Introduction

A *Dynamical System* is any system that evolves in time.[2] This encompasses quite a few phenomena familiar to us. Of course, at this level of generality, there exists a manifold of very relevant questions that can be asked. In some sense the type of questions investigated through the centuries has shaped what today is meant by Dynamical Systems. Although this is not the proper occasion for an historical excursus, it is worthwhile to stress that the first Dynamical Systems widely investigated have been the planetary motions. Not surprisingly the main emphasis in such investigations was accurate prediction of future positions. Nevertheless, exactly from the effort of predicting accurately future motions stemmed the consciousness of the existence of very serious obstructions to such a program. Specifically, in the work of Poincaré [72] appeared for the first time the phenomena of instability with respect to initial conditions, a central concept in the understanding of modern Dynamical Systems. In fact, we will see briefly that such instability phenomena can be already observed in very simple systems–such as a periodically forced pendulum–that exhibit a so called "homoclinic tangle" [67, 70].

The realization that many relevant systems are very sensitive with respect to the initial conditions dealt a strong blow to the idea that it is always possible to predict the future behavior of a system,[3] yet the work of many physicist (and we must mention at least Boltzmann) and mathematicians (in particular, the so called *Russian School* with people like Kolmogorov, Anosov, Sinai, but also some western mathematicians, like Ruelle and Bowen, gave important contributions) led to the understanding that, although precise predictions where not possible, it was possible and, at times, even easy to make

[2] That is, a set X consisting of all possible configurations of the system and maps $\phi^t : X \to X$ associating to each configuration $x \in X$ the configuration $\phi^t x$ representing the configuration reached by the system at time t when starting from the configuration x at time 0.

[3] Without going to the extreme of some authors of the eighteen century arguing that, given the present state of the universe, a sufficiently powerful mind (maybe God) could predict all the future. Think, more reasonably, of an isolated system and imagine to use some numerical scheme to try to solve the equations of motion for an arbitrarily long time with an arbitrary precision.

statistical predictions. The concept of statistical properties of a Dynamical System is exactly at the base of the material treated in this book.

This introduction is dedicated to making precise, in a simple example, the nature of the above mentioned instability.

0.1 A pendulum–The model and a question

We will study a seemingly trivial example: a forced pendulum. To be more concrete, let us imagine a pendulum of length $l = 1$ meter, mass $m = 1$ kilogram and remember that the gravitational constant (on the earth surface) is approximately $g = 9.8$ meters per second squared. The Hamiltonian of the system reads [41]

$$H = \frac{1}{2l^2 m} p^2 - mgl\cos\theta, \tag{0.1.1}$$

where θ is the angle, counted counterclockwise, formed by the pendulum with the vertical direction ($\theta = 0$ corresponds to the configuration in which the pendulum assumes the lowest possible position) and $p = l^2 m\dot{\theta}$ is the associated momentum. Thus (θ, p) are the coordinates of the pendulum. The phase space $\mathcal{M}$ where the motion takes place consists of $\mathbb{T}^1 \times \mathbb{R}$.

The equations of motion associated to the Hamiltonian (0.1.1) represent the motion of an ideal pendulum in the vacuum feeling only the force of gravity. Clearly, this is an highly idealized situation with no counterpart in realty. Every system interacts with the rest of the universe. Thus the only hope for the idea of *isolated systems* to be fruitful is that the interaction with the exterior does not affect significantly the behavior of the system. Let us try to see what this can mean in reality.

The first issue is clearly friction. Let us imagine that we have set up the pendulum in a reasonable vacuum and reduced the friction at the suspension point so that the loss of energy is negligible on the time scale of few minutes. Does such a system behaves as an isolated pendulum within such a time frame? One problem is that the suspension point is still in contact with the rest of the world. If the pendulum is in a lab not so distant from an street (a rather common situation), then the traffic will induce some vibrations. It is then natural to ask: what happens if the suspension point of the pendulum vibrates?

In fact, nothing much happens for small pendulum oscillations (this is a consequence of Komogorv-Arnold-Moser theory, an highly non trivial fact), but if we start close to the vertical configuration it is conceivable that a motion that would be oscillatory for the unperturbed pendulum could gather enough energy from the external force as to change its nature and become rotatory, this would create a substantial difference between the unperturbed (ideal) and the perturbed (more realistic) case.

This is exactly the question we want to address:

Question: *Can we really predict the motion for a reasonable time if the initial condition is close to the vertical ?*

We will assume that the frequency of vibration ω is of the order of one hertz[4] and the amplitude of the oscillations is very very small. Hence, as good mathematicians, we will call such an amplitude ε. In other words, the suspension point moves vertically according to the law $\varepsilon \cos \omega t$.

The Hamiltonian of the vibrating pendulum is then given by (see Problem 0.1)

$$H_\varepsilon(\theta, p, t) = \frac{1}{2l^2 m} p^2 - mgl\cos\theta - \varepsilon m\omega^2 l \cos\omega t \cos\theta. \tag{0.1.2}$$

Accordingly the equation of motion are (see Problem 0.1)[5]

$$\begin{aligned} \dot\theta &= \frac{\partial H_\varepsilon}{\partial p} = \frac{p}{l^2 m} \\ \dot p &= -\frac{\partial H_\varepsilon}{\partial \theta} = -mgl\sin\theta - \varepsilon m\omega^2 l\cos\omega t \sin\theta. \end{aligned} \tag{0.1.3}$$

It is well known that the function H is an integral of motion for the solutions of (0.1.3) for $\varepsilon = 0$, that is: H computed along the solutions of the associated equations of motion is constant.[6] The physical meaning of H is the energy of the system. Clearly, the energy H_ε is not constant in general since the vibration can add or subtract energy to the pendulum.

0.2 Instability–unperturbed case

Let us first recall few basic facts about the unperturbed pendulum. The equation of motions are given by the (0.1.3) setting $\varepsilon = 0$. It is obvious that there exists two fixed points: $(0, 0)$ which corresponds to the pendulum at rest and is clearly stable , and $(\pi, 0)$ which corresponds to the pendulum in the vertical position and is certainly unstable. Our interest here is to analyze the motions that start close to the unstable equilibrium and to make more precise what it is meant by *instability*.

[4]One hertz corresponds to one oscillation every second, and it can be the order of magnitude for the frequency of a vibration transmitted through the ground (R waves) at a reasonable distance. Thus we are assuming $\omega = 2\pi$.

[5]Here we write the Hamilton equations associated to the Hamiltonian, see [5, 41] for the general theory.

[6]See [5, 41] for this general fact or do Problem 0.4 for the simple case at hand.

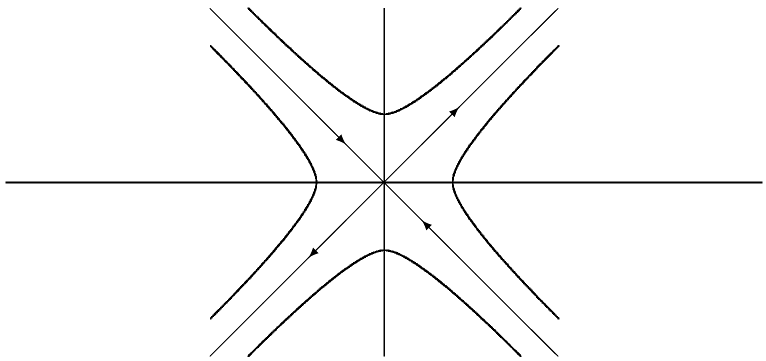

Figure 0.1: Unstable fixed point (phase portrait)

0.2.1 Unstable equilibrium

If we want to have an idea of how the motion looks like near a fixed point the natural first step is to study the linearization of the equation of motion near such a point. In our case, using the coordinates $(\theta_0, p) = (\theta - \pi, p)$, they look like

$$\begin{aligned} \dot{\theta}_0 &= \frac{p}{l^2 m} \\ \dot{p} &= mgl\theta_0. \end{aligned} \tag{0.2.1}$$

Let $\omega_p = \sqrt{\frac{g}{l}}$, the general solution of (0.2.1) is

$$(\theta_0(t), p(t)) = (\alpha e^{\omega_p t} + \beta e^{-\omega_p t}, ml^2\omega_p\{\alpha e^{\omega_p t} - \beta e^{-\omega_p t}\}),$$

where α and β are determined by the initial conditions. Note that if the initial condition has the form $\alpha(1,\, ml\sqrt{gl})$ it will evolve as $\alpha e^{\omega_p t}(1,\, ml\sqrt{gl})$. While if the initial condition is of the form $\beta(1,\, -ml\sqrt{gl})$ it will evolve as $\beta e^{-\omega_p t}(1,\, -ml\sqrt{gl})$. In other words the directions $(1,\, ml\sqrt{gl})$ and $(1,\, -ml\sqrt{gl})$ are invariant for the linear dynamics. The first direction is expanded (and because of this is called *unstable direction*) while the second is contracted (*stable direction*).

Let us imagine to start the motion from an initial condition of the type $(\pi + \theta_0, 0)$, $\theta_0 \in [-\delta, \delta]$, where $\delta \leq 10^{-4}$ represents the precision with which we are able to set the initial condition (one tenth of a millimeter); what will happen under the linear dynamics?

Our initial condition correspond to choosing, at time zero, $\alpha = \beta \leq \frac{\delta}{2}$. As time goes on the coefficient of β becomes exponentially small while the coefficient of α increases exponentially, thus a good approximation of the position of the pendulum after some time is given by

$$\theta_0(t) \approx \alpha e^{\omega_p t}. \tag{0.2.2}$$

Since $\omega_p \approx 3.13$ seconds^{-1}, it follows that after about 2.5 seconds the position of the pendulum can be anywhere up to a distance of about 10 centimeters from the unstable position.

This means that the unstable position is really unstable and if we tray, as best as we can, to put the pendulum in the unstable equilibrium (always imagining that the friction has been properly reduced) it will typically fall after few seconds and it will fall in a direction that we are not able to predict (since it depend on the sign of δ, our unknown mistake). Nevertheless, after the ideal pendulum starts falling in one direction the subsequent motion is completely predictable, as we will see shortly.

An obvious objection to the above analysis is that I did not show that the linearized equation describes a motion really close to the one of the original equations. The answer to this question is particularly simple in this setting and is addressed in the next subsection.

0.2.2 The unstable trajectories (separatrices)

Given the already noted fact that, for $\varepsilon = 0$, H is a constant of motion, the phase space $\mathcal{M}$ is naturally foliated in the level curves of H, on which the motion must take place. This allows us to obtain a fairly accurate picture of the motions of the unperturbed pendulum. In fact, the level curves are given by the equations

$$\frac{p^2}{2l^2m} - mgl\cos\theta = E$$

where E is the energy of the motion. It is easy to see that $E = -mgl$ corresponds to the stable fixed point $(\theta, p) = (0, 0)$; $-mgl < E < mgl$ corresponds to oscillations of amplitude $\arccos\left[\frac{E}{mgl}\right]$; $E > mgl$ corresponds to rotatory motions of the pendulum. The last case $E = mgl$ is of particular interest to us: obviously it corresponds to the unstable fixed point $(\pi, 0)$, yet there are other two solution that travel on the two curves

$$p = \pm ml\sqrt{2lg(1 + \cos\theta)}.$$

This two curves are the ones that separate the oscillatory motions from the rotatory ones and, for this reasons, are called *separatrices*. It is very important to understand the motion along such trajectories, luckily the two differential equations

$$\dot\theta = \pm\sqrt{2\frac{g}{l}(1 + \cos\theta)}. \tag{0.2.3}$$

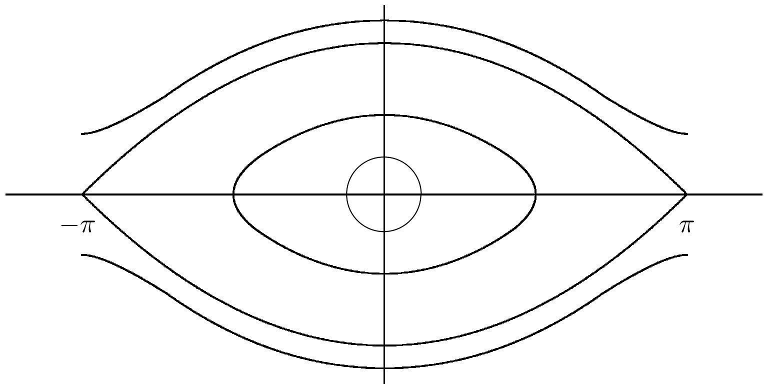

Figure 0.2: Unperturbed pendulum (phase portrait)

can be integrated explicitly (see Problem 0.5) yielding, for $\theta(0) = 0$,

$$\theta(t) = 4 \arctan e^{\pm \omega_p t} - \pi. \tag{0.2.4}$$

This orbits are asymptotic to the unstable fixed point both at $t \to +\infty$ and at $-\infty$ and, for $|t|$ large, agree with the linear behaviour of section 0.2.1. This situation is somewhat atypical as we will see briefly.

0.3 The perturbed case

0.3.1 Reduction to a map

The motion of the above system takes place on the cylinder $\mathcal{M} = S^1 \times \mathbb{R}$. By the theorem of existence and uniqueness for the solutions of differential equations [31, 6] follows immediately the possibility to define the maps ϕ^t_ε : $\mathcal{M} \to \mathcal{M}$ associating to the point (θ, p) the point reached by the solution of (0.1.3) at time t, when starting at time 0 from the initial condition (θ, p). In such a way we define the *flow* ϕ^t_ε associated to the (0.1.3).

Clearly $\phi^0_\varepsilon(\theta, p) = (\theta, p)$, that is the map corresponding at time zero is the identity. Moreover, if $\varepsilon = 0$ the system is autonomous (the vector field does not depend on the time) hence the flow defines a group: for each $t, s \in \mathbb{R}$

$$\phi^{t+s}_0(\theta, p) = \phi^t_0(\phi^s_0(\theta, p)).$$

This corresponds to the obvious fact that the motion for a time $t + s$ can be obtained first as the motion from time 0 to time s, and then pretending that the time s is the initial time and following the motion for time t.

Of course, the above fact does not hold anymore when $\varepsilon \neq 0$. In this case, the maps ϕ_ε^t depend from our choice of the initial time (if we define them by starting from time 1 instead then time 0, in general we obtain different maps). Nevertheless, due to the fact that the external force is periodic something can be saved of the above nice property.

Let us define the map $T_\varepsilon : \mathcal{M} \to \mathcal{M}$ by

$$T_\varepsilon = \phi_\varepsilon^{\frac{2\pi}{\omega}},$$

then (see Problem 0.3), for each $n \in \mathbb{Z}$,

$$T_\varepsilon^n = \phi_\varepsilon^{\frac{2n\pi}{\omega}}. \tag{0.3.1}$$

The interest of (0.3.1) is that, for many purposes, we can study the map T_ε instead than the more complex object ϕ_ε^t. Morally, it means that if we look at the system *stroboscopically*, that is only at the times $\frac{2\pi}{\omega}n$ with $n \in \mathbb{Z}$, then it behaves like an autonomous (time independent) system.[7] Another interesting fact is that the flow ϕ_ε^t (and hence also the map T_ε) is area preserving (see Problem 0.7).[8]

0.3.2 Perturbed pendulum, $\varepsilon \neq 0$

The situation for the case $\varepsilon \neq 0$ is more complex and no easy way exists to study these motions.

