ZETA FUNCTIONS AND DYNAMICAL SYSTEMS

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ABSTRACT. In this brief note we present a very simple strategy to investigate dynamical determinants for uniformly hyperbolic systems. The construction builds on the recent introduction of suitable functional spaces which allow to transform simple heuristic arguments in rigorous ones. Although the results so obtained are not exactly optimal the straightforwardness of the argument makes it noticeable.

1. INTRODUCTION

The goal of the paper is to investigate the properties of the dynamical Fredholm determinants of uniformly hyperbolic systems and to relate them to the statistical properties of such systems. This subject has been widely investigated and there exists a large literature where many partial results are obtained. We refer the reader to [1] for references and an introduction to the subject, to [8, 3] for a more recent account of the situation and to [4] for an in depth discussion of the physical relevance of these issues.

The basic idea presented in this paper is to study the action of the dynamics on an appropriate singular functional kernel, as suggested in Dmitry Dolgopyat's thesis (Princeton 1997), to obtain results on the radius of convergence of the dynamical Fredholm determinant, its relation to the spectral properties of the transfer operator and the Ruelle resonances. The new ingredient allowing to carry out such a program is the possibility, after [6] and [2], to introduce spaces in which such singular kernels are legal object. To clarify matters we start with a folklore explanation.

Let X be a d-dimensional $\mathcal{C}^{r+1}$ Riemannian manifold and $T : X \to X$ a $\mathcal{C}^{r+1}$ diffeomorphism which satisfies some hyperbolicity condition. (We assume at least that all the periodic points of T are hyperbolic.) For each $g \in \mathcal{C}^r(X, \mathbb{C})$ we define the Ruelle transfer operator $T_g : \mathcal{C}^r(X, \mathbb{C}) \to \mathcal{C}^r(X, \mathbb{C})$ by

$$T_g h := g \cdot h \circ T. \tag{1.1}$$

The dynamical Fredholm determinant of this operator T_g is formally defined by[1]

$$d^\flat_{T,g}(z) = \exp\left[-\sum_{n\geq 1}\frac{z^n}{n}\sum_{x\in \text{Fix } T^n}\frac{g_n(x)}{|\det(Id - DT^n(x))|}\right], \tag{1.2}$$

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The first named author would like to thank D.Dolgopyat for pointing out to him the wonders of tensor products.

[1] Note that if the number of periodic points does not grow more than exponentially, then $d^\flat_{T,g}(z)$ is well defined and holomorphic in a sufficiently small disk.

where $g_n(x) := \prod_{i=0}^{n-1} g(T^i(x))$.

Let us first note that this can be heuristically regarded as the determinant $\det(Id - z \cdot T_g)$. Indeed, let δ be the distribution on $X^2 = X \times X$ defined by

$$\delta(h) = \int_X h(x,x)dx.$$

Then the "kernel" of the operator T_g^n is given by $(Id \otimes T_g)(\delta)$.[2] Thus, as in the case of operators with smooth kernel, it would be natural to define

$$\operatorname{Tr} T_g = \langle \delta, (Id \otimes T_g)(\delta) \rangle.$$

Though the product of two distribution is not defined in general, we will be able to give an appropriate meaning to the right hand side above since the singular supports (in $\mathcal{C}^r$ sense) of δ and $(Id \otimes T_g)(\delta)$ do not intersect. We find [3]

$$\langle \delta, (Id \otimes T_g)^n(\delta) \rangle = \sum_{x \in \text{Fix } T^n} \frac{g_n(x)}{|\det(Id - DT^n(x))|}.$$

The definition (1.2) of the dynamical Fredholm determinant follows then via the formal relation $\det(Id - zA) = \exp(-\sum_{n=1}^{\infty}(z^n \operatorname{Tr} A^n)/n)$.

It is thus natural to expect that the properties of the dynamical Fredholm determinant as holomorphic function are closely related to the spectral properties of the operator T_g. In this paper, we present an argument providing exactly such a relation, although in a slightly more restrictive setting. The argument is rigorous, yet it follows the above simple ideas very closely.

