = 1$ then $1 = \omega(DT^n v, DT^n w) \sim \|DT^n v\|\|DT^n w\|$.)

4.4 Check that $\inf_{v\in\mathcal{C}_+} \sqrt{\frac{Q(Lv)}{Q(v)}} = \left[\inf_{v\in\mathcal{C}_-} \sqrt{\frac{Q(L^{-1}v)}{Q(v)}}\right]^{-1}$, remember that $\mathcal{C}_- = \overline{(\mathcal{C}_+)^c}$. (Hint: see [49])

4.5 Consider $\mathbb{R}^2$ endowed with the scalar product $\langle v, w\rangle_G := \langle v, Gw\rangle$, where $\langle\cdot,\cdot\rangle$ is the standard scalar product and $G > 0$. Show that there exists a change of coordinates $M : \mathbb{R}^2 \to \mathbb{R}^2$ such that, in the new coordinates $\langle\cdot,\cdot\rangle_G$ becomes the standard scalar product.

4.6 Consider the cone $\mathcal{C}$ defined by the two transversal vectors $v_1, v_2 \in \mathbb{R}^2$. This means that $v \in \mathbb{R}^2$ belongs to the cone iff $v = \alpha v_1 + \beta v_2$ with $\alpha\beta \geq 0$. Show that there is a linear change of coordinates $M : \mathbb{R}^2 \to \mathbb{R}^2$ such that $M\mathcal{C} = \mathcal{C}_+$ and $\det M = 1$.

4.7 Show that, in a two dimensional area preserving systems, if the LE are different from zero then there exists and eventually strictly invariant cone family.(Hint: By Oseledec and Problem ?? there exists the unstable distributions, then construct the cones around it.)

4.8 Prove that if M is the two by two matrix

$$M = \begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

with $a, b, c \in \mathbb{Z}$, then $M > 0$ iff $a, c > 0$ and $c > \frac{b^2}{a}$.

4.9 Show that a $2d \times 2d$ matrix L of the form

$$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C, D are $d \times d$ blocs, is symplecticsymplectic matrices, iff $C^* A = A^* C$, $D^* B = B^* D$ and $A^* D - C^* B = \mathbb{1}$. (Hint: Note that, by introducing the matrix

$$J = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$$

the standard symplectic form **??** can be written in terms of the usual scalar product as

$$\omega((\xi^1, \xi^2),\, (\eta^1, \eta^2)) = \langle (\xi^1, \xi^2),\, J(\eta^1, \eta^2) \rangle.$$

From this point of view the definition of symplectic matrix 4.2.10 can be written as

$$L^* J L = J.$$

A trivial algebraic computation yields now the result.)

4.10 Prove that is L is symplect then $\det L = 1$. (Hint: The determinant of a matrix is nothing else than the volume of the parallelepiped of sides $(Le_1, \dots, Le_{2d})$ (where $e_1, \dots, e_{2d}$ is the standard orthonormal basis of $\mathbb{R}^{2d}$). On the other hand the volume form can be written has $\wedge^d \omega$ (since that is a $2d$ form with the right normalization and the space of $2d$ forms is one dimensional). Thus $\det L = \wedge^d \omega(Le_1, \dots, Le_{2d}) = \wedge^d \omega(e_1, \dots, e_{2d}) = 1$ where we have used the fact that $\omega(Lv, Lu) = \omega(v, u)$. The reader that wants to appreciate the power of the above geometrical interpretation of the determinant and of the external forms can try to prove the statement by purely algebraic means.)