As a general strategy, to study the behavior of a system (in our case the map T_ε) it is a good idea to start by investigating simple cases and then move on from there. In our systems the simplest motion consists of the equilibrium solutions. These are the time independent solutions.[9] Because of the special type of perturbation chosen the fixed points of the system for the case $\varepsilon = 0$ remain unchanged when $\varepsilon \neq 0$ (see Problem 0.8 for a brief discussion of a more general case).

Next, we can study the infinitesimal nature of the fixed points. It is natural to expect that the nature of the two fixed points does not change if ε is small, yet to verify this requires some checking. We will discuss explicitly only the fixed point $(\pi, 0)$.

The first step is to make precise the sense in which the case $\varepsilon \neq 0$ is a perturbation of the case $\varepsilon = 0$. This can be achieved by obtaining an explicit

[7] This is a very simple case of a very fruitful an general strategy: to look at the system only when some special event happens–in our case at each time in which the suspension point has its maximum height. See 1.2 if you want to know more.

[8] This also is a special instance of a more general fact: the Hamiltonian nature of the system, see [41], [5] if you want to know more.

[9] That is, equilibrium solutions for the map T_ε. These are *periodic* solutions for the flows of period $\frac{2\pi}{\omega}$. In fact, $T_\varepsilon x = x$ means $\phi^{\frac{2\pi}{\omega}} x = x$.

estimate on the size of

$$R_\varepsilon = \varepsilon^{-1}(T_0 - T_\varepsilon).$$

Let $z(t) = (z_1(t), z_2(t)) = \phi_0^t(x) - \phi_\varepsilon^t(x)$, then substituting in (0.1.3) and subtracting the general case to the case $\varepsilon = 0$ it yields

$$\begin{aligned} |\dot z_1| &\leq \frac{|z_2|}{ml^2} \\ |\dot z_2| &\leq mgl|z_1| + \varepsilon m\omega^2 l. \end{aligned}$$

In order to get better estimates it is convenient to define the new variables $\zeta_1 = z_1$ and $ml^2\omega_p\zeta_2 = z_2$. In these new variables the preceding equations read

$$\begin{aligned} |\dot\zeta_1| &\leq \omega_p|\zeta_2| \\ |\dot\zeta_2| &\leq \omega_p|\zeta_1| + \varepsilon\frac{\omega^2}{\omega_p l}. \end{aligned} \tag{0.3.2}$$

Which implies $\|\dot\zeta\| \leq \omega_p\|\zeta\| + \varepsilon m\omega^2 l$. Taking into account that, in our situation, $ml^2\omega_p > 1$, it follows (see Problem 0.9)

$$\|R\|_{\mathcal{C}^0} \leq \frac{m\omega^2}{l\omega_p}(e^{2\pi\frac{\omega_p}{\omega}} - 1) \leq 69.$$

Unfortunately, the above norm does not suffice for our future needs. We will see quite soon that it is necessary to estimate also the first derivatives of R, that is the $\mathcal{C}^1$ norm.

To do so the easiest way is to use the differentiability with respect to the initial conditions of the solutions of our differential equation (see [6, 31]). Fixing any point $x \in \mathcal{M}$ and calling $\xi^\varepsilon(t) = d_x\phi_\varepsilon^t\xi(0)$ we readily obtain:[10]

$$\begin{aligned} \dot\xi_1^\varepsilon &= \frac{\xi_2^\varepsilon}{l^2 m} \\ \dot\xi_2^\varepsilon &= -mgl\cos\theta\,\xi_1^\varepsilon - \varepsilon m\omega^2 l\cos\omega t\cos\theta\,\xi_1^\varepsilon \end{aligned} \tag{0.3.3}$$

One can then estimate the $\mathcal{C}^1$ norm of R by estimating $\|\xi^\varepsilon(\frac{2\pi}{\omega}) - \xi^0(\frac{2\pi}{\omega})\|$, since $\xi^\varepsilon(\frac{2\pi}{\omega}) = D_{(\theta,p)}T_\varepsilon\xi^\varepsilon(0)$. Doing so one obtains[11]

$$\|R\|_{\mathcal{C}^1} \leq \frac{2m\omega^2}{l\omega_p}e^{3\pi\frac{\omega_p}{\omega}} := d_1 \leq 690. \tag{0.3.4}$$

[10]The vector $\xi_\varepsilon(t)$ is nothing else than the derivative $\frac{d\phi_\varepsilon^t(x+s\xi(0))}{ds}|_{s=0}$, the following equation is then obtained by exchanging the derivative with respect to t with the derivative with respect to s.

[11]The following bounds are not sharp, working more one can obtain better estimates but this would not make much of a difference in the sequel.

0.4 Infinitesimal behavior (linearization)

As a first application of the above considerations let us study the linearization of T_ε at $x_f = (\pi, 0)$. From (0.3.3) follows (see Problem 0.12)

$$
\begin{aligned}
D_{x_f}T_0 &= \begin{pmatrix} \cosh\frac{2\pi\omega_p}{\omega} & \frac{\sinh\frac{2\pi\omega_p}{\omega}}{ml^2\omega_p} \\ ml^2\omega_p\sinh\frac{2\pi\omega_p}{\omega} & \cosh\frac{2\pi\omega_p}{\omega} \end{pmatrix} \\
D_{x_f}T_\varepsilon &= D_{x_f}T_0 + \mathcal{O}(d_1\varepsilon)
\end{aligned} \tag{0.4.1}
$$

The eigenvalues of $D_{x_f}T_\varepsilon$ are then $\lambda_\varepsilon = e^{\frac{2\pi\omega_p}{\omega}} + \mathcal{O}(d_2\varepsilon)$,[12] λ_ε^{-1}, where $d_2 = 2d_1\omega_p ml^2 \simeq 4400$. In addition, calling v_ε, $\langle v_\varepsilon, v_0\rangle = 1$, the eigenvector associate to λ_ε, holds true $\|v_0 - v_\varepsilon\| \leq d_3\varepsilon$, $d_3 = 4\lambda_0^{-1}\omega_p^2\omega^2 l^4 d_1 \simeq 1200$.[13]

Clearly, if ε is sufficiently small, then $\lambda_\varepsilon > 1$. This means that the hyperbolic nature of the unstable fixed point remains unchanged under small perturbations (see Problem 0.13 for a case when the perturbation is not so small).[14]

If one does a similar analysis at the fixed point $(0,0)$ one finds that the eigenvalues have modulus one: that is the infinitesimal motion is a rotation around the fixed point, exactly as in the $\varepsilon = 0$ case.

Hence the comments made at the end of subsection 0.2.1 for the unperturbed pendulum hold for the perturbed pendulum as well. Only now the is no longer an integral of motion (the energy) that controls globally the behavior of the system.

Imagining that the map is linear (which is clearly false but, as we will see, qualitatively not so wrong) this would mean that the distance between two trajectories can be expanded by almost a factor 23 in a second. Initial conditions that are δ close at time zero will be about 23δ far apart after 1 second. If such a state of affair could persist (and we will see it may) after one minute the two configurations would differ roughly by a factor $10^{80}\delta$, which means that not even knowing the initial condition plus or minus a quark could we predict the final one. This is certainly a rather worrisome perspective but much more work it is needed to decide if this may be indeed the case.

[12] In this chapter we will adopt the strict convention that $\mathcal{O}(x)$ means a quantity bounded, in absolute value, by x.

[13] This follows by the fact that the eigenvalues of $D_{x_f}T_0$ are $e^{\pm\frac{2\pi\omega_p}{\omega}} \simeq (23)^{\pm 1}$, a simple perturbation theory of matrices (see Problems 0.10, 0.11) and the already mentioned fact that the map T_ε is area preserving, thus the determinant of its derivative must be one.

[14] As we will see later in detail, hyperbolicity means that there is a direction in which the maps expands (the eigenvector v_ε^u associated to the eigenvalue λ_ε) and a direction in which the map contracts (the eigenvector v_ε^s associated to the eigenvalue λ_ε^{-1})

0.5 Local behavior (Hadamard-Perron Theorem)

The next step is to try to go from the above infinitesimal analysis to a local picture in a small neighborhood of the fixed points.

It is natural to expects that the two fixed points are still stable and unstable respectively, yet this is a far from trivial fact.

The stability of the point $(0,0)$ can be proven by invoking the so called KAM Theorem (this exceeds the scope of the present book and we will not discuss such matters, see [41] for such a discussion).[15]

The study of the local behavior around the point x_f is instead a bit easier and can be performed by applying the Hadamard-Perron Theorem 3.3.2 to conclude that, in a neighborhood of $(\pi, 0)$, there exists two curves $x^u_\varepsilon(s) = (\theta^u_\varepsilon(s), p^u_\varepsilon(s))$, $x^s_\varepsilon(s)$ that are invariant with respect to the map T_ε. Namely, there exists $\delta_\varepsilon > 0$ such that $T_\varepsilon x^s_\varepsilon([-\delta_\varepsilon, \delta_\varepsilon]) \subset x^s_\varepsilon([-\delta_\varepsilon, \delta_\varepsilon])$ and $T_\varepsilon^{-1} x^u_\varepsilon([-\delta_\varepsilon, \delta_\varepsilon]) \subset x^u_\varepsilon([-\delta_\varepsilon, \delta_\varepsilon])$; this are called the local stable and unstable manifold of zero, respectively. Essentially δ_ε is determined by the requirement that the non-linear part of T_ε be smaller than the linear part.

Clearly, for $\varepsilon = 0$ $x^s_0 = x^u_0 = x_0$ and it coincides with the homoclinic orbit of the unperturbed pendulum. In addition, by Hadarmd-Perron and the estimates of the previous section, we can choose δ_ε such that

$$\|x^u_\varepsilon - x_0\| \leq 2 d_3 \varepsilon \|x_0\|. \tag{0.5.1}$$

and the analogous for the stable manifold. We have so obtained a local picture of the behavior of the map T_ε, yet this does not suffice to answer to our original question. To do so we need to follow the motion for at least a full oscillation: this requires really a global information.

To gain a more global knowledge we can try to construct larger invariant set for the map T_ε. A natural way to do so is to iterate: define $W^u = \cup_{n=0}^\infty T_\varepsilon^n x^u([-\delta_\varepsilon, \delta_\varepsilon])$. Since $T_\varepsilon x^u([-\delta_\varepsilon, \delta_\varepsilon]) \supset x^u([-\delta_\varepsilon, \delta_\varepsilon])$, it is clear that each time we iterate we get a longer and longer curve. The set W^u is then clearly a manifold and it is called the global unstable manifold.[16]

The global manifold, as the name clearly states, it is a global object: it carries information on the dynamics for arbitrarily long times. Yet, the procedure by which it has been defined is far from constructive and the truth

[15] In some sense this implies that we can indeed predict the motion for an extremely long time if we consider only oscillations close to the configuration $(0,0)$, so in that case the assumption that the pendulum is isolated is legitimate. Yet, this depends on the precision we are interested in and tends to degenerate if the amplitude of the oscillations is rather large. A complete analysis would be a very complicated matter but we will have an idea of the type of problems that can arise by considering extremely large oscillations, close to a full rotation of the pendulum.

[16] Applying the above procedure to the unperturbed problem yields the full separatrix.

is that, besides the sketchy considerations above, at the moment we know very little of it. The next step is to gain some more detailed understanding of a large portion of W^u.

0.6 A more global understanding (the Melnikov method)

From the above considerations follows that the stable and unstable manifolds $(\theta_\varepsilon^s(s),\, p_\varepsilon^s(s))$, $(\theta_\varepsilon^u(s),\, p_\varepsilon^u(s))$, $|s| \leq \delta_\varepsilon$, of T_ε at 0, are ε close to the homoclinic orbit of the unperturbed pendulum, $(\theta_0(t),\, p_0(t))$, $\theta_0(0) = 0$.

Note, however, that while $x_0 = (\theta_0, p_0)$ is invariant under the unperturbed flow, the same does not apply to $(\theta_\varepsilon^{s,u}(s),\, p_\varepsilon^{s,u}(s))$ under ϕ_ε^t. Indeed the invariant object is the time-space surface $(\tau, x_\varepsilon^{s,u}(s,\tau)) := (\tau, \phi_\varepsilon^\tau(\theta_\varepsilon^s(s),\, p_\varepsilon^s(s)))$ where $(s,\tau) \in [-\delta_\varepsilon, \delta_\varepsilon] \times [0, \frac{2\pi}{\omega}]$ and and $\tau = t \mod \frac{2\pi}{\omega}$.[17]

Clearly, we can choose freely the parameterization of our curves in such a surface and some are more convenient than others. The separatrix of the unperturbed pendulum is most conveniently parametrized by time, hence $\phi^t(\theta_0(s), p_0(s)) = (\theta_0(s+t), p_0(s+t))$. We wish to parameterize the perturbed manifold in a convenient way, one simple possibility could be to impose $\theta_\varepsilon^u(-s) = \theta_0(-s)$, $\theta_\varepsilon^s(s) = \theta_0(s)$, yet this happens to be not very helpful for our goals. To find a more convenient parameterization it is necessary to do first some preliminary considerations.