Let $T : X \to X$ be an Anosov diffeomorphism, i.e. there exists a DT-invariant decomposition $TM = E^u \oplus E^s$ and constants $\lambda \in (0,1)$ and $C > 0$ such that:

$$\|DT^n|_{E^s}\| \le C\lambda^n, \qquad \|DT^{-n}|_{E^u}\| \le C\lambda^n \qquad \text{for all } \ n \ge 0.$$

In [6] and [2], Banach spaces $\mathcal{B}$ of distributions on X are defined so that the operator T_g extends to a bounded operator $T_g : \mathcal{B} \to \mathcal{B}$ whose essential spectral radius is bounded by $\|g\|_{L^\infty} \cdot \lambda^{\alpha_r}$, where $\alpha_r := \min\{[r/2], r - [r/2]\}$, $[a] \in \mathbb{N}$ being the closest integer to $a \in \mathbb{R}$.[4] In addition, it is shown that the eigenvalues outside the essential spectral radius have a well defined dynamical meaning (Ruelle resonances). Let $\rho_* = \|g\|_{L^\infty} \cdot \lambda^{\alpha_r/2}$. Our result is as follows:

Theorem 1. *$d^\flat_{T,g}(z)$ extends holomorphically to $D(\rho_*^{-1}) = \{|z| < \rho_*^{-1}\}$ and the zeros of such an extension are in one-one correspondence, with multiplicity, to the inverse of the eigenvalues of $T_g : \mathcal{B} \to \mathcal{B}$ in the region $\{|z| > \rho_*\}$.*

This result is not new nor optimal. Kitaev [7] has given a stronger result for the extendibility part of the former claim,[5] while the spectral interpretation, albeit for a smaller radius, appeared already in [8] and, more recently, in [3] it has been obtained

[2] Formally, δ is $\delta(x - y)$ (where now δ is the physicists *delta function*) and the kernel is given by $g(x)\delta(T_g x - y)$. As we shall see later, the action of $Id \otimes T_g$ can in fact be extended to an operator on the space of distributions.

[3] One can easily guess this formula by approximating δ by a sequence of $\mathcal{C}^\infty$ functions, see [1] or [8] if details are really needed.

[4] Actually, [2] allows the better bound $\alpha_r = r/2$. Also, [6] deals explicitly only with the adjoint of T_g in the case $g \equiv 1$ (SRB measures), yet the extension to the present setting is straightforward.

[5] Essentially, instead of the bound ρ_*^{-1}, Kitaev has the more natural bound $(\|g\|_{L^\infty} \cdot \lambda^{r/2})^{-1} \sim (\|g\|_{L^\infty} \cdot \lambda^{\alpha_r})^{-1}$, that is the inverse of the bound for the essential spectral radius of $T_g : \mathcal{B} \to \mathcal{B}$.

for a domain corresponding to the result of Kitaev. Nevertheless, the proofs yielding such sharper results are far more complex than the present argument.

The non-optimality of the above theorem is the price for considering, in the following, the operator $T_g^* \otimes T_g$ instead of the operator $Id \otimes T_g$ used in the previous heuristic argument. Unfortunately, we do not know how to treat the transfer operator $Id \otimes T_g$ directly as the mapping $Id \times T$ is not hyperbolic and the extension of the results in [6], [2] to the partially hyperbolic setting is far from trivial (if possible at all).

2. Basic definitions

For $g \in \mathcal{C}^r(X, \mathbb{C})$ we define the transfer operator $T_g : \mathcal{C}^r(X, \mathbb{C}) \to \mathcal{C}^r(X, \mathbb{C})$ by

$$T_g h := g \cdot (h \circ T).$$

Its formal adjoint (in fact, dual) $T_g^* : \mathcal{C}^r(X, \mathbb{C}) \circlearrowleft$ is given by the transfer operator

$$T_g^* f = |\det DT^{-1}| \cdot (g \cdot f) \circ T^{-1}.$$

That is, $\langle \overline{T_g h}, f \rangle_{L^2(X)} = \langle \bar{h}, T_g^* f \rangle_{L^2(X)}$.