4.11 Show that all symplectic Q-isometrys L (that is $Q(Lv) = Q(v)$) have the form

$$L = \begin{pmatrix} A & 0 \\ 0 & A^{*-1} \end{pmatrix}.$$

(Hint: Start by considering the vector $(0, u)$, $U \in \mathbb{R}^d$, clearly $Q((0, u)) = 0$ thus $Q(L(0, u)) = 0$ if L is a Q-isometry. But if

$$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

it follows $\langle Bu, Du \rangle = 0$ for each $u \in \mathbb{R}^d$, that is $B^* D = 0$. The same argument applied to the vector $(u, 0)$ yields $A^* C = 0$. Accordingly, by symplecticity (see Problem 4.9),

$$\begin{aligned} Q(L(v, u)) =& \langle Au + Bv,\, Cu + Dv \rangle = \langle u, (A^* D + C^* B) v \rangle \\ =& \langle u, (\mathbb{1} + 2C^* B) v \rangle \end{aligned}$$

thus $Q(L(v, u)) = Q(v, u)$ iff $C^* B = 0$ which implies $A^* D = \mathbb{1}$.)

4.12 show that if the matrix

$$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is symplectic then

$$L^{-1} = \begin{pmatrix} D^* & -B^* \\ -C^* & A^* \end{pmatrix}$$

(Hint: multiply and use Problem 4.9.)

4.13 Show that the symplectic matrices form a multiplicative group. (Hint: Use the definition and the above problems.)

4.14 A symplectic map L is a Q-isometry iff $L\mathcal{C} = \mathcal{C}$. (Hint: One direction is trivial. On the other hand, if $L\mathcal{C} = \mathcal{C}$ it follows that L maps the boundary, of $\mathcal{C}$, to the boundary. Accordingly, if $\langle v, u\rangle = 0$ it must be

$$0 = \langle Av + bu,\, Cv + Du\rangle. \tag{4.6.18}$$

Choosing in 4.6.18 $u = 0$ yields $A^*C = 0$, choosing $v = 0$ shows that it must be $B^*D = 0$. Thus 4.6.18 yields

$$0 = \langle u,\, (A^*D + C^*B)v\rangle = 2\langle u,\, C^*Bv\rangle.$$

The above equality shows that C^*Bv is parallel to v for each $v \in R^d$, that is $C^*B = \alpha\mathbb{1}$ for some $\alpha \in \mathbb{R}$. If $\alpha = 0$, then $A^*D = \mathbb{1}$ and thus $C = 0$ which is the wanted result. If $\alpha \neq 0$, then B is invertible and $C = \alpha B^{*1}$. But this implies $A = 0$ and hence $-\mathbb{1} = C^*B = \alpha\mathbb{1}$, that is $\alpha = -1$. Accordingly the matrix would have the form

$$L = \begin{pmatrix} 0 & B \\ -B^{*-1} & 0 \end{pmatrix}$$

which sends $\mathcal{C}$ in its complement, contrary to our requirement.)

4.15 Show that a strictly monotone symplectic matrix can be put into the form

$$\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ M & \mathbb{1} + M \end{pmatrix}$$

by multiplying it by Q-isometries on the left and on the right.

4.16 Show that all the Lagrangian subspaces transversal to $V = \{(0, \eta) \in \mathbb{R}^{2d} \mid \eta \in \mathbb{R}^d\}$ can be represented as $\{(\xi,\, U\xi \in \mathbb{R}^{2d} \mid \xi \in \mathbb{R}^d\}$ for some symmetric matrix U. (Hint: Let $V_U := \{(\xi,\, U\xi \in \mathbb{R}^{2d} \mid \xi \in \mathbb{R}^d\}$, then $\omega((\xi,\, U\xi),\, (\zeta,\, U\zeta)) = 0$, thus V_U is Lagrangian. On the other hand, if $\tilde{V}$ is Lagrangian, then it is a d dimensional space (??). Let $\{(\xi_i, \eta_i)\}_{i=1}^d$ be a base for $\tilde{V}$, then $\xi_i \neq 0$ by the transversality assumption and we can define the matrix U via $U\xi := \eta_i$. It is immediate that $\tilde{V}$ Lagrangian implies $U = U^*$.)

4.17 Show that $V_U := \{(\xi,\, U\xi \in \mathbb{R}^{2d} \mid \xi \in \mathbb{R}^d\}$, $U = U^*$, belongs to the standard cone iff $U \geq 0$.