To grow the above manifolds, as explained in the previous section, we can start from some remote time $-S_n := 2\pi\omega^{-1}n$, $n \in \mathbb{N}$, (S_n for the stable) and then iterate forward the unstable manifold and backward the stable. This is better done by using the flow and the equations of motion. To this end, it turns out to be specially smart to first use global coordinates similar to the ones used to simplify equation (0.3.2) and then to consider local coordinates adapted to the separatrix of the unperturbed pendulum. Namely, let us introduce $p =: ml^2\omega_p \tilde{p}$, $\theta =: \tilde{\theta}$. Note that such a change of coordinate is not symplectic, hence we have to compute the resulting Hamiltonian in the new coordinates. It is easy to verify that the Hamiltonian becomes

$$\tilde{H}_\varepsilon := \frac{\omega_p}{2}\tilde{p}^2 - \omega_p \cos\tilde{\theta} - \varepsilon\frac{\omega^2}{l\omega_p}\cos\omega t \cos\tilde{\theta} =: H_0 + \varepsilon H_1 \tag{0.6.1}$$

[17] A standard way to bring the present non-autonomous setting in the more familiar autonomous one is to introduce the fake variables $(\varphi, \eta) \in S^1 \times \mathbb{R}$ and the new, time independent, Hamiltonian $\bar{H}_\varepsilon(\theta, p, \varphi, \eta) := H_\varepsilon(\theta, p, \varphi) + \frac{2\pi}{\omega}\eta$. The Hamilton equations yield $\varphi(t) = \frac{2\pi}{\omega}t + \varphi(0)$ and hence the equations for θ, p reduce to (0.1.3). Since $\bar{H}_\varepsilon$ is now conserved under the motion we can restrict the system to the three dimensional manifold $\bar{H}_\varepsilon = 0$. In such a manifold we have the *weak* stable and unstable manifolds (now flow invariant) $(x_\varepsilon^{s,u}(s,\varphi), \varphi, -\frac{2\pi}{\omega}H_\varepsilon((x_\varepsilon^{s,u}(s,\varphi),\varphi))$.

which yields the corrects equations of motion.

$$\begin{aligned}\dot{\tilde{\theta}} &= \omega_p \tilde{p} \\ \dot{\tilde{p}} &= -\omega_p \sin\tilde{\theta} - \varepsilon \frac{\omega^2}{l\omega_p}\cos\omega t \sin\tilde{\theta}\end{aligned} \tag{0.6.2}$$

We will use the vector notation $x := (\tilde{\theta}, \tilde{p})$.[18] In such coordinates we consider the stable and unstable manifolds $x^s_\varepsilon(s), x^u_\varepsilon(s)$ for the perturbed pendulum and the separatrix $x_0(s)$ for the unperturbed pendulum and we define

$$x^{s,u}_\varepsilon(s,\, t) = \phi^t_\varepsilon x^{s,u}_\varepsilon(s). \tag{0.6.3}$$

If we call ϕ^t_ε the flow started at the time $-S_n$ (S_n, respectively),[19] and we consider $t = S_m$ ($t = -S_m$), $m < n$, we obtain new curves that are much longer than the original ones and still describe the unstable and stable manifolds (albeit with a different parameterization). Next, we define the vectors

$$\eta_1(s) := \frac{\dot{x}_0(s)}{\|\dot{x}_0(s)\|} = \frac{J\nabla_{x_0(s)}H_0}{\|\nabla_{x_0(s)}H_0\|} \quad \text{and} \quad \eta_2(s) := \frac{\nabla_{x_0(s)}H_0}{\|\nabla_{x_0(s)}H_0\|}.$$

This form an orthonormal basis of $\mathbb{R}^2$ (see Problem 0.14). We can then consider the map $F(a,b) := x_0(a) + b\eta_2(a)$. One can check that $\det DF_{(a,0)} \neq 0$, hence F defines a change of coordinates in a neighborhood of x_0. Note that in the new coordinates the unperturbed separatrix x_0 reads $\{(a,0)\}$.

In analogy with a standard approach to the Hadamard-Perron Theorem (see 3.3.2) it seems natural to have our curves parametrized so that, in the new coordinates, they have the same first component. This means that we would like to have $\langle x^u_\varepsilon(s) - x_0(s), \eta_1(s)\rangle = 0$. We can obviously arrange such a property for the original curve at the tine $-S_n$, but can we keep it thruought the growth process? A simple possibility is to flow different points for different times as to maintain the wanted property. That is to look for a τ such that,[20]

$$G(s,t,\tau) := \langle x^u_\varepsilon(s, t+\tau) - x_0(s+t), \eta_1(s+t)\rangle = 0. \tag{0.6.4}$$

Since, by construction, $G(s,0,0) = 0$ we can apply the implicit function theorem, to prove the existence of the wanted function $\tau(s,t)$. The necessary

[18] Using such a notation equations (0.6.2) take the more compact form

$$\dot{x} = J\nabla_x \tilde{H}_\varepsilon \; ; \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

[19] Remember that the flow started at such times is exactly the same than the flow started at time zero, see subsection 0.3.1.

[20] Note that, in so doing, we will construct an object different from the starting one associated to a fixed Poincarè section.

condition to do so is a lower bound on $|\partial_\tau G|$. Next, setting $x^u_\varepsilon(s,t+\tau) =: x_0(s+t) + \varepsilon x^u_1(s,t,\tau)$,

$$\begin{aligned}\partial_\tau G(s,t,\tau) &= \langle J\nabla_{x^u_\varepsilon(s,t+\tau)}\tilde{H}_\varepsilon, \eta_1(s+t)\rangle \\ &= \|\nabla_{x^u_0(s+t)}H_0\| + \varepsilon\mathcal{O}(\|D^2\tilde{H}_\varepsilon\|\,\|x^u_1\| + \|\nabla_{x^u_0}H_1\|).\end{aligned} \tag{0.6.5}$$

By (0.5.1) we have $\|x^u_\varepsilon(s) - x_0(s)\|\,\|x_0(s)\|^{-1} \le 2d_3\varepsilon$, for $s \le -T_{n_0}$. In addition, from (0.2.4) and Problem 0.6 follows $\sin\tilde{\theta}_0(t) = 2\frac{\sinh\omega_p t}{(\cosh\omega_p t)^2} \simeq 2e^{\omega_p t}$, for $t \ll 0$. Moreover $\tilde{p}_0 = \sqrt{2(1+\cos\tilde{\theta}_0)} = 2(\cosh\omega_p t)^{-1}$. Then $\|\nabla_{x_0(t)}H_0\| \ge \frac{\omega_p}{\sqrt{2}}e^{-\omega_p|t|}$.

Accordingly, remembering equations(0.5.1) and (0.6.5) we can apply the Implicit Function Theorem provided $\|x^u_1(s,t,\tau)\| \le 4d_3e^{-\omega_p|s+t|}$ and $\varepsilon \le (8d_3)^{-1} \simeq 10^{-4}$. Hence the wanted function $\tau(s,t)$ is well defined and[21]

$$\frac{\partial\tau}{\partial t} = -\frac{\partial_t G}{\partial_\tau G} = \mathcal{O}(64d_3\varepsilon). \tag{0.6.6}$$

It is then convenient to define

$$\Delta^u(s,\,t) = \varepsilon^{-1}\langle x^u_\varepsilon(s,t,\tau) - x_0(s,t),\,\nabla_{x_0(s+t)}H_0\rangle = \|x^u_1\|\,\|\nabla_{x_0}H_0\|.$$

Using (0.1.3) we can differentiate Δ^u with respect to t and since $J\nabla_{x^u_\varepsilon}H_\varepsilon = J\nabla_{x_0}H_\varepsilon + \varepsilon JD^2_{x_0}H_\varepsilon x^u_1 + \mathcal{O}(\frac{\varepsilon^2}{2}\|D^3H_0\|\,|x^u_1|^2)$, we have

$$\begin{aligned}&\frac{d\Delta^u}{dt}(s,\,t) = \varepsilon^{-1}\langle J\nabla_{x^u_\varepsilon}H_\varepsilon(1+\dot{\tau}), \nabla_{x_0}H\rangle + \langle x^u_1, D^2_{x_0}H_0 J\nabla_{x_0}H_0\rangle \\ &= \left\{\langle J\nabla_{x_0}H_1, J\nabla_{x_0}H_0\rangle + \mathcal{O}(2\varepsilon\omega_p d_3 e^{\omega_p(t+s)}|\Delta^u_1|)\right\}(1+|\dot{\tau}|_\infty) \\ &\quad + \mathcal{O}(|\dot{\tau}|_\infty|\Delta^u_1|\omega_p).\end{aligned} \tag{0.6.7}$$

We can thus integrate the Gronwald type inequality (0.6.7), (if in doubt, see Problem 0.9), and, assuming $256\omega_p d_3\varepsilon < 1$ (roughly $\varepsilon \le 10^{-5}$),

$$|\Delta^u(s,t)| \le \frac{8\omega^2}{l\omega_p}e^{2\omega_p(t+s)}.$$

Hence, $\|x^u_1\| \le \frac{24\omega^2 e^{\omega_p(t+s)}}{l\omega_p^2} < 4d_3e^{-\omega_p|t+s|}$, provided it holds true $t+s \le (2\omega_p)^{-1}\ln\left[\frac{d_3 l\omega_p^2}{6\omega^2}\right] =: t_0 \simeq 0.6$.

To gain complete control on the stable manifold we need only to discuss the issue of the time shift. On the one hand, all is needed is to change

[21] Indeed, $\partial_t G = \langle J\nabla_{x^u_\varepsilon}\tilde{H}_\varepsilon - J\nabla_{x_0}H_0, \eta_1\rangle + \varepsilon\langle x^u_1, \dot{\eta}_1\rangle = \varepsilon\mathcal{O}(3\omega_p\|x^u_1\| + \|\nabla_{x_0}H_1\|)$.

$t+\tau(s,t)$ to zero ($\mod \frac{2\pi}{\omega}$). On the other hand if $\rho \in [0, \frac{2\pi}{\omega}]$, then $\zeta(\rho) := \phi_\varepsilon^\rho(x) - \phi_0^\rho(y)$ can be estimated, slightly refining (0.3.2), by integrating $\|\dot{\zeta}\| \leq (\omega_p + \varepsilon\frac{\omega^2}{\omega_p l})\|\zeta\| + \varepsilon\frac{\omega^2}{\omega_p l}|\theta_0(t+s+\rho)|$. This shows that we can extend the unstable manifold till a neighborhood of $x_0(-S_2)$ and still keep the an inequality of the type $\|x_\varepsilon^u - x_0\| \leq 3d_3\|x_0\|$.

Finally, substituting the above estimate in (0.6.7), yields

$$\frac{d\Delta^u}{dt}(s,\, t) = \langle J\nabla_{x_0}H_1, J\nabla_{x_0}H_0\rangle + \mathcal{O}\left(544 \cdot l^{-1}\omega^2 d_3 \varepsilon e^{2\omega_p(t+s)}\right).$$

Integrating from 0 to S_m, $m \in \mathbb{N}$ for $s + S_m \leq t_0$, yields

$$\begin{aligned}\Delta^u(s, S_m) &= \int_0^{S_m} \{H_1(\cdot, t_1,\, H\}_{x_0(s+t_1)} dt_1 + \Delta^u(s,0) + \mathcal{O}\left(\varepsilon d_4 e^{2\omega_p(s+S_m)}\right) \\ &= \int_{-S_m}^0 \{H_1(\cdot, t_1,\, H\}_{x_0(s+S_n+t_1)} dt_1 + \Delta^u(s,0) + \mathcal{O}\left(\varepsilon d_4 e^{2\omega_p(s+S_m)}\right)\end{aligned} \tag{0.6.8}$$

where $d_4 := 272 \cdot \frac{\omega^2}{\omega_p l} d_3 \simeq 4 \cdot 10^6$ and the curly brackets stand for the so called *Poisson brackets* ($\{f,\, g\}_x = \langle J\nabla_x f,\, \nabla_x g\rangle$).

The stable manifold can be studied similarly, yet it is faster to define the transformation $\Psi(\theta, p) = (-\theta, p)$, and note that $\phi_\varepsilon^{-t}(\Psi(x)) = \Psi(\phi_\varepsilon^t(x))$. Accordingly, $x_\varepsilon^s(s,-t) = \Psi(x_\varepsilon^u(-s,t))$. Also, one easily checks that, calling $\tau^s(s,t)$ the time shift arising from the analogous of (0.6.4), $\tau^s(s,-t) = \tau(-s,t)$. In addition, $|\tau(s, S_m)| \leq 65 d_3 \varepsilon S_m$.

Setting $\Delta(\sigma) := \Delta^u(-s-S_m, S_m) - \Delta^s(s+S_m, -S_m)$, for all $\sigma \in [-t_0, t_0]$, we finally have

$$\begin{aligned}&\|x_\varepsilon^u(-\sigma - S_m, S_m) - x_\varepsilon^s(S_m - \sigma, S_m)\| \leq 64\frac{4\pi\omega - p}{\omega} d_3 \varepsilon S_m + \Delta(\sigma) \\ &\Delta(\sigma) = \int_{-\infty}^{\infty} \{H_1,\, H\}_{x_0(t+\sigma)} dt + \mathcal{O}\left(\varepsilon 2 d_4 e^{2\omega_p|\sigma|}\right),\end{aligned} \tag{0.6.9}$$

provided $m > 2$. The integral in (0.6.9) is called *Melnikov integral* and provides an expression, at first order in ε, of the distance between the stable and the unstable manifold. All we are left with is to compute the integrals in (0.6.9). This turns out to be an exercise in complex analysis and it is left to the reader (see Problem 0.15), the result is:[22]

$$\int_{-\infty}^{\infty} \{H_1(\cdot, t),\, H\}_{x_0(t+\sigma)} dt = 8\pi m l \frac{\omega^4 e^{-\frac{\pi\omega}{2\omega_p}}}{\omega_p^2(e^{\frac{\pi\omega}{\omega_p}} - 1)} \sin\omega\sigma.$$

[22] A simple computation yields:

$$\{H_1,\, H\}_{x_0(t+s)} = -\frac{\omega^2}{l} p(t+s)\cos\omega t \sin\theta(t+s).$$

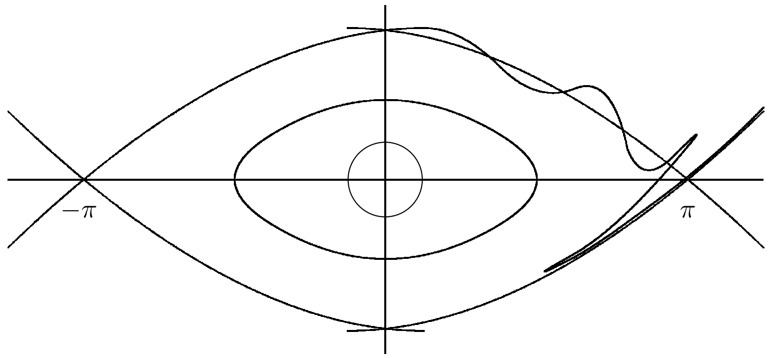

Figure 0.3: Perturbed pendulum

We have thus gained a very sharp control on the shape of the above manifolds.[23] In particular, $\Delta(\pm 1/4) \simeq \pm 76 + \mathcal{O}(4 \cdot 10^7 \varepsilon) \neq 0$ provided $\varepsilon \leq 1.5 \cdot 10^{-6}$, that is the two manifolds intersect. To understand a bit better such an intersection (we would like to know that in the region $\sigma \in [-1/4, 1/4]$ there is only one *transversal* intersection) its suffices to notice that (0.6.7) provides a control on the angle between x^u_ε and x_0.

This intersections are called *homoclinic* intersection and their very existence is responsible for extremely interesting phenomena as can be readily seen by trying to draw the stable and unstable manifolds (see Figure 0.3 for an approximate first idea); we will discuss this issue in detail shortly.[24]

We have gained much more global information on the map T_ε, yet it does not suffice to answer to our question. The next section is devoted to obtaining a really global picture. Up to now we have used mainly analytic tools. Next, geometry will play a much more significant rôle.[25]

Then, by using (0.2.4) and looking at Problem 0.6, one readily obtains:

$$\{H_1, H\}_{x_0(t)} = 4\frac{\omega^2}{l}\frac{\cos\omega(t-s)\sinh\omega_p t}{(\cosh\omega_p t)^3}.$$

Finally, use Problem 0.15.

[23] Note that ε must be exponentially small with respect to ω. In many concrete problems (notably the so called *Arnold diffusion* [1]) it happens that this it is not the case. One can try to solve such an obstacle by computing the next terms of the ε expansion of Δ. In fact, it turns out that it is possible to express Δ as a power series in ε with all the terms exponentially small in ω [1]. Yet this is a quite complex task far beyond our scopes.