Let $\mathcal{D}'_r(X)$ be the space of distribution on X of order r.[6] Using the above formal relation, we can extends the operators T_g and T_g^* to continuous operators $T_g : \mathcal{D}'_r(X) \to \mathcal{D}'_r(X)$ and $T_g^* : \mathcal{D}'_r(X) \to \mathcal{D}'_r(X)$, respectively.

Next we define $T_g^* \otimes T_g : \mathcal{C}^r(X^2, \mathbb{C}) \to \mathcal{C}^r(X^2, \mathbb{C})$ as the unique extension of $T_g^* \otimes T_g : \mathcal{C}^r(X, \mathbb{C}) \otimes \mathcal{C}^r(X, \mathbb{C}) \circlearrowleft$. The latter operator reads

$$T_g^* \otimes T_g(\varphi)(x, y) := g(T^{-1}x) \cdot |\det D_x T^{-1}| \cdot g(y) \cdot \varphi(T^{-1}x, Ty). \tag{2.1}$$

So it can be interpreted as the transfer operator associated to the hyperbolic mapping $T^{-1} \times T$ with the weight $\widetilde{g}(x, y) = g(T^{-1}x) \cdot |\det D_x T^{-1}| \cdot g(y)$. Note that the formal adjoint of $T_g^* \otimes T_g$ is $T_g \otimes T_g^*$. As above, we can extend these operators to

$$T_g \otimes T_g^*(\varphi) : \mathcal{D}'_r(X^2) \to \mathcal{D}'_r(X^2) \quad \text{and} \quad T_g^* \otimes T_g(\varphi) : \mathcal{D}'_r(X^2) \to \mathcal{D}'_r(X^2). \tag{2.2}$$

3. The Banach spaces of distribution

The spaces of distributions on which we have defined the transfer operators are not appropriate for a study of the dynamics. Many recent works (e.g. [6, 2, 5]) have focussed on the problem of finding adapted functional spaces and different choices have different advantages. Accordingly, it may better not to focus on a particular choice but to enlightened which are the properties needed to carry out the study of the Zeta functions. We will list the properties of the function spaces that suffice for our argument. Yet, for definiteness of exposition we will comment explicitly the Banach spaces introduced in [6] and remark that they indeed satisfy such properties.

In [6], S.Gouëzel and the first-named-author introduced a scale of Banach spaces $\mathcal{B}^{p,q}$ with $q \in \mathbb{R}_+$, $p \in \mathbb{N}$ and $p + q < r$ adapted to $\mathcal{C}^{r+1}$ Anosov diffeomorphisms

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[6]Here, by $\mathcal{D}'_r(X)$ we mean the dual of the space $\mathcal{C}^r(X)$ defined as follows. For $r \geq 0$, let $\lfloor r \rfloor$ be its integer part. We denote by $\bar{\mathcal{C}}^r$ the set of functions which are $\lfloor r \rfloor$ times continuously differentiable, and whose $\lfloor r \rfloor$-th derivative is Hölder continuous of exponent $r - \lfloor r \rfloor$ if r is not an integer. To fix notation, in this paper we choose, for each $r \in \mathbb{R}_+$, a norm on $\bar{\mathcal{C}}^r$ functions so that $|\varphi_1 \varphi_2|_{\mathcal{C}^r} \leq |\varphi_1|_{\mathcal{C}^r} |\varphi_2|_{\mathcal{C}^r}$. We will denote by $\mathcal{C}^r$ the closure in $\bar{\mathcal{C}}^r$ of the set of $\mathcal{C}^\infty$ functions. It coincides with $\bar{\mathcal{C}}^r$ if r is an integer, but is strictly included in it otherwise. In any case, it contains $\bar{\mathcal{C}}^{r'}$ for all $r' > r$.