4.18 Find a symplectic change of coordinates that transforms the standard form Q into the form Q_h defined by:

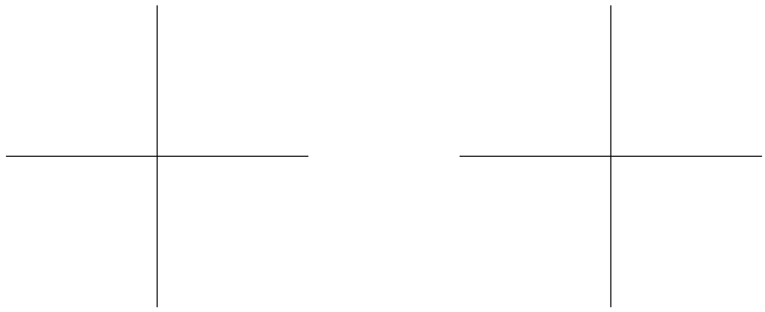
$$Q_h((x,y)) = \frac{1}{2}(\langle x, x\rangle - \langle y, y\rangle),$$

and draw the associate cone. (Hint: Consider

$$\begin{aligned} x &= \tfrac{x'-y'}{\sqrt{2}} \\ y &= \tfrac{x'+y'}{\sqrt{2}}. \end{aligned}$$

✘ shadow the two cones, the second is narrow arround the x axis.

See the figure for the shape of the cones.)



$Q \geq 0$ $\qquad\qquad$ $Q_h \geq 0$

Figure 4.1: Cones

4.19 relation between cones and order structures (Lattices lattice) and properties of $\preceq$ w.r.t. algebraic structure.

✘ Convexity is equivalent to $v \succeq w$, $w \succeq z$ implies $v \succeq z$!

4.20 Let $C \in \mathbb{R}^n$ be a strictly convex compact set. For each two point $x, y \in C$ consider the line $\ell = \{\lambda x + (1-\lambda y) \mid \lambda \in \mathbb{R}\}$ passing through x and y. Let $\{u, v\} = C \cap \ell$ and define

$$\Theta(x,y) = \left| \ln \frac{\|x-u\|\|y-v\|}{\|x-v\|\|y-v\|} \right|$$

(the logarithm of the cross ratio). Show that Θ is a metric in C (the Hilbert metric). Show that the distance of $x \in \operatorname{int} C$ from ∂C is infinite. (Hint: The only non trivial task is to check the triangle inequality. Consider three points $x, y, z \in C$. If the points are collinear then the proof is easy. If they are not the consider the plane defined by them, we have now a two dimensional problem, thus it suffices to prove the result in $\mathbb{R}^2$. Consider the Figure 20 and remember that the cross ratio

$$R(x,y,u,v) = \frac{\|x-u\|\|y-v\|}{\|x-v\|\|y-v\|}$$

is a projective invariant. Then

$$\begin{aligned} R(x,z,u,v) = R(x,w,x',y') \geq R(x,w,\alpha,\beta) \\ R(y,z,a,b) = R(w,y,x',y') \geq R(w,y,\alpha,\beta) \end{aligned}$$

and the result follows since $R(x,w,\alpha,\beta)R(w,y,\alpha,\beta) = R(x,y,\alpha,\beta)$.)