[24] Note that the intersection corresponds to an homoclinic orbit for the map T_ε (that is, an orbit which approaches the fixed point x_f both in the future and in the past). This is what it is left of the homoclinic orbit of the unperturbed pendulum.

[25] What comes next is the first example in this book of what is loosely called *a dynamical argument*.

0.7 Global behavior (an horseshoe)

We want to explicitly construct trajectories with special properties. A standard way to do so is to start by studying the evolution of appropriate regions and to use judiciously the knowledge so gained. Let us see what this does mean in practice.

The starting point is to note that we understand the shape of the invariant manifold but not very well the dynamics on them, this is our next task. Since points on the unstable manifolds are pulled apart by the dynamics, the estimate must be done with a bit of care. In fact, we will use a way of arguing which it typical when instabilities are present, we will see many other instances of this type of strategy in the sequel.

For each x in the unstable manifold (zero included) let us call $D^u_xT_\varepsilon := D_xT_\varepsilon v^u(x)$, where $v^u(0) = v^u$ and if $x = x^u_\varepsilon(t)$ then $v^u(x) = \|\dot{x}^u_\varepsilon(t)\|^{-1}\dot{x}^u_\varepsilon(t)$, that is the derivative of the map computed along the unstable manifold. A useful idea in the following is the concept of *fundamental domain*. Define $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ by $x^u_\varepsilon(t) = x^u_\varepsilon(\alpha(t))$. Then $[t, \alpha(t)]$ is a fundamental domain and has the property that, setting $t_i := \alpha^i(t)$, the sets $\alpha^i[t_0, t_1]$ intersect only at the boundary.

Lemma 0.7.1 (Distortion) *For each x, y in the same fundamental domain of the unstable manifold, $\delta_0 > 0$, and $n \in \mathbb{N}$ such that $\|T^n_\varepsilon x\| \leq \delta_0$, holds*[26]

$$e^{-\delta_0 C_2} \leq \left|\frac{D^u_xT^n_\varepsilon}{D^u_yT^n_\varepsilon}\right| \leq e^{\delta_0 C_2},$$

where $C_2 = \sup_{t\leq 0}\left|\frac{\ddot{\alpha}(t)}{\dot{\alpha}(t)}\right|$.

Proof. The proof is a direct application of the chain rule:

$$\left|\frac{D^u_xT^n_\varepsilon}{D^u_yT^n_\varepsilon}\right| = \prod_{i=1}^n \left|\frac{D^u_{T^ix}T_\varepsilon}{D^u_{T^iy}T_\varepsilon}\right| \leq \operatorname{Exp}\left[\sum_{i=1}^n |\log(|D^u_{T^ix}T_\varepsilon) - \log(|D^u_{T^iy}T_\varepsilon|)|\right]$$
$$\leq \operatorname{Exp}\left[\sum_{i=1}^n C_2\|T^ix - T^iy\|\right] = \operatorname{Exp}\left[\sum_{i=1}^n C_2\|x^u_\varepsilon(t_i) - x^u_\varepsilon(t_{i-1})\|\right] \leq e^{C_2\delta_0}.$$

The other inequality is obtained by exchanging the rôle of x and y. □

Next we would like to consider the evolution of a small box constructed around the fix point.

[26]This quantity is commonly called *Distortion* because it measures how much the map differs from a linear one (notice that if T is linear then $\frac{D_xT}{D_yT} = 1$). Although apparently an innocent quantity, it is hard to overstate its importance in the study of hyperbolic dynamics.

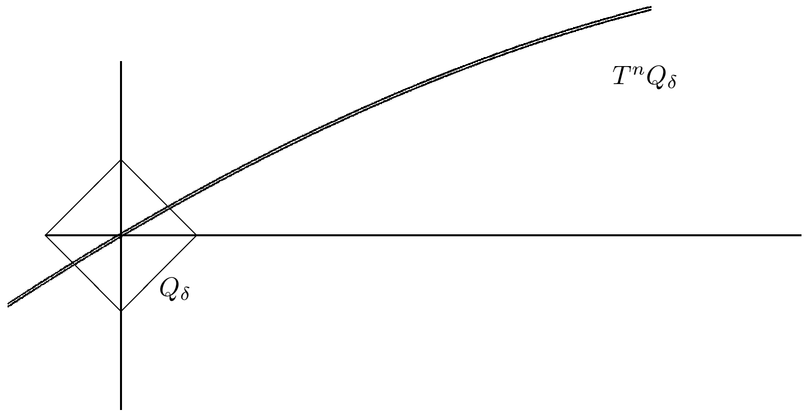

Figure 0.4: The evolution of the small box Q_δ

Consider the following small parallelogram: $Q_\delta := \{\xi \in \mathbb{R}^2 \mid \xi = av^u + bv^s$ for some $a, b \in [-\frac{\delta}{2}, \frac{\delta}{2}]\}$, $\delta \ll \delta_0$. Next consider the first $n \in \mathbb{N}$ such that $T_\varepsilon^n Q_\delta \cap \{\theta = 0\} \neq \emptyset$. Our first task is to understand the shape of $T_\varepsilon^n Q_\delta$ near $\{\theta = 0\}$. Since a fundamental domain in the latter region is of order one, while at the boundary of Q_ε is of order δ, Lemma 0.7.1 implies that the expansion is proportional to $C\delta^{-1}$. By the area preserving of the map it follows that $T_\varepsilon^n Q_\delta$ must be contained din a $C\delta^2$ neighborhood of the unstable manifold, see Figure 0.4.

By the previous section considerations on the shape of the invariant manifolds $T^n Q_\delta \cap T^n Q_\delta \neq \emptyset$, moreover they intersect *transversally.*[27]

This is all is needed to construct an horseshoe (see section ???). In particular, in our case it means that $T^{2n_0} Q_\delta \cap Q_\delta \neq \emptyset$, in fact the intersection are transversal and consist of three strips almost parallel to the unstable sides. One contains zero, and it is the lest interesting for us, the other two cross above and below the unstable manifold respectively. The with of such strip is about δ^{-3}. We will discuss in the next chapters all the implications of this situation, here it suffices to notice that if we have two initial conditions in $T^{-2n_0} Q_\delta \cap Q_\delta$ at a distance h, after $2n_0$ iterations the two points will be in Q_δ again but at a distance $h\varepsilon^{-1}$. Since to decide if after that there will be a rotation or an oscillation we need to know the final position with a precision of order δ, we need to know the initial position with a precision $\mathcal{O}(\delta\varepsilon) = \mathcal{O}(\delta^3)$.

[27] The meaning of *transversally* is the following: the square Q_δ has two sides parallel to v^u (the unstable direction), which we will call unstable sides, and two sides parallel to v^s (the stable direction), which we will call stable sides. Then the intersection is transversal if it consists of a region with again four sides: two made of the image of the unstable sides and two made of images of stable sides of Q_δ.

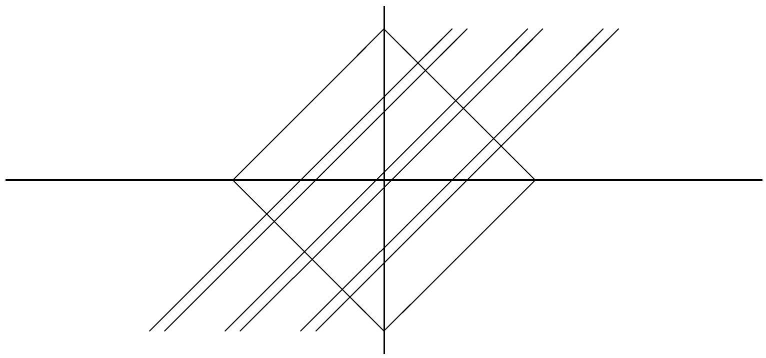

Figure 0.5: Horseshoe construction

Note that in the above construction we have lost almost all the points, only the ones that come back to Q_δ at time $2n_0$ are under control. Nevertheless, we can consider the set $\Lambda := \cup_{k\in\mathbb{Z}} \bigcap T_\varepsilon^{2kn_0} Q_\delta$. This is clearly a measure zero set, yet it is far from empty (it contains uncountably many points) and it is made of points that at times multiple of $2n_0$ are always in Q_δ. When they arrive in Q_δ they will rotate if they are above the separatrices and oscillate otherwise. Let us call this two subset of Q_δ R and O. Given a point $\xi \in Q_\delta$ we can associate to it the doubly infinite sequence $\sigma \in \{0,1\}^{\mathbb{Z}}$ by the rule $\sigma_i = 1$ iff $T^{2n_0 i}\xi \in R$. The reader can check that the correspondence is onto.

0.8 Conclusion–an answer

If $\varepsilon = 10^{-6}$ and δ is a millimeter then we need to know the initial condition with a precision of 10^{-9} meters if we want to decide if the point will come back or rotate when it will get almost vertical again (this will happen in about 6 seconds). By the same token if we want to answer the same question, but for the second time the pendulum get close to the unstable position, we need to know the initial condition with a precision of the order 10^{-15} meters, and this just to predict the motion for about 12 seconds.[28]

We can finally answer to our original question:

[28] Remark that it is not just a matter of precision on the initial condition, it is also a matter of how one actually does the prediction. If the method is to integrate numerically the equation of motion, then one has to insure that the precision of the algorithm is of the order of 10^{-15}. This maybe achieved by working in double precision but if one wants to make predictions of the order of one minute it is quite clear that the numerical problem becomes very quickly intractable.

Answer: *NO!*

Nevertheless, as we mentioned at the beginning, the above answer it is not the end of the story. In fact, there exists many other very relevant questions that can be answered.[29] The rest of the book deals with a particular type of question: can we meaningfully talk about the *statistical behavior* of a system?

Problems

0.1 Derive the Lagrangian, Hamiltonian and equations of motions for a pendulum attached to a point vibrating with frequency ω and amplitude ε. (Hint: see [59, 41] on how to do such things. Remember that two Lagrangian that differ by a total time derivative give rise to the same equation of motions and are thus equivalent.)

0.2 Consider the systems of differential equations $\dot{x} = f(x)$, $x \in \mathbb{R}^n$ and f smooth and bounded. Prove that the associated flow form a group. (Hint: use the uniqueness of the solutions of the ordinary differential equation)

0.3 Consider the systems of differential equations $\dot{x} = f(x,t)$, $x \in \mathbb{R}^n$ and f smooth, bounded and periodic in t of period τ. Let ϕ^t be the associated flow. Define $T = \phi^\tau$, prove that $T^n = \phi^{n\tau}$.

0.4 Show that the Hamiltonian is a constant of motion for the pendulum. (Hint: Compute the time derivative)

0.5 Prove (0.2.4). (Hint: Write (0.2.3) in the integral form

$$t = \int_0^t \frac{\dot{\theta}(s)}{\sqrt{\frac{2g}{l}(1+\cos\theta(s))}} ds.$$

Using some trigonometry and changing variable obtain

$$t = \int_0^{\theta(t)} \frac{1}{2\omega_p \cos\frac{\theta}{2}} d\theta.$$

and compute it.)

[29] For example: which type of motions are possible? This is a *qualitative* question. Such type of questions give rise to the qualitative theory of Dynamical Systems [70, 51], an extremely important part of the theory of dynamical systems, although not the focus here.

0.6 If $\theta(t)$ is the motion obtained in the previous problem, show that

$$\sin\theta(t) = 2\frac{\sinh\omega_p t}{(\cosh\omega_p t)^2}; \quad \cos\theta(t) = \frac{2}{(\cosh\omega_p t)^2} - 1;$$
$$\cos^2\frac{\theta(t)+\pi}{4} = \frac{1}{1+e^{2\omega_p t}}.$$

0.7 Consider the systems of differential equations $\dot{x} = f(x,t)$, $x \in \mathbb{R}^n$ and f smooth. Suppose further that $div f = 0$ (that is $\sum_{i=1}^n \frac{\partial f_i}{\partial x_i} = 0$). Show that the associated flow preserves the volume. (Hint: note that this is equivalent to saying that $|\det d\phi^t| = 1$, moreover by the group property and the chain rule for differentiating it suffices to check the property for small t. See that $d\phi^t = \mathbb{1} + Dft + \mathcal{O}(t^2) = e^{Dft+\mathcal{O}(t^2)}$. Finally, remember the formula $\det e^A = e^{\mathrm{Tr}A}$.)

0.8 Let $T, T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth maps such that $T0 = 0$ and $\det(\mathbb{1} - D_0T) \neq 0$. Consider the map $T_\varepsilon = T + \varepsilon T_1$ and show that, for ε small enough, there exists points $x_\varepsilon \in \mathbb{R}^2$ such that $T_\varepsilon x_\varepsilon = x_\varepsilon$. (Hint: Consider the function $F(x,\varepsilon) = x - T_\varepsilon x$ and apply the Implicit Function Theorem to $F = 0$.)

0.9 Let $x(t) \in \mathbb{R}^n$ be a smooth curve satisfying $\|\dot{x}(t)\| \leq a(t)\|x(t)\| + b(t)$, $x(0) = x_0$, $a, b \in \mathcal{C}^0(\mathbb{R}, \mathbb{R}_+)$, prove that

$$\|x(t) - x_0\| \leq \int_0^t e^{\int_s^t a(\tau)d\tau} \left[a(s)\|x_0\| + b(s)\right] ds.$$

(Hint: Note that $\|x(t) - x_0\| \leq \int_0^t \|\dot{x}(s)\| ds$. Transform then the differential inequality into an integral inequality. Show that if $z(t) \leq 0$ and $z(t) \leq \int_0^t z(s)ds$, then $z(t) \leq 0$ for each t. Use the last fact to compare a function satisfying the obtained integral inequality with the solution of the associated integral equation.)

0.10 Given two by two matrices A, B such that A has eigenvalues $\lambda \neq \mu$, show that the matrix $A_\varepsilon = A + \varepsilon B$, for ε small enough, has eigenvalues $\lambda_\varepsilon, \mu_\varepsilon$ analytic as functions of ε. Show that the same holds for the eigenvectors. (Hint:[30] consider z in the resolvent of A, that is $(z-A)^{-1}$ exists. Then $(z - A_\varepsilon) = (z - A)(\mathbb{1} - \varepsilon(z-A)^{-1}B)$. Accordingly, if ε is small enough, $(z - A_\varepsilon)^{-1} = \left\{\sum_{n=0}^\infty \varepsilon^n \left[(z-A)^{-1}B\right]^n\right\}(z-A)^{-1}$.

[30]Of course for matrices one could argue more directly by looking at the characteristic polynomial. Yet the strategy below has the advantage to work even in infinitely many dimensions (that is, for operators over Banach spaces).