$T : X \to X$. The parameters p and q will be fixed at the end of the argument. We denote $\mathcal{B} = \mathcal{B}^{p,q}$ and set

$$\rho = \rho_{p,q} = \lambda^{\min\{p,q\}}\|g\|_{L^\infty}, \quad \widetilde{\rho} = \rho\|g\|_{L^\infty}.$$

The basic properties of $\mathcal{B}$ are the following (see [6] for a proof):

- **(P1)** $\mathcal{C}^r(X)$ is continuously embedded in $\mathcal{B}$ and its image is a dense subset.
- **(P2)** $\mathcal{B}$ is continuously embedded in $\mathcal{D}'_r(X)$.
- **(P3)** $T_g : \mathcal{B} \to \mathcal{B}$ is a bounded operator with essential spectral radius bounded by ρ.

Remark 3.1. *The meaning of* **(P3)** *is that we use* **(P2)** *to identify $\mathcal{B}$ with a subspace of $\mathcal{D}'_r(X)$ and consider the restriction of T_g to $\mathcal{B}$. With such an identification the embedding in* **(P1)** *is required to be the standard embedding of $\mathcal{C}^\infty$ in $\mathcal{D}'_r(X)$. In the following we will use the embeddings* **(P2)**, *(and* **(P5)***) to identify the elements of $\mathcal{B}$ (and $\widetilde{\mathcal{B}}$) with distributions without making further remarks.*

As already noted in (2.1) the operator $T_g^* \otimes T_g$ is a transfer operator for the Anosov diffeomorphism $T^{-1} \times T : X^2 \to X^2$ with the same hyperbolicity constant λ of T. So, as above, we can introduce a Banach space $\widetilde{\mathcal{B}}$ with the following properties

- **(P4)** $\mathcal{C}^r(X^2)$ is continuously embedded in $\widetilde{\mathcal{B}}$ and its image is a dense subset.
- **(P5)** $\widetilde{\mathcal{B}}$ is continuously embedded in $\mathcal{D}'_r(X^2)$.
- **(P6)** $T_g^* \otimes T_g : \widetilde{\mathcal{B}} \to \widetilde{\mathcal{B}}$ is a bounded operator with essential spectral radius bounded by $\widetilde{\rho}$.

The reader should be aware that usually there is some freedom in the definition of the Banach spaces. For example, in [6] they depend on a family Σ of admissible leafs, that is, $\mathcal{C}^{r+1}$ embedded compact $\dim E^s$ dimensional submanifolds with boundary close to local stable manifolds. By taking the family appropriately,[7] we can insure that $\widetilde{\mathcal{B}}$ enjoys the following extra properties

- **(P7)** $(T_g^* \otimes T_g)^{n_0}(\delta)$ is contained in $\widetilde{\mathcal{B}}$ for some $n_0 \in \mathbb{N}$.
- **(P8)** The functional $\bar{\delta} : \mathcal{C}^\infty(X^2) \to \mathbb{C}$, $\bar{\delta}(\varphi) = \delta(\varphi)$, extends to a bounded functional $\bar{\delta} : \widetilde{\mathcal{B}} \to \mathbb{C}$.

Finally, as it should be apparent from the previous heuristic argument, we need some control on how to approximate singular kernels by smooth ones. Let $\{(U_i, \Psi_i : U \to \mathbb{R}^n)\}_{i=1}^k$ be a $\mathcal{C}^{r+1}$ atlas of X, and let $\{\rho_i\}_{i=1}^k$ be a $\mathcal{C}^\infty$ partition of unity subordinated to such an atlas. Next, define the functions $j_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}_+)$ so that $\int_{\mathbb{R}^d} j_\varepsilon = 1$ and $\operatorname{supp}(j_\varepsilon) \subset \{x \in \mathbb{R}^d \ : \ \|x\| \le \varepsilon\}$. We then define [8]

$$J_\varepsilon f(x) := \sum_i \int_{\mathbb{R}^d} \rho_i \circ \Psi_i^{-1}(y) j_\varepsilon(\Psi_i(x) - y) f \circ \Psi_i^{-1}(y) dy \quad \text{for } f \in \mathcal{C}^r(X).$$

Let J_ε^* be the formal adjoint of J_ε. Clearly these extend to bounded operators

$$J_\varepsilon : \mathcal{D}'_r(X) \to \mathcal{C}^\infty(X) \quad \text{and} \quad J_\varepsilon^* : \mathcal{D}'_r(X) \to \mathcal{C}^\infty(X).$$

We also define

$$\widetilde{J}_\varepsilon : \mathcal{D}'_r(X^2) \to \mathcal{C}^\infty(X^2)$$

[7] In the definition of $\widetilde{\mathcal{B}}$, one can take Σ so that the diagonal in X^2 is covered by finitely many elements in Σ. This immediately implies **(P8)**.