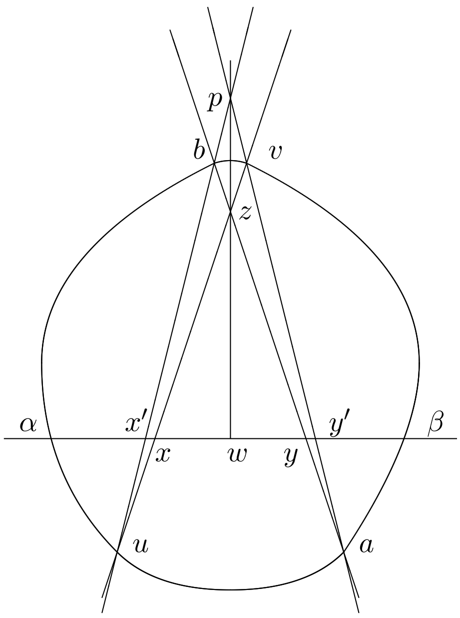


Figure 4.2: Hilbert metric

4.21 Prove the same as the previous Problem for convex sets in a projective space.

4.22 Check that the metric defined in Problem 4.20 and Problem 4.21 applied to the projectivization of convex cones yields to the same metric defined in (4.3.11).(Hint: Let $\mathcal{C} \subset \mathbb{V}$ be a convex cone. The projectivization consists in considering the space $\mathbb{P}$ of the equivalence classes $[v]$ with respect to the equivalence relation $v \sim w$ iff $v = \lambda w$ for some $\lambda \in \mathbb{R}^+$. Let $\tilde{\mathcal{C}} = \{[v] \in \mathbb{P} \mid v \in \mathcal{C}\}$. Clearly $\tilde{C}$ is convex in the projective space $\mathbb{P}$.[21] So, if $[x], [y] \in \tilde{\mathcal{C}}$, the Hilbert metric is defined by the points $[u], [v] \in \partial\tilde{\mathcal{C}}$ on the line determined by $[x]$, $[y]$. By normalizing properly one obtains $[u] = [-\alpha x + y]$ and $[v] = [\beta x - y]$ form which the result follows.)

4.23 Hilbert metric for a disc and the half plane–hyperbolic geometry.

4.24 Kobayashi and Caratheodory metrics....

4.25 Show that the Hilbert metric for the cone $\mathcal{C}_\alpha$ defined in (4.3.17) is given by ???

4.26 Show that the Perron-Frobenius operator associated to a smooth expanding map of the circle has a spectral gap as an operator on $Lip(\mathbb{T}^2)$. (Hint: Check that there exists $b \in \mathbb{R}^+$ such that the norm

$$\|h\| := \|h\|_\infty + b\|h\|_{Lip}$$

is adapted to the cone. Define $\mathbb{V} = \{h \in Lip(\mathbb{T}^2) \mid \int h = 0\}$, notice that $\mathcal{L}\mathbb{V} = \mathbb{V}$. Then, for each $h \in \mathbb{V}$ there exists$\rho \in \mathbb{R}^+$ such that $h + \rho h_* \in \mathcal{C}_\alpha$, so

$$\|\mathcal{L}^n h\| = \|\mathcal{L}^n(h + \rho h_*) - \rho h_*\| \leq K\Lambda^n \rho.$$

Thus the spectral radius of $\mathcal{L}|_\mathbb{V}$ is less than Λ.)

[21]The lines in $\mathbb{P}$ are given by $[\alpha v + \beta w]$ where $\alpha, \beta \in \mathbb{R}$ where $[v] \neq [w]$.

4.27 Estimate the rate of mixing for Lipschitz functions for a smooth expanding map of the circle (Hint: use the spectral gap of the previous Problem.)

4.28 Prove that any continuous fraction of the form

$$\cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}$$

$a_i > 0$ is convergent provided the series $\sum_{n=1}^{\infty} a_n$ is divergent. (Hint: Let

$$\prod_{i=1}^{n} \begin{pmatrix} 1 & a_{2(n-i)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_{2(n-1)+1} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ u \end{pmatrix} = \begin{pmatrix} \beta_n \\ \alpha_n \end{pmatrix}$$

and verify, by induction, that $\frac{\alpha_n}{\beta_n}$ is exactly the $2n$ truncation of the continuous fraction. Thus the continuous fraction is a projective coordinate for the vector (α_n, β_n). Consider the cone $\mathcal{C}_+ = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0;\ y \geq 0\}$. Then, for each $a, b \in \mathbb{R}^+$, holds

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \mathcal{C}_+ \subset \mathcal{C}_+.$$

The result follows by computing the Hilbert metric contraction.

For a different approach see [68][Th14.1].)

4.29 Prove ??(4.3.9) and 4.3.16.

Notes

When cones first appeared?

The point of W. th. is that no estimate on the cone contraction is needed. To appreciate the advantage one can try to prove the existence of the LE via direct estimates for the Levovich map in example 4.1.1

Generalization of metrics on cones, see [67]

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