Finally, if γ, γ' are curves on the complex plane containing λ and μ, respectively, verify that

$$\Pi_\varepsilon := \frac{1}{2\pi i}\int_\gamma (z - A_\varepsilon)^{-1} dz \quad \Pi'_\varepsilon := \frac{1}{2\pi i}\int_{\gamma'} (z - A_\varepsilon)^{-1} dz$$

are commuting projectors and $A_\varepsilon = \lambda_\varepsilon \Pi_\varepsilon + \mu_\varepsilon \Pi'_\varepsilon$. Finally verify that

$$\lambda_\varepsilon \Pi_\varepsilon := \frac{1}{2\pi i}\int_\gamma z(z - A_\varepsilon)^{-1} dz \quad \mu_\varepsilon \Pi'_\varepsilon := \frac{1}{2\pi i}\int_{\gamma'} z(z - A_\varepsilon)^{-1} dz.$$

The statement follows then from the fact that the right hand side of the above equalities is written as a power series in ε.[31])

0.11 Given two by two matrices A, B such that A has eigenvalues $\lambda \neq \mu$, show that the matrix $A_\varepsilon = A + \varepsilon B$ has eigenvalues $\lambda_\varepsilon, \mu_\varepsilon$ such that $|\lambda_\varepsilon - \lambda| \leq C\varepsilon\|B\|$ and $|\mu_\varepsilon - \mu| \leq C\varepsilon\|B\|$. Compute C. (Hint: By Problem 0.10 we know that $\lambda_\varepsilon, \mu_\varepsilon$ are differentiable function of ε and the same holds for the corresponding eigenvector $v_\varepsilon, \tilde{v}_\varepsilon$. Let us discuss λ_ε since the other eigenvalues can be treated in the same way. One possibility is to use the above formula for $\lambda_\varepsilon \Pi_\varepsilon$ to obtain the wanted estimates.

In alternative, let v, w, $\langle w, v\rangle = 1$ and $\|v\| = 1$, be the eigenvectors of A, with eigenvalue λ and of A^*, with eigenvalue $\bar{\lambda}$, respectively. Hence $\Pi_0 = v \otimes w$ and $\|\Pi_0\| = \|w\|$. Normalize v_ε such that $\langle v_\varepsilon, w\rangle = 1$. Differentiate then the above constraint and the defining equation $(A + \varepsilon B)v_\varepsilon = \lambda_\varepsilon v_\varepsilon$ obtaining (the prime refers to the derivative with respect to ε)

$$\begin{aligned} &Av'_\varepsilon + Bv_\varepsilon + \varepsilon Bv'_\varepsilon = \lambda'_\varepsilon v_\varepsilon + \lambda_\varepsilon v'_\varepsilon \\ &\langle v'_\varepsilon, w\rangle = 0. \end{aligned}$$

Multiplying the first for w yields $\lambda'_\varepsilon = \langle w, Bv_\varepsilon\rangle + \varepsilon\langle w, Bv'_\varepsilon\rangle$. Setting $\tilde{A} := A - \lambda\Pi_0$ we have

$$v'_\varepsilon = (\lambda - \tilde{A})^{-1}\left[Bv_\varepsilon + \varepsilon Bv'_\varepsilon - \lambda'_\varepsilon v_\varepsilon - (\lambda - \lambda_\varepsilon)v'_\varepsilon\right].$$

Next, consider ε_0 such that, for $\varepsilon < \varepsilon_0$ holds

$$\|v'_\varepsilon\| \leq 4\|(\lambda - \tilde{A})^{-1}\|\,\|B\|\,\|w\| = 4\|(\lambda - \tilde{A})^{-1}\|\,\|B\|\,\|\Pi_0\| =: C_0, \quad (0.8.1)$$

then $\|v_\varepsilon - v\| \leq \varepsilon C_0$ and $|\lambda'_\varepsilon| \leq \|B\|\,\|w\|(1 + 2\varepsilon C_0)$. If $4\varepsilon_0 C_0 < 1$, then, indeed, (0.8.1) holds true.)

[31] This is a very simple case of the very general problem of perturbation of point spectrum, see [49] (Kato) if you want to know more.

0.12 Compute D_0T. (Hint: solve (0.3.3) for $\varepsilon = 0$, $\theta = \pi$, $p = 0$ and $t = \frac{2\pi}{\omega}$.)

0.13 Compute D_0T_ε and see that, if ω is sufficiently large, the eigenvalues have modulus one (the unstable point becomes stable!). (Hint: setting $\xi := \xi_1$ equation (0.3.3) yields $\ddot{\xi} = \omega_p^2\xi + \varepsilon\frac{\omega^2}{l}\cos\omega t\xi$. It is then convenient to write $\xi := \bar{\xi} + \varepsilon\eta + \varepsilon^2\zeta$ where $\ddot{\bar{\xi}} = \omega_p^2\bar{\xi}$ and $\ddot{\eta} = \omega_p^2\eta + \frac{\omega^2}{l}\cos\omega t\,\bar{\xi}$. One can look for a solution of the latter equation of the form

$$\bar{\eta} = Ae^{\omega_p t}\cos\omega t + Be^{\omega_p t}\sin\omega t + Ce^{-\omega_p t}\cos\omega t + De^{-\omega_p t}\sin\omega t.$$

This allows to compute $D_0T_\varepsilon(\alpha,\beta) = (\xi_1(\frac{2\pi}{\omega}), \xi_2(\frac{2\pi}{\omega})) + \mathcal{O}(\varepsilon^2)$, where $(\xi_1(0), \xi_2(0)) = (\alpha, \beta)$. Finally one can verify that, for ε small and ω large enough the eigenvalues of D_0T_ε are imaginary, hence the equilibrium is linearly stable.)

0.14 Given an Hamiltonian $H : \mathbb{R}^2 \to \mathbb{R}$, for each solution $x(t)$ of the associated equations of motion show that $\langle \nabla_{x(t)}H, \dot{x}(t)\rangle = 0$.

0.15 Compute the following integrals (0.6.9):

$$\int_{\mathbb{R}} e^{iat}(\cosh t)^{-n}\sinh t\,dt,$$

$a \in \mathbb{R}$ and $n \in \mathbb{N}$, $n > 1$.[32] (Hint: By a change of variable one can consider only the case $a > 0$. Consider the integral on the complex plane, show that the integral on the half circle $Re^{i\phi}$, $\phi \in [0,\pi]$, goes to zero as $R \to \infty$, then check that the poles of the integrand, on the complex plane, lie on the imaginary axis, finally use the residue theorem to compute the integrals.)

0.16 Do the same analysis carried out for the pendulum with a vibrating suspension point in the case of a pendulum subject to an external force $\varepsilon\cos\omega t$ and in presence of a small friction $-\varepsilon^2\gamma\dot{\theta}$.

[32] The result, for $a > 0$, is:

$$\int_{\mathbb{R}} e^{iat}(\cosh t)^{-n}\sinh t = 2\pi i\sum_{k=0}^{\infty}\frac{\phi_{n,k}^{(n-1)}(i\frac{2k+1}{2}\pi)}{(n-1)!},$$

where

$$\phi_{n,k}(z) = e^{iza}\sinh z\left(\frac{z - i\frac{2k+1}{2}\pi}{\cosh z}\right)^n.$$

For $n = 3$ the above formula yields

$$\int_{\mathbb{R}} e^{iat}(\cosh t)^{-3}\sinh t = \pi a^2 e^{-\frac{\pi}{2}a}(1 - e^{-\pi a})^{-1}.$$

Notes

As already mentioned in the text the first to realize that the motions arising from differential equations can be very complex was probably Poincaré [72]. At the time the main problem in celestial mechanics (the famous n-body problem) was to find all the integral of motion. Dirichlet and Weierstrass worked on this problem, but Poincaré was the first to rise serious doubt on the existence of such integrals (which would have implied regular motions). For more historical remarks see [67]. In fact, all the content of this chapter is inspired by the more sophisticated, but more qualitative, analysis in [67].

CHAPTER 1

General facts and definitions

Before entering in the hart of the matter it is necessary the knowledge of some general facts concerning (measurable) Dynamical Systems. This chapter is intended for readers with no previous knowledge of Dynamical Systems. The chapter contains few basic facts, some of which will be used in the following while others are meant to provide a wider context to the material actually discussed. For a much more complete discussion of the relevant concepts the reader is referred to [65], [51].

1.1 Basic Definitions and examples

Definition 1.1.1 *By Dynamical System*[1] *with discrete time we mean a triplet* (X, T, μ) *where* X *is a measurable space,*[2] μ *is a measure and* T *is a measurable map from* X *to itself that preserves the measure (i.e.,* $\mu(T^{-1}A) = \mu(A)$ *for each measurable set* $A \subset X$*).*

An equivalent characterization of invariant measure is $\mu(f \circ T) = \mu(f)$ for each $f \in L^1(X, \mu)$ since, for each measurable set A, $\mu(\chi_A \circ T) = \mu(\chi_{T^{-1}A}) = \mu(T^{-1}A)$, where χ_A is the characteristic function of the set A.

[1] To be really precise this is the definition of "Measurable Dynamical Systems," hopefully the reader will excuse this abuse of language. More generally a Dynamical System can be defined as a set X together with a map $T : X \to X$ or, even more generally, an algebra $\mathcal{A}$ (e.g., the algebra of the continuous functions on X) and an isomorphism $\tau : \mathcal{A} \to \mathcal{A}$ (e.g., $\tau f := f \circ T$). This last definition is so general as to include Stochastic Processes and Quantum Systems. A further generalization consists in realizing that the above setting can be view as the action of the semigroup $\mathbb{N}$ (or the group $\mathbb{Z}$ if T is invertible) on the algebra $\mathcal{A}$. One can then consider other groups (already in the next definition the group is $\mathbb{R}$), for example $\mathbb{Z}^n$ or $\mathbb{R}^n$, this goes in the direction of the Statistical Mechanics and it has receive a lot of attention lately [1]. Of course, such a generality is excessive for the task at hand.

[2] By measurable space we simply mean a set X together with a σ-algebra that defines the measurable sets.

Remark 1.1.2 *In this book we will always assume $\mu(X) < \infty$ (and quite often $\mu(X) = 1$, i.e. μ is a probability measure). Nevertheless, the reader should be aware that there exists a very rich theory pertaining to the case $\mu(X) = \infty$, see [3].*

Definition 1.1.3 *By Dynamical System with continuous time we mean a triplet (X, ϕ^t, μ) where X is a measurable space, μ is a measure and ϕ^t is a measurable group ($\phi^t(x)$ is a measurable function for each t, $\phi^t(x)$ is a measurable function of t for almost all $x \in X$; $\phi^0 =$identity and $\phi^t \circ \phi^s = \phi^{t+s}$ for each $t, s \in \mathbb{R}$) or semigroup ($t \in \mathbb{R}^+$) from X to itself that preserves the measure (i.e., $\mu((\phi^t)^{-1}A) = \mu(A)$ for each measurable set $A \subset X$).*

The above definitions are very general, this reflects the wideness of the field of Dynamical Systems. In the present book we will be interested in much more specialized situations.

In particular, X will always be a topological compact space. The measures will alway belong to the class $\mathcal{M}^1(X)$ of Borel probability measures on X.[3] For future use, given a topological space X and a map T let us define $\mathcal{M}_T$ as the collection of all Borel measures that are T invariant.[4]

Often X will consist of finite unions of smooth manifolds (eventually with boundaries). Analogously, the dynamics (the map or the flow) will be smooth in the interior of X.

Let us see few examples to get a feeling of how a Dynamical System can look like.

1.1.1 Examples

1.1.1.a *Rotations*

Let $\mathbb{T}$ be $\mathbb{R}$ mod 1. By this we mean $\mathbb{R}$ quotiented with respect to the equivalence relations $x \sim y$ if and only if $x - y \in \mathbb{Z}$. $\mathbb{T}$ can be though as the interval $[0, 1]$ with the points 0 and 1 identified. We put on it the topology induced by the topology of $\mathbb{R}$ via the defined equivalence relation. Such a topology is the usual one on $[0, 1]$, apart from the fact that each open set containing 0 must contain 1 as well. Clearly, from the topological point of view, $\mathbb{T}$ is a circle. We choose the Borel σ-algebra. By μ we choose the Lebesgue measure m, while $T : \mathbb{T} \to \mathbb{T}$ is defined by

$$Tx = x + \omega \mod 1,$$

for some $\omega \in \mathbb{R}$. In essence, T translates, or rotates, each point by the same quantity ω. It is easy to see that the measure μ is invariant (Problem 1.4).

[3] Remember that a Borel measure is a measure defined on the Borel σ-algebra, that is the σ-algebra generated by the open sets.

[4] Obviously, for each $\mu \in \mathcal{M}_T$, (X, T, μ) is a Dynamical System.

1.1.1.b *Bernoulli shift*

A Dynamical System needs not live on some differentiable manifold, more abstract possibilities are available.

Let $\mathbb{Z}_n = \{1, 2, ..., n\}$, then define the set of two sided (or one sided) sequences $\Sigma_n = \mathbb{Z}_n^{\mathbb{Z}}$ ($\Sigma_n^+ = \mathbb{Z}_n^{\mathbb{Z}+}$). This means that the elements of Σ_n are sequences $\sigma = \{..., \sigma_{-1}, \sigma_0, \sigma_1, ...\}$ ($\sigma = \{\sigma_0, \sigma_1, ...\}$ in the one sided case) where $\sigma_i \in \mathbb{Z}_n$. To define the measure and the σ-algebra a bit of care is necessary. To start with, consider the *cylinder sets*, that is the sets of the form

$$A_i^j = \{\sigma \in \Sigma_n \mid \sigma_i = j\}.$$

Such sets will be our basic objects and can be used to generate the algebra $\mathcal{A}$ of the cylinder sets via unions and complements (or, equivalently, intersections and complements). We can then define a topology on Σ_n (the product topology, if $\{1, \dots, n\}$ is endowed by the discrete topology) by declaring the above algebra made of open sets and a basis for the topology. To define the σ-algebra we could take the minimal σ-algebra containing $\mathcal{A}$, yet this it is not a very constructive definition, neither a particular useful one, it is better to invoke the Carathèodory construction.

Let us start by defining a measure on $\mathbb{Z}_n$, that is n numbers $p_i > 0$ such that $\sum_{i=1}^n p_i = 1$. Then, for each $i \in \mathbb{Z}$ and $j \in \mathbb{Z}_n$,

$$\mu(A_i^j) = p_j.$$

Next, for each collection of sets $\{A_{i_l}^{j_l}\}_{l=1}^s$, with $i_l \neq i_k$ for each $l \neq k$, we define

$$\mu(A_{i_1}^{j_1} \cap A_{i_2}^{j_2} \cap ... \cap A_{i_s}^{j_s}) = \prod_{l=1}^s p_{j_l}.$$

We now know the measure of all finite intersection of the sets A_i^j. Obviously $\mu(A^c) := 1 - \mu(A)$ and the measure of the union of two sets A, B obviously must satisfy $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$. We have so defined μ on $\mathcal{A}$. It is easy to check that such a μ is σ-additive on $\mathcal{A}$; namely: if $\{A_i\} \subset \mathcal{A}$ are pairwise disjoint sets and $\cup_{i=1}^\infty A_i \in \mathcal{A}$, then $\mu(\cup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \mu(A_i)$. The next step is to define an outer measure[5]

$$\mu^*(A) := \inf_{\substack{B \in \mathcal{A} \\ B \supset A}} \mu(B) \quad \forall A \subset \Sigma_n.$$

Finally, we can define the σ-algebra as the collection of all the sets that satisfy the *Carathèodory's criterion*, namely A is measurable (that is belongs to the σ-algebra) iff

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \quad \forall E \subset \Sigma_n.$$

[5]An outer measure has the following properties: i) $\mu^*(\emptyset) = 0$; ii) $\mu^*(A) \leq \mu^*(B)$ if $A \subset B$; iii)$\mu^*(\cup_{i=1}^\infty A_i) \leq \sum_{i=1}^\infty \mu^*(A_i)$. Note that μ^* need not be additive on all sets.