[8] Note that J_ε is well defined for ε small enough.

as the unique extension of $J_\varepsilon \otimes J_\varepsilon : \mathcal{C}^r(X) \otimes \mathcal{C}^r(X) \to \mathcal{C}^\infty(X^2)$. Then we have[9]

(P9) For $u \in \mathcal{B}$, $J_\varepsilon^* J_\varepsilon u \to u$ in $\mathcal{B}$, as $\varepsilon \to +0$.

(P10) For $u \in \widetilde{\mathcal{B}}$, we have $\widetilde{J}_\varepsilon u \to u$ in $\widetilde{\mathcal{B}}$, as $\varepsilon \to +0$.

Remark 3.2. *From now on we will use only the above properties, regardless of the way the spaces are actually constructed.*

Here are two consequence of properties (**P1**)-(**P10**) that show the relevance of the above objects for the problem at hand.

Lemma 3.3. *Let $\{\ell_i\} \subset \mathcal{B}'$ and $\{e_i\} \subset \mathcal{B}$. If $\sum_{i=1}^k \ell_i \otimes e_i$ belongs to $\widetilde{\mathcal{B}}$, then we have*

$$\bar{\delta}\left(\sum_{i=1}^k \ell_i \otimes e_i\right) = \sum_{i=1}^k \ell_i(e_i).$$

Proof. Define $J'_\varepsilon : \mathcal{B}' \to \mathcal{C}^\infty$ by the duality relation $J'_\varepsilon \ell(h) := \ell(J_\varepsilon^* h)$. By (**P10**), $\widetilde{J}_\varepsilon(\sum_{i=1}^k \ell_i \otimes e_i) = \sum_{i=1}^k J'_\varepsilon(\ell_i) \otimes J_\varepsilon(e_i) \in \mathcal{C}^\infty(X^2)$ converges to $\sum_{i=1}^k \ell_i \otimes e_i$ in $\widetilde{\mathcal{B}}$, as $\varepsilon \to +0$. Since

$$\begin{aligned}\bar{\delta}\left(\sum_{i=1}^k J'_\varepsilon(\ell_i) \otimes J_\varepsilon(e_i)\right) &= \sum_{i=1}^k \langle \overline{J'_\varepsilon(\ell_i)}, J_\varepsilon(e_i)\rangle_{L^2(X)} = \sum_{i=1}^k J'_\varepsilon \ell_i(J_\varepsilon(e_i)) \\ &= \sum_{i=1}^k \ell_i(J_\varepsilon^* J_\varepsilon(e_i))\end{aligned}$$

the claim of the lemma holds by (**P8**) and (**P9**). □

Lemma 3.4. *Set $\delta_T = (T_g^* \otimes \mathbf{Id})\delta$. For each $n \ge n_0$, we have*

$$\bar{\delta}((T_g^* \otimes T_g)^n \delta) = \sum_{x \in \mathrm{Fix}\, T^{2n}} \frac{g_{2n}(x)}{|\det(\mathbf{Id} - DT^{2n}(x))|}$$

$$\bar{\delta}((T_g^* \otimes T_g)^n \delta_T) = \sum_{x \in \mathrm{Fix}\, T^{2n+1}} \frac{g_{2n+1}(x)}{|\det(\mathbf{Id} - DT^{2n+1}(x))|}$$