The reader can check that the sets in $\mathcal{A}$ are indeed measurable.

The Carathèodory Theorem then asserts that the measurable sets form a σ-algebra and that on such a σ-algebra μ^* is numerably additive, thus we have our measure μ (simply the restriction of μ^* to the σ-algebra).[6] The σ-algebra so obtained is nothing else than the completion with respect to μ of the minimal σ-algebra containing $\mathcal{A}$ (all the sets with zero outer measure are measurable).

The map $T : \Sigma_n \to \Sigma_n$ (usually called *shift*) is defined by

$$(T\sigma)_i = \sigma_{i+1}.$$

We leave to the reader the task to show that the measure is invariant (see Problem 1.12).

To understand what's going on, let us consider the function $f : \Sigma \to \mathbb{Z}_n$ defined by $f(\sigma) = \sigma_0$. If we consider T^t, $t \in \mathbb{N}$, as the time evolution and f as an observation, then $f(T^t\sigma) = \sigma_t$. This can be interpreted as the observation of some phenomenon at various times. If we do not know anything concerning the state of the system, then the probability to see the value j at the time t is simply p_j. If $n = 2$ and $p_1 = p_2 = \frac{1}{2}$, it could very well be that we are observing the successive outcomes of tossing a fair coin where 1 means head and 2 tail (or vice versa); if $n = 6$ it could be the outcome of throwing a dice and so on.

1.1.1.c *Dilation*

Again $X = \mathbb{T}$ and the measure is Lebesgue. T is defined by

$$Tx = 2x \mod 1.$$

This map it is not invertible (similarly to the one sided shift). Note that, in general, $\mu(TA) \neq \mu(A)$ (e.g., $A = [0, \frac{1}{2}]$).

1.1.1.d *Toral automorphism (Arnold cat)*

This is an automorphism of the torus and gets its name by a picture draw by Arnold [10]. The space X is the two dimensional torus $\mathbb{T}^2$. The measure is again Lebesgue measure and the map is

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \mod 1 := L\begin{pmatrix} x \\ y \end{pmatrix} \mod 1.$$

Since the entries of L are integers numbers it is clear that T is well defined on the torus; in fact, it is a linear toral automorphism. The invariance of the measure follows from $\det L = 1$.

[6] See [62] if you want a quick look at the details of the above Theorem or consult [74] if you want a more in depth immersion in measure theory. If you think that the above construction is too cumbersome see Problem 1.14.

1.1.1.e *Hamiltonian Systems*

Up to now we have seen only examples with discrete time. Typical examples of Dynamical Systems with continuous time are the solutions of an ODE or a PDE. Let us consider the case of an Hamiltonian system. The simplest case is when $X = \mathbb{R}^{2n}$, the σ-algebra is the Borel one and the measure μ is the Lebesgue measure m. The dynamics is defined by a smooth function $H : X \to \mathbb{R}$ via the equations

$$\frac{dx}{dt} = J\mathsf{grad}H(x)$$

where $\mathsf{grad}(H)_i = (\nabla H)_i = \frac{\partial H}{\partial x_i}$ and J is the block matrix

$$J = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}.$$

The fact that m is invariant with respect to the Hamiltonian flow is due to the Liouville Theorem (see [5] or Problem 0.7).

Such a dynamical system has a natural decomposition. Since H is an integral of the motion, for each $h \in \mathbb{R}$ we can consider $X_h = \{x \in X \mid H(x) = h\}$. If $X_h \neq \emptyset$, then it will typically consist of a smooth manifold,[7] let us restrict ourselves to this case. Let σ be the surface measure on X_h, then $\mu_h = \frac{\sigma}{\|\mathsf{grad}H\|}$ is an invariant measure on X_h and (X_h, ϕ_t, μ_h) is a Dynamical System (see Problem 1.6).

1.1.1.f *Geodesic flow*

Along the same lines any geodesic flow on a compact Riemannian manifold naturally defines a dynamical system.

1.2 Return maps and Poincaré sections

Normally in Dynamical Systems there is a lot of emphasis on the discrete case. One reason is that there is a general device that allows to reduce the study of many properties of a continuous time Dynamical System to the study of an appropriate discrete time Dynamical System: Poincaré sections (we have already seen an instance of this in the introduction). Here we want to make few comments on this precious tool that we will largely employ in the study of billiards.

Let us consider a smooth Dynamical System (X, ϕ^t, μ) (that is a Dynamical Systems in continuous time where X is a smooth manifold and ϕ^t is a

[7] By the implicit function theorem this is locally the case if $\nabla H \neq 0$.

smooth flow). Then we can define the vector field $V(x) := \frac{d\phi^t(x)}{dt}|_{t=0}$.[8]

Consider a smooth compact submanifold (possibly with boundaries) Σ of codimension one such that $\mathcal{T}_x\Sigma$ (the tangent space of Σ at the point x) is transversal to $V(x)$.[9] We can then define the *return time* $\tau_\Sigma : \Sigma \to \mathbb{R}^+ \cup \{\infty\}$ by

$$\tau_\Sigma = \inf\{t \in \mathbb{R}^+ \backslash \{0\} \mid \phi^t(x) \in \Sigma\},$$

where the inf is taken to be ∞ if the set is empty. Next we define the *return map* $T_\Sigma : D(T) \subset \Sigma \to \Sigma$, where $D(T) = \{x \in \Sigma | \tau_\Sigma(x) < \infty\}$, by

$$T_\Sigma(x) = \phi^{\tau_\Sigma(x)}(x).$$

It is easy to check that there exists $c > 0$ such that $\tau_\Sigma \geq c$ (Problem 1.9).

To define the measure, the natural idea is to project the invariant measure along the flow direction: for all measurable sets $A \subset \Sigma$, define[10]

$$\nu_\Sigma(A) := \lim_{\delta \to 0} \frac{1}{\delta} \mu(\phi^{[0,\,\delta]}(A)). \tag{1.2.1}$$

See Problem 1.8 for the existence of the above limit; see Problem 1.9 for the proof that τ_Σ is finite almost everywhere and Problem 1.10 for the proof that $(\Sigma,\, T_\Sigma,\, \nu_\Sigma)$ is a dynamical system. The reader is invited to meditate on the relation between this Dynamical System and the original one.

1.3 Suspension flows

A natural question is if it is possible to construct a flow with a given Poincaré section, the answer is that there are infinitely many flows with a given section. Let us construct some of them. Given a dynamical system (Σ, T, ν) consider $\tilde{X} := \Sigma \times R^+$. Define the flow $\phi_t((x, s)) = (x, s + t)$. We then define in $\tilde{X}$ the equivalence relation $(x, t) \sim (y, s)$ iff $s = t + n$ and $y = T^n x$ or $t = s + n$ and $x = T^n y$ for some $n \in \mathbb{N}$. A moment of reflection shows that the set X of equivalence classes is nothing else than the set $\Sigma \times [0, 1]$ with the points $(x, 1)$ and $(Tx, 0)$ identified. Clearly the flow is naturally quotiented over the equivalence classes and yields a quotient flow on X, such a flow is called a *suspension flow*.

A more general construction can by obtained by applying a time change to the above example. Alternatively, one can can choose any smooth function $\tau : \Sigma \to \mathbb{R}^+$, that will be called a *ceiling function* and consider the set $X_\tau = \{(x, t) \in \Sigma \times \mathbb{R}^+ \mid t \in [0, \tau(x)]\}$ with the points $(x, \tau(x))$ and $(Tx, 0)$ identified.

[8] Very often it is the other way around: the vector field is given first and then the flow–as we saw in the introduction.

[9] That is $\mathcal{T}_x\Sigma \oplus V(x)$ form the full tangent space at x.

[10] We use the notation: $\phi^I(A) := \cup_{t \in I} \phi^t(A)$ for each $I \subset \mathbb{R}$.

A moment of reflection should show that the topology of X_τ does not depend on τ and is then the same than the suspension defined above. The flow is again defined by $\phi_t(x,s) = (x, s+t)$ for $t \leq \tau(x) - s$. Such flows are called *special flows*.

1.4 Invariant measures

A very natural question is: given a space X and a map T does there always exists an invariant measure μ? A non exhaustive, but quite general, answer exists: Krylov-Bogoluvov Theorem.

First of all we need a useful characterization of invariance.

Lemma 1.4.1 *Given a compact metric space X and map T continuous apart from a compact set K,*[11] *a Borel measure μ, such that $\mu(K) = 0$, is invariant if and only if $\mu(f \circ T) = \mu(f)$ for each $f \in \mathcal{C}^{(0)}(X)$.*

PROOF. To prove that the invariance of the measure implies the invariance for continuous functions is obvious since each such function can be approximate uniformly by simple functions–that is, sum of characteristic functions of measurable sets–for which the invariance it is immediate.[12] The converse implication is not so obvious.

The first thing to remember is that the Borel measures, on a compact metric space, are regular [75]. This means that for each measurable set A the following holds[13]

$$\mu(A) = \inf_{\substack{G \supset A \\ G = \mathring{G}}} \mu(G) = \sup_{\substack{C \subset A \\ C = \overline{C}}} \mu(C). \tag{1.4.1}$$

Next, remember that for each closed set A and open set $G \supset A$, there exists $f \in \mathcal{C}^{(0)}(X)$ such that $f(X) \subset [0,1]$, $f|_{G^c} = 0$ and $f|_A = 1$ (this is Urysohn Lemma for Normal spaces [74]). Hence, setting $B_A := \{f \in \mathcal{C}^{(0)}(X) \mid f \geq \chi_A\}$,

$$\mu(A) \leq \inf_{f \in B_A} \mu(f) \leq \inf_{\substack{G \supset A \\ G = \mathring{G}}} \mu(G) = \mu(A). \tag{1.4.2}$$

Accordingly, for each A closed, we have

$$\mu(T^{-1}A) \leq \inf_{f \in B_A} \mu(f \circ T) = \inf_{f \in B_A} \mu(f) = \mu(A).$$

[11] This means that, if $C \subset X$ is closed, then $T^{-1}C \cup K$ is closed as well.

[12] This is essentially the definition of integral.

[13] This is rather clear if one thinks of the Carathéodory construction starting from the open sets.

In addition, using again the regularity of the measure, for each A Borel holds[14]

$$\begin{aligned}\mu(T^{-1}A) &= \inf_{\substack{U\supset K\\ U=\mathring{U}}} \mu(T^{-1}A\backslash U) \le \inf_{\substack{U\supset K\\ U=\mathring{U}}} \sup_{\substack{C\subset T^{-1}A\backslash U\\ C=\overline{C}}} \mu(T^{-1}(TC))\\ &\le \inf_{\substack{U\supset K\\ U=\mathring{U}}} \sup_{\substack{C\subset A\backslash TU\\ C=\overline{C}}} \mu(T^{-1}C) \le \sup_{\substack{C\subset A\\ C=\overline{C}}} \mu(T^{-1}C) = \sup_{\substack{C\subset A\\ C=\overline{C}}} \mu(C) = \mu(A).\end{aligned}$$

Applying the same argument to the complement A^c of A it follow that it must be $\mu(T^{-1}A) = \mu(A)$ for each Borel set. □

Proposition 1.4.2 (Krylov–Bogoluvov) *If X is a metric compact space and $T : X \to X$ is continuous, then there exists at least one invariant (Borel) measure.*

PROOF. Consider any Borel probability measure ν and define the following sequence of measures $\{\nu_n\}_{n\in\mathbb{N}}$:[15] for each Borel set A

$$\nu_n(A) = \nu(T^{-n}A).$$

The reader can easily see that $\nu_n \in \mathcal{M}^1(X)$, the sets of the probability measures. Indeed, since $T^{-1}X = X$, $\nu_n(X) = 1$ for each $n \in \mathbb{N}$. Next, define

$$\mu_n = \frac{1}{n}\sum_{i=0}^{n-1} \nu_i.$$

Again $\mu_n(X) = 1$, so the sequence $\{\mu_i\}_{i=1}^\infty$ is contained in a weakly compact set (the unit ball) and therefore admits a weakly convergent subsequence $\{\mu_{n_i}\}_{i=1}^\infty$; let μ be the weak limit.[16] We claim that μ is T invariant. Since μ is a Borel measure it suffices to verify that for each $f \in \mathcal{C}^{(0)}(X)$ holds $\mu(f\circ T) = \mu(f)$ (see Lemma 1.4.1). Let f be a continuous function, then by the weak convergence we have[17]

[14] Note that, by hypothesis, if C is compact and $C\cap K = \emptyset$, then TC is compact.

[15] Intuitively, if we chose a point $x \in X$ at random, according to the measure ν and we ask what is the probability that $T^n x \in A$, this is exactly $\nu(T^{-n}A)$. Hence, our procedure to produce the point $T^n x$ is equivalent to picking a point at random according to the evolved measure ν_n.

[16] This depends on the Riesz-Markov Representation Theorem [75] that states that $\mathcal{M}(X)$ is exactly the dual of the Banach space $\mathcal{C}^{(0)}(X)$. Since the weak convergence of measures in this case correspond exactly to the weak-* topology [75], the result follows from the Banach-Alaoglu theorem stating that the unit ball of the dual of a Banach space is compact in the weak-* topology. But see Problem 1.17 if you want a more elementary proof.

[17] Note that it is essential that we can check invariance only on continuous functions: if we would have to check it with respect to all bounded measurable functions we would need that μ_n converges in a stronger sense (*strong convergence*) and this may not be true. Note as well that this is the only point where the continuity of T is used: to insure that $f\circ T$ is continuous and hence that $\mu_{n_j}(f\circ T) \to \mu(f\circ T)$.

$$\mu(f \circ T) = \lim_{j\to\infty} \frac{1}{n_j} \sum_{i=0}^{n_j-1} \nu_i(f \circ T) = \lim_{j\to\infty} \frac{1}{n_j} \sum_{i=0}^{n_j-1} \nu(f \circ T^{i+1})$$
$$= \lim_{j\to\infty} \frac{1}{n_j} \left\{ \sum_{i=0}^{n_j-1} \nu_i(f) + \nu(f \circ T^{n_j}) - \nu(f) \right\} = \mu(f).$$

□

The reason why the above theorem is not completely satisfactory is that it is not constructive and, in particular, does not provide any information on the nature of the invariant measure. On the contrary, in many instances the interest is focused not just on any Borel measure but on special classes of measures, for example measures connected to the Lebesgue measure which, in some sense, can be thought as reasonably physical measures (if such measures exists).

In the following examples we will see two main techniques to study such problems: on the one hand it is possible to try to construct explicitly the measure and study its properties in the given situations (expanding maps, strange attractors, solenoid, horseshoe); on the other hand one can try to *conjugate*[18] the given problem with another, better understood, one (logistic map, circle maps). In view of the second possibility the last example is very important (Markov measures). Such an example gives just a hint to the possibility to construct a multitude of invariant measures for the shift which, as we will see briefly, is a standard system to which many other can be conjugated.

1.4.1 Examples

1.4.1.a *Contracting maps*

Let $X \subset \mathbb{R}^n$ be compact and connected, $T : X \to X$ differentiable with $\|DT\| \leq \lambda^{-1} < 1$ and $T0 = 0 \in X$. In this case 0 is the unique fixed point and the delta function at zero is the only invariant measure.[19]

1.4.1.b *Expanding maps*

The simplest possible case is $X = \mathbb{T}$, $T \in \mathcal{C}^{(2)}(\mathbb{T})$ with $|DT| \geq \lambda > 1$, (see Figure 1.1 for a pictorial example).[20]

[18] See Definition 1.8.2 for a precise definition and Problem 1.40 and 1.41 for some insight.

[19] The reader will hopefully excuse this physicist language, naturally we mean that the invariant measure is defined by $\delta_0(f) = f(0)$. The property that there exists only one invariant measure is called *unique ergodicity*, we will see more of it in the sequel, e.g. see example 1.5.1.a.

[20] Note that this generalizes Examples 1.1.1.c.

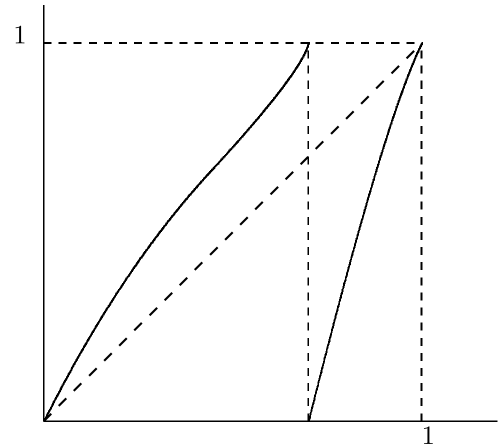

Figure 1.1: Graph of an expanding map on $\mathbb{T}$

We would like to have an invariant measure absolutely continuous with respect to Lebesgue. Any such measure μ has, by definition, the Radon-Nikodym derivative $h = \frac{d\mu}{dm} \in L^1(\mathbb{T}, m)$, [74]. In Proposition 1.4.2 we saw how a measure evolves by defining the operator

$$T_*\mu(f) = \mu(f \circ T) \tag{1.4.3}$$

for each $f \in \mathcal{C}^{(0)}$ and $\mu \in \mathcal{M}(X)$ (see also footnote 16 at page 32). If we want to study a smaller class of measures we must first check that T_* leaves such a class invariant. Indeed, if μ is absolutely continuous with respect to Lebesgue then $T_*\mu$ has the same property. Moreover, if $h = \frac{d\mu}{dm}$ and $h_1 = \frac{dT_*\mu}{dm}$ then (Problem 1.15)

$$h_1(x) = \mathcal{L}h(x) := \sum_{y \in T^{-1}(x)} |D_yT|^{-1}h(y).$$

The operator $\mathcal{L} : L^1(\mathbb{T}, m) \to L^1(\mathbb{T}, m)$ is called *Transfer operator* or Ruelle-Perron-Frobenius operator, and has an extremely important rôle in the study of the statistical properties of the system. Notice that $\|\mathcal{L}h\|_1 \le \|h\|_1$. The key property of $\mathcal{L}$, in this context, is given by the following inequality (this type of inequality is commonly called of Lasota-York type) (Problem 1.16)

$$\|\frac{d}{dx}\mathcal{L}h\|_1 \le \lambda^{-1}\|h'\|_1 + C\|h\|_1 \tag{1.4.4}$$

where $C = \frac{\|D^2T\|_\infty}{\|DT\|_\infty^2}$.

The above inequality implies immediately $\|(\mathcal{L}^n h)'\|_1 \leq \frac{C}{1-\lambda^{-1}}\|h\|_1 + \|h'\|_1$, for all $n \in \mathbb{N}$. This, in turn, implies that the $\sup_{n\in\mathbb{N}} \|\mathcal{L}^n h\|_\infty < \infty$. Consequently, the sequence $h_n := \frac{1}{n}\sum_{i=0}^{n-1} \mathcal{L}^i h$ is compact in L^1 (this is a consequence of standard embedding theorems [62] but see Problem 1.17 for an elementary proof). In analogy with Lemma 1.4.2, we have that there exists $h_* \in L^1$ such that $\mathcal{L}h_* = h_*$. Thus $d\mu := h_* dm$ is an invariant measure of the type we are looking for.[21]

1.4.1.c *Logistic maps*

Consider $X = [0, 1]$ and

$$T(x) = 4x(1 - x).$$

This map is not an everywhere expanding map ($D_{\frac{1}{2}}T = 0$), yet it can be conjugate with one, [89].

To see this consider the continuous change of variables $\Psi : [0, 1] \to [0, 1]$ defined by

$$\Psi(x) = \frac{2}{\pi} \arcsin \sqrt{x},$$

thus $\Psi^{-1}(x) = \left(\sin \frac{\pi}{2} x\right)^2$. Accordingly,

$$\begin{aligned}\tilde{T}(x) \quad &:= \Psi \circ T \circ \Psi^{-1}(x) = \Psi(4 \sin^2 \tfrac{\pi}{2} x \cos^2 \tfrac{\pi}{2} x) \\ &= \Psi([\sin \pi x]^2) = \tfrac{2}{\pi} \arcsin[\sin \pi x]\end{aligned}$$

which yields[22]

$$\tilde{T}(x) = \begin{cases} 2x & \text{for } x \in [0, \frac{1}{2}] \\ 2 - 2x & \text{for } x \in [\frac{1}{2}, 1]. \end{cases}$$

The map $\tilde{T}$ is called *tent map* for its characteristic shape, see figure 1.2. What is more interesting is that the Lebesgue measure is invariant for $\tilde{T}$, as the reader can easily check. This means that, if we define $\mu(f) := m(f \circ \Psi^{-1})$, it holds true

$$\mu(f \circ T) = m(f \circ T \circ \Psi^{-1}) = m(f \circ \Psi^{-1} \circ \tilde{T}) = m(f \circ \Psi^{-1}) = \mu(f).$$

Hence, $([0, 1], T, \mu)$ is a Dynamical System. In addition, a trivial computation shows

$$\mu(dx) = \frac{1}{\pi\sqrt{x(1-x)}} dx,$$

thus μ is absolutely continuous with respect to Lebesgue.

[21] In fact, there exists only one such measure, see Examples 4.3.1.c.

[22] Remember that the domain of arcsin is $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\sin \pi x = \sin \pi(1 - x)$.

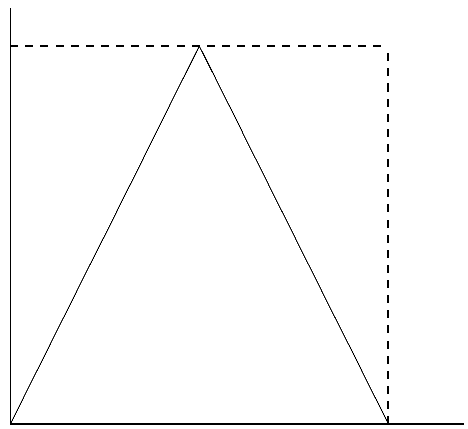

Figure 1.2: Graph of tent map

1.4.1.d *Circle maps*

A circle map is an order preserving continuous map of the circle. A simple way to describe it is to start by considering its lift. Let $\hat{T}:\mathbb{R}\to\mathbb{R}$, such that $\hat{T}(0)\in[0,1]$, $\hat{T}(x+1)=\hat{T}(x)+1$ ad it is monotone increasing. The circle map is then defined as $T(x)=\hat{T}(x) \mod 1$. Circle maps have a very rich theory that we do not intend to develop here, we confine ourselves to some facts (see [51] for a detailed discussion of the properties below). The first fact is that the *rotation number*

$$\rho(T)=\lim_{n\to\infty}\frac{1}{n}\hat{T}^n(x).$$

is well defined and does not depend on x.

We have already seen a concrete example of circle maps: the rotation R_ω by ω. Clearly $\rho(R_\omega)=\omega$. It is fairly easy to see that if $\rho(T)\in\mathbb{Q}$ then the map has a periodic orbit. We are more interested in the case in which the rotation number is irrational. In this case, with the extra assumption that T is twice differentiable (actually a bit less is needed) the Denjoy theorem holds stating that there exists a continuous invertible function h such that $R_{\rho(T)}\circ h=h\circ T$, that is T is *topologically conjugated* to a rigid rotation. Since we know that the Lebesgue measure is invariant for the rotations, we can obtain an invariant measure for T by pushing the Lebesgue measure by h, namely define

$$\mu(f)=m(f\circ h^{-1}).$$

The natural question if the measure μ is absolutely continuous with respect to Lebesgue is rather subtle and depends, once again, on KAM theory. In essence the answer is positive only if T has more regularity and the rotation number is not very well approximated by rational numbers (in some sense it is 'very irrational') [1].

1.4.1.e *Strange Attractors*

We have seen the case in which all the trajectories are attracted by a point. The reader can probably imagine a case in which the attractor is a curve or some other simple set. Yet, it has been a fairly recent discovery that an attractor may have a very complex (strange) structure. The following is probably the simplest example. Let $X = Q = [0,1]^2$ and

$$T(x,\, y) = \begin{cases} (2x,\, \frac{1}{8}y + \frac{1}{4}) & \text{if } x \in [0,\, 1/2] \\ (2x-1,\, \frac{1}{8}y + \frac{3}{4}) & \text{if } x \in]1/2,\, 1]. \end{cases}$$

We have a map of the square that stretches in one direction by a factor 2 and contract in the other by a factor 8.

Note that T it is not continuous with respect to the normal topology, so Proposition 1.4.2 cannot be applied directly. This problem can be solved in at least two ways: one is to *code* the system and we will discuss it later (see Examples 1.8.1), the other is to study more precisely what happens iterating a measure in special cases.

In our situation, since T^nQ consists of a multitude of thinner and thinner strips, it is clear that there can be no invariant measure absolutely continuous with respect to Lebesgue.[23] Yet, it is very natural to ask what happens if we iterate the Lebesgue measure by the operator T_*. It is easy to see that T_*m is still absolutely continuous with respect to Lebesgue. In fact, T_* maps absolutely continuous measures into absolutely continuous measures. Once we note this, it is very tempting to define the transfer operator. An easy computation yields

$$\mathcal{L}h(x) = \chi_{TQ}(x) \sum_{y \in T^{-1}(x)} |\det(D_yT)|^{-1} h(y) = 4\chi_{TQ}(x)h(T^{-1}(x)).$$

Since the map expands in the unstable direction, it is quite natural to investigate, in analogy with the expanding case, the *unstable derivative* D^u, that is the derivative in the x direction, of the iterate of the density.

$$\|D^u\mathcal{L}h\|_1 \leq \frac{1}{2}\|D^uh\|_1 \quad \forall h \in \mathcal{C}^{(1)}(Q) \tag{1.4.5}$$

[23] In fact, if μ is an invariant measure, $T_*\mu = \mu$, it follows

$$\mu(\chi_{T^nQ}) = T_*^n\mu(\chi_{T^nQ}) = \mu(\chi_Q) = 1,$$

so μ must be supported on $\Lambda = \cap_{n=0}^{\infty} T^nQ$.

To see the consequences of the above estimate, consider $f \in \mathcal{C}^{(1)}(Q)$ with $f(0,y) = f(1,y) = 0$ for each $y \in [0,1]$, then if μ is a measure obtained by the measure hdm ($h \in \mathcal{C}^{(1)}$) with the procedure of Proposition 1.4.2,[24] we have

$$\begin{aligned}\mu(D^u f) &= \lim_{j\to\infty} \frac{1}{n_j}\sum_{i=0}^{n_j-1}(T_*)^i m(hD^u f) = \lim_{j\to\infty}\frac{1}{n_j}\sum_{i=0}^{n_j-1} m(\mathcal{L}^i h D^u f)\\ &= -\lim_{j\to\infty}\frac{1}{n_j}\sum_{i=0}^{n_j-1} m(fD^u\mathcal{L}^i h)\end{aligned}$$

where we have integrated by part. Remembering (1.4.5) we have

$$\mu(D^u f) = 0,$$

for all $f \in \mathcal{C}^{(1)}_{\mathrm{per}}(Q) = \{f \in \mathcal{C}^{(1)}(Q) \mid f(0,y) = f(1,y)\}$. The enlargement of the class of functions is due to the obvious fact that, if $f \in \mathcal{C}^{(1)}_{\mathrm{per}}(Q)$, then $\tilde f(x,y) = f(x,y) - f(0,y)$ is zero on the vertical (stable) boundary and $D^u\tilde f = D^u f$.

This means that the measure μ, when restricted to the horizontal direction, is μ-a.e. constant (see Problem 1.32). Such a strong result is clearly a consequence of the fact that the map is essentially linear, one can easily imagine a non linear case (think of dilations and expanding maps) and in that case the same argument would lead to conclude that the measure, when restricted to unstable manifolds, is absolutely continuous with respect to the restriction of Lebesgue (these type of measures are commonly called SRB from Sinai, Ruelle and Bowen).

We can now prove that indeed the measure μ is invariant. The discontinuity line of T is $\{x = \frac{1}{2}\}$. Points close to $\{x = \frac{1}{2}\}$ are mapped close to the boundary of Q, so if $f(0,y) = f(1,y) = 0$, then $f \circ T$ is continuous. Hence, the argument of Proposition 1.4.2 proves that $\mu(f\circ T) = \mu(f)$ for all f that vanish at the stable boundary. Yet, the characterization of μ proves that $\mu(\{(x,y) \in Q \mid x \in \{0,1\}\}) = 0$, thus we can obtain $\mu(f\circ T) = \mu(f)$ for all continuous functions via the Lebesgue dominated convergence theorem and the invariance follows by Lemma 1.4.1.

1.4.1.f *Horseshoe*

This very famous example consists of a map of the square $Q = [0,1]^2$, the map is obtained by stretching the square in the horizontal direction, bending it in the shape of an horseshoe and then superimposing it to the original square in such a

[24] As we noted in the proof of Proposition 1.4.2, the only part that uses the continuity of T is the proof of the invariance. Thus, in general we can construct a measure by the averaging procedure but its invariance is not automatic.

way that the intersection consists of two horizontal strips.[25] Such a description is just topological, to make things clearer let us consider a very special case:

$$T(x,\,y) = \begin{cases} (5x \mod 1,\, \frac{1}{4}y) & \text{if } x \in [1/5,\, 2/5] \\ (5x \mod 1,\, \frac{1}{4}y + \frac{3}{4}) & \text{if } x \in [3/5,\, 4/5]. \end{cases}$$

Note that T is not explicitly defined for $x \in [0,\, 1/5[\cup[\frac{2}{3},\, \frac{3}{5}[\cup]4/5,\, 1]$ since for this values the horseshoe falls outside Q, so its actual shape is irrelevant. Since the map from Q to Q is not defined on the full square, we can have a Dynamical System only with respect to a measure for which the domain of definition of T, and all of its powers, has measure one. We will start by constructing such a measure.