Proof. Let $\delta_\varepsilon = \widetilde{J}_\varepsilon((T_g^* \otimes T_g)^{n_0}\delta)$. By (**P6**), (**P7**) and (**P10**) we have $(T_g^* \otimes T_g)^{n-n_0}\delta_\varepsilon \to (T_g^* \otimes T_g)^n \delta$ in $\widetilde{\mathcal{B}}$. We leave it to the reader to check that the right hand side of the first equality above is obtained as the limit $\lim_{\varepsilon \to +0} \bar{\delta}((T_g^* \otimes T_g)^{n-n_0}\delta_\varepsilon)$.[10] The second equality is obtained in a parallel manner. □

4. The proof

Take $\sigma > \max\{\rho, \widetilde{\rho}\}$ arbitrarily. Then properties (**P3**) and (**P6**) imply

$$T_g^* = P + R : \mathcal{B} \to \mathcal{B}, \qquad T_g^* \otimes T_g = \widetilde{P} + \widetilde{R} : \widetilde{\mathcal{B}} \to \widetilde{\mathcal{B}} \tag{4.1}$$

where P and $\widetilde{P}$ are of finite rank, the spectral radius of R and $\widetilde{R}$ are bounded by σ, and $PR = RP = 0$, $\widetilde{P}\widetilde{R} = \widetilde{R}\widetilde{P} = 0$. Notice that, for each $h, f \in \mathcal{C}^\infty(X)$, $n \in \mathbb{N}$,

$$\begin{aligned}((T_g^* \otimes T_g)^n \delta)(h \otimes f) &= \delta((T_g)^n h \otimes (T_g^*)^n f) = \langle \overline{(T_g)^n h}, (T_g^*)^n f\rangle_{L^2(X)} \\ &= \langle \bar{h}, (T_g^*)^{2n} f\rangle_{L^2(X)} = (T_g^{2n} h)(f),\end{aligned} \tag{4.2}$$

[9] These properties are essentially proven in [6, section 7], or see [8].

[10] If in trouble, see [8] for details

where, in the last expression, h is interpreted as an element of $\mathcal{D}'_r$ (and hence $\mathcal{B}$) via the natural embedding. It follows, using the Neumann series for $|z|$ small enough,

$$(\mathbf{Id} - z(T_g^* \otimes T_g))^{-1}(T_g^* \otimes T_g)^{n_0}\delta(h \otimes f) = ((\mathbf{Id} - z(T_g)^2)^{-1}T_g^{2n_0}h)(f) \tag{4.3}$$

where n_0 is the integer in the condition (**P7**). Note that, from (4.1) and the arbitrariness of σ, both sides of (4.3) have a meromorphic extensions to an closed disk $\{z \in \mathbb{C} \;;\; |z| \le \sigma^{-1}\}$, and the equality must holds for those extensions. Since R^{2n} and $\widetilde{R}^n$ can be written as[11]

$$\begin{aligned} R^{2n} &= \frac{1}{2\pi i}\int_{|\zeta|=\sigma} z^n (z\cdot \mathbf{Id} - (T_g)^2)^{-1}dz \;; \\ \widetilde{R}^n &= \frac{1}{2\pi i}\int_{|\zeta|=\sigma} z^n (z\cdot \mathbf{Id} - (T_g^* \otimes T_g))^{-1}dz \end{aligned}$$

we have, for all $n > n_0$ and each $h, f \in \mathcal{C}^\infty(X)$,

$$\begin{aligned} \widetilde{R}^n\delta(h\otimes f) &:= \widetilde{R}^{n-n_0}(T_g^*\otimes T_g)^{n_0}\delta(h\otimes f) = (R^{2n}(h))(f), \\ \widetilde{P}^n\delta(h\otimes f) &:= \widetilde{P}^{n-n_0}(T_g^*\otimes T_g)^{n_0}\delta(h\otimes f) = (P^{2n}h)(f) \end{aligned} \tag{4.4}$$