The first step is to notice that the set

$$\Lambda = \cap_{n\in\mathbb{Z}} T^n Q \tag{1.4.6}$$

of the points which trajectories are always in Q is $\neq \emptyset$. Second, note that $\Lambda = T\Lambda = T^{-1}\Lambda$, such an invariant set is called *hyperbolic set* as we will see in ???. We would like to construct an invariant measure on Λ. Since Λ is a compact set and T is continuous on it we know that there exist invariant measures; yet, in analogy with the previous examples, we would like to construct one *coming from Lebesgue*.

As already mentioned we must start by constructing a measure on $\Lambda_- = \cap_{n\in\mathbb{N}\cup\{0\}} T^{-n}Q$ since $T^k\Lambda_- \subset \Lambda_-$. To do so it is quite natural to construct a measure by *subtracting* the mass that leaks out of Q. namely, define the operator $\tilde{T} : \mathcal{M}(X) \to \mathcal{M}(X)$ by

$$\tilde{T}\mu(A) := \mu(TA \cap Q).$$

Again we consider the evolution of measures of the type $d\mu = hdm$. For each continuous f with $\text{supp}(f) \subset Q$ holds

$$\tilde{T}\mu(f) = \mu(f \circ T^{-1}\chi_Q) = \int_{T^{-1}Q} fh \circ T|\det DT|dm.$$

We can thus define the operator $\mathcal{L}$ that evolves the densities:

$$\mathcal{L}h(x) = \frac{5}{4}\chi_{T^{-1}Q\cap Q}(x)h(Tx).$$

Clearly $\tilde{T}\mu(f) = m(f\mathcal{L}h)$.

Note that $\tilde{T}m(1) = \frac{1}{2}$, thus $\tilde{T}$ does not map probability measures into probability measures; this is clearly due to the mass leaking out of Q. Calling D^s

[25] We have already seen something very similar in the introduction.

(stable derivative) the derivative in the y direction, follows easily

$$\|D^s\mathcal{L}h\|_1 \leq \frac{1}{4}\|D^s h\|_1$$

for each h differentiable in the stable direction.

On the other hand, if $\|D^s h\|_1 \leq c$ and $\Delta = [0, 1/4] \cup [3/4, 1]$,

$$\begin{aligned}|\tilde{T}\mu(1)| &= \int_{Q\cap TQ} h = \int_\Delta dy \int_0^1 dx h(x,y)\\ &= \int_\Delta dy \int_0^1 dx \int_0^1 d\xi h(x,\xi) + \mathcal{O}(\|D^s h\|_1)\\ &= |\Delta|\|h\|_1 + \mathcal{O}(\|D^s h\|_1) = \frac{1}{2}\mu(1) + \mathcal{O}(\|D^s h\|_1).\end{aligned}$$

It is then natural to define $\hat{\mathcal{L}}h := 2\mathcal{L}h$ and $\hat{T} = 2\tilde{T}$. Thus $\|D^s\hat{\mathcal{L}}h\|_1 \leq \frac{1}{2}\|D^s h\|_1$. This means that $\{\frac{1}{n}\sum_{i=0}^{n-1}\hat{T}^i\mu\}$ are probability measures. Accordingly, there exists an accumulation point μ_* and $\mu_*(D^s f) = 0$ for each f periodic in the y direction. By the same type of arguments used in the previous examples, this means that μ_* is constant in the y direction, it is supported on Λ_- by construction and $\tilde{T}\mu_* = \frac{1}{2}\mu_*$ (*conformal invariance*) : just the measure we where looking for.

We can now conclude the argument by evolving the measure as usual:

$$T_*\mu_*(f) = \mu_*(f \circ T)$$

for all continuous f with the support in Q. Now the standard argument applies. In such a way we have obtained the invariant measure supported on Λ.

1.4.1.g *Markov Measures*

Let us consider the shift (Σ_n^+, T). We would like to construct other invariant measures bedside Bernoulli. As we have seen it suffices to specify the measure on the algebra of the cylinders. Let us define

$$A(m; k_1, \dots, k_l) = \{\sigma \in \Sigma_n^+ \mid \sigma_{i+m} = k_i \ \forall\, i \in \{1, \dots, l\}\};$$

this are a basis for the algebra of the cylinders.

For each $n \times n$ matrix P, $P_{ij} \geq 0$, $\sum_j P_{ij} = 1$ by the Perron-Frobenius theorem (see Example (4.3.1.a)) there exists $\{p_i\}$ such that $pP = p$. Let us define

$$\mu(A(m; k_1, \dots, k_l)) = p_{k_1}P_{k_1k_2}P_{k_2k_3}\dots P_{k_{l-1}k_l}.$$

The reader can easily verify that μ is invariant over the algebra $\mathcal{A}$ and thus extends to an invariant measure. This is called Markov because it is nothing else than a Markov chain together with its stationary measure [1].[26]

These last examples (strange attractor, solenoid, horseshoe) show only a very dim glimpse of a much more general and extremely rich theory (the study of SRB measures) while the last (Markov measures) points toward another extremely rich theory: Gibbs (or equilibrium) measures. Although this it is not the focus here, we will see a bit more of this in the future.

One of the main objectives in dynamical systems is the study of the long time behavior (that is the study of the trajectories $T^n x$ for large n). There are two main cases in which it is possible to study, in some detail, such a long time behavior. The case in which the motion is rather regular[27] or close to it (the main examples of this possibility are given by the so called KAM [6] theory and by situations in which the motions is attracted by a simple set); and the case in which the motion is very irregular.[28] This last case may seem surprising since the irregularity of the motion should make its study very difficult. The reason why such systems can be studied is, as usual, because we ask the right questions,[29] that is we ask questions not concerning the fine details of the motion but only concerning its statistical or qualitative properties.

The first example of such properties is the study of the invariant sets.

1.5 Ergodicity

Definition 1.5.1 *A measurable set A is invariant for T if $T^{-1}A \subset A$.*

A dynamical system (X, T, μ) is ergodic if each invariant set has measure zero or one.

The definition for continuous dynamical systems being exactly the same.

Note that if A is invariant then $\mu(A \backslash T^{-1}A) = \mu(A) - \mu(T^{-1}A) = 0$, moreover $\Lambda = \cap_{n=0}^{\infty} T^{-n}A \subset A$ is invariant as well. In addition, by definition, $\Lambda = T\Lambda$, which implies $\Lambda = T^{-1}\Lambda$ and $\mu(A \backslash \Lambda) = 0$. This means that, if A is invariant, then it always contains a set Λ invariant in the stronger (maybe more natural) sense that $T\Lambda = T^{-1}\Lambda = \Lambda$. Moreover, Λ is of full measure in A. Our definition of invariance is motivated by its greater flexibility and the

[26] The probabilistic interpretation is that the probability of seeing the state k at time one, given that we saw the state l at time zero, is given by P_{lk}. So the process has a bit of memory: it remembers its state one time step before. Of course it is possible to consider processes that have a longer–possibly infinite–memory. Proceeding in this direction one would define the so called *Gibbs measures*.

[27] Typically, quasi periodic motion, remember the small oscillation in the pendulum.

[28] Remember the example in the introduction.

[29] Of course, the “right questions” are the ones that can be answered.

fact that, from a measure theoretical point of view, zero measure sets can be discarded.

In essence, if a system is ergodic then most trajectories explore all the available space. In fact, for any A of positive measure, define $A_b = \cup_{n\in\mathbb{N}\cup\{0\}} T^{-n}A$ (this are the points that eventually end up in A), since $A_b \supset A$, $\mu(A_b) > 0$. Since $T^{-1}A_b \subset A_b$, by ergodicity follows $\mu(A_b) = 1$. Thus, the points that never enter in A (that is, the points in A_b^c) have zero measure. Actually, if the system has more structure (topology) more is true (see Problem 1.21).

The reader should be aware that there are many equivalent definitions of ergodicity, see Problems 1.25, 1.27, 1.28 and Theorem 1.6.6 for some possibilities.

1.5.1 Examples

1.5.1.a *Rotations*

The ergodicity of a rotation depends on ω. If $\omega \in \mathcal{Q}$ then the system is not ergodic. In fact, let $\omega = \frac{p}{q}$ $(p, q \in \mathbb{N})$, then, for each $x \in \mathbb{T}$ $T^q x = x + p \mod 1 = x$, so T^q is just the identity. An alternative way of saying this is to notice that all the points have a periodic trajectory of period q. It is then easy to exhibit an invariant set with measure strictly larger than 0 but strictly less than 1. Consider $[0, \varepsilon]$, then $A = \cup_{i=1}^{q-1} T^{-i}[0, \varepsilon]$ is an invariant set; clearly $\varepsilon \leq \mu(A) \leq q\varepsilon$, so it suffices to choose $\varepsilon < q^{-1}$.

The case $\omega \notin \mathcal{Q}$ is much more interesting. First of all, for each point $x \in \mathbb{T}$ we have that the closure of the set $\{T^n x\}_{i=0}^{\infty}$ is equal to $\mathbb{T}$, which is to say that the orbits are dense.[30] The proof is based on the fact that there cannot be any periodic orbit. To see this suppose that $x \in \mathbb{T}$ has a periodic orbit, that is there exists $q \in \mathbb{N}$ such that $T^q x = x$. As a consequence there must exist $p \in \mathbb{Z}$ such that $x + p = x + q\omega$ or $\omega \in \mathcal{Q}$ contrary to the hypothesis. Hence, the set $\{T^k 0\}_{k=0}^{\infty}$ must contain infinitely many points and, by compactness, must contain a convergent subsequence k_i. Hence, for each $\varepsilon > 0$, there exists $m > n \in \mathbb{N}$:

$$|T^m 0 - T^n 0| < \varepsilon.$$

Since T preserves the distances, calling $q = m - n$, holds

$$|T^q 0| < \varepsilon.$$

Accordingly, the trajectory of $T^{jq}0$ is a translation by a quantity less than ε, therefore it will get closer than ε to each point in $\mathbb{T}$ (i.e., the orbit is dense). Again by the conservation of the distance, since zero has a dense orbit the same will hold for every other point.

[30] A system with a dense orbit called *Topologically Transitive*.

Intuitively, the fact that the orbits are dense implies that there cannot be a non trivial invariant set, henceforth the system is ergodic. Yet, the proof it is not trivial since it is based on the existence of Lebesgue density points [74] (see Problem 1.43). It is a fact from general measure theory that each measurable set $A \subset \mathbb{R}$ of positive Lebesgue measure contains, at least, one point $\bar{x}$ such that for each $\varepsilon \in (0, 1)$ there exists $\delta > 0$:

$$\frac{m(A \cap [\bar{x} - \delta, \bar{x} + \delta])}{2\delta} > 1 - \varepsilon.$$

Hence, given an invariant set A of positive measure and $\varepsilon > 0$, first choose δ such that the interval $I := [\bar{x} - \delta, \bar{x} + \delta]$ has the property $m(I \cap A) > (1 - \varepsilon) m(I)$. Second, we know already that there exists $q, M \in \mathbb{N}$ such that $\{T^{-kq}x\}_{k=1}^{M}$ divides $[0, 1]$ into intervals of length less that $\frac{\varepsilon}{2}\delta$. Hence, given any point $x \in \mathbb{T}$ choose $k \in \mathbb{N}$ such that $m(T^{-kq}I \cap [x - \delta, x + \delta]) > m(I)(1 - \varepsilon)$ so,

$$\begin{aligned} m(A \cap [x - \delta, x + \delta]) &\geq m(A \cap T^{-kq} I) - m(I)\varepsilon \\ &\geq m(A \cap I) - m(I)\varepsilon \geq (1 - 2\varepsilon) 2\delta. \end{aligned}$$

Thus, A has density everywhere larger than $1 - 2\varepsilon$, which implies $\mu(A) = 1$ since ε is arbitrary.

The above proof of ergodicity it is not so trivial but it has a definite dynamical flavor (in the sense that it is obtained by studying the evolution of the system). Its structure allows generalizations to contexts whit a less rich algebraic structure. Nevertheless, we must notice that, by taking advantage of the algebraic structure (or rather the group structure) of $\mathbb{T}$, a much simpler and powerful proof is available.

Let $\nu \in \mathcal{M}_T^1$, then define

$$F_n = \int_{\mathbb{T}} e^{2\pi i n x} \nu(dx), \quad n \in \mathbb{N}.$$

A simple computation, using the invariance of ν, yields

$$F_n = e^{2\pi i n \omega} F_n$$

and, if ω is irrational, this implies $F_n = 0$ for all $n \neq 0$, while $F_0 = 1$. Next, consider $f \in \mathcal{C}^{(2)}(\mathbb{T}^1)$ (so that we are sure that the Fourier series converges uniformly, see Problem 1.31), then

$$\nu(f) = \sum_{n=0}^{\infty} \nu(f_n e^{2\pi i n \cdot}) = \sum_{n=0}^{\infty} f_n F_n = f_0 = \int_{\mathbb{T}} f(x) dx.$$

Hence m is the unique invariant measure (unique ergodicity). This is clearly much stronger than ergodicity (see Problem 1.25)

1.5.1.b *Baker*

This transformation gets its name from the activity of bread making, it bears some resemblance with the horseshoe. The space X is the square $[0,\, 1]^2$, μ is again Lebesgue, and T is a transformation obtained by squashing down the square into the rectangle $[0,\, 2] \times [0,\, \frac{1}{2}]$ and then cutting the piece $[1,\, 2] \times [0,\, \frac{1}{2}]$ and putting it on top of the other one. In formulas

$$T(x,\, y) = \begin{cases} (2x,\, \dfrac{1}{2}y) \mod 1 & \text{if } x \in [0,\, \dfrac{1}{2}) \\ (2x,\, \dfrac{1}{2}(y+1)) \mod 1 & \text{if } x \in [\dfrac{1}{2},\, 1]. \end{cases}$$

This transformation is ergodic as well, in fact much more. We will discuss it later.

1.5.1.c *Translations ($\mathbb{T}^1$)*

Let us consider the flow $(\mathbb{T}^1, \phi_t, m)$ where $\phi_t(x) = x + \omega t \mod 1$, for some $\omega \in \mathbb{R} \setminus \{0\}$. This is just a translation on the unit circle. The proof of ergodicity is trivial and it is left to the reader.

We conclude the chapter with a theorem very helpful to establish the ergodicity of a flow.

Theorem 1.5.2 *Consider a flow (X, ϕ_t, μ) and a Poincarè section Σ such that the set $\{x \in X \mid \cup_{t\in\mathbb{R}} \phi_t(x) \cap \Sigma = \emptyset\}$ has zero measure. Then the ergodicity of the flow (X, ϕ_t, μ) is equivalent to the ergodicity of the section $(\Sigma, T_\Sigma, \mu_\Sigma)