Next, write the finite rank operator P^{2n} as $P^{2n}g = \sum_{i=1}^k \ell_i^{(n)}(g)e_i^{(n)}$ where $\ell_i^{(n)} \in \mathcal{B}'$ and $e_i^{(n)} \in \mathcal{B}$. By (4.4), for each $h, f \in \mathcal{C}^\infty(X)$ and $n > n_0$,

$$\widetilde{P}^n\delta(h\otimes f) = \sum_{i=1}^k \ell_i^{(n)}(h)\cdot e_i^{(n)}(f) = \sum_{i=1}^k \ell_i^{(n)} \otimes e_i^{(n)}(h\otimes f).$$

Since $\mathcal{C}^\infty(X)\otimes\mathcal{C}^\infty(X)$ is dense in $\mathcal{C}^p(X^2)$ in the $\mathcal{C}^p$ topology,[12] and since the elements of $\widetilde{\mathcal{B}}$ are distributions of order p by hypothesis, it follows, for each $n > n_0$,

$$\widetilde{\mathcal{B}} \ni \widetilde{P}^n\delta = \sum_{i=1}^k \ell_i^{(n)} \otimes e_i^{(n)} \in \mathcal{B}' \times \mathcal{B}. \tag{4.5}$$

We can thus apply Lemma 3.3 obtaining

$$\bar{\delta}\left(\widetilde{P}^n\delta\right) = \sum_{i=1}^k \ell_i^{(n)}(e_i^{(n)}) = \operatorname{Tr} P^{2n}.$$

In conclusion we have, for $n > n_0$,

$$\bar{\delta}\left((T_g^*\otimes T_g)^n\delta\right) = \bar{\delta}\left(\widetilde{P}^n\delta\right) + \bar{\delta}\left(\widetilde{R}^n\delta\right) = \operatorname{Tr} P^{2n} + \mathcal{O}(\sigma^n).$$

By replacing δ by $\delta_T = (T_g^*\otimes\mathbf{Id})\delta$ in the argument above, we obtain also

$$\bar{\delta}\left((T_g^*\otimes T_g)^n\delta_T\right) = \bar{\delta}\left(\widetilde{P}^n\delta_T\right) + \bar{\delta}\left(\widetilde{R}^n\delta_T\right) = \operatorname{Tr} P^{2n+1} + \mathcal{O}(\sigma^n).$$

[11] Here one has to choose σ so that no eigenvalues belong to the circle $\{z \in \mathbb{C} \;:\; |z| = \sigma\}$.

[12] This is a direct consequence of Stone-Weierstrass theorem.

Using Lemma 3.4 we can conclude

$$\begin{aligned} d^\flat_{T,g}(z) :&= \operatorname{Exp}\left(-\sum_{n=1}^\infty \frac{z^n}{n}\sum_{x\in\operatorname{Fix}\,T^n}\frac{g_n(x)}{|\det(\mathbf{Id}-D_xT^n)|}\right)\\ &= \operatorname{Exp}\left(-\sum_{n=1}^\infty\frac{z^n}{n}\operatorname{Tr}P^n + p_0(z)+\sum_{n=1}^\infty\mathcal{O}(\sigma^{n/2})z^n\right)\\ &= \det(\mathbf{Id}-zP)\operatorname{Exp}\left(-p_0(z)-\sum_{n=1}^\infty\mathcal{O}(\sigma^{n/2})z^n\right), \end{aligned} \tag{4.6}$$

where $p_0(z)$ is a polynomial of order $2n_0+1$.

Since the last series is convergent for $|z|<\sigma^{-1/2}$, we have that the $d^\flat_{T,g}(z)$ is holomorphic in such a disk and the zeroes correspond to the eigenvalues of P, that is to the eigenvalues of $T_g : \mathcal{B}\to\mathcal{B}$ in the region $\{|z|>\sigma^{1/2}\}$.

Applying the argument above to the case where $p=[r/2]$ and q is arbitrarily close to $r-[r/2]$ we obtain Theorem 1. Let us conclude by reiterating the generality of the approach.

Remark 4.1. *Given any Banach spaces $\mathcal{B}$ and $\widetilde{\mathcal{B}}$ satisfying the properties* **(P1-10)** *the proof above applies, hence we obtain Theorem 1 with $\rho_*=\max\{\rho^{-1},\widetilde{\rho}^{-1/2}\}$.*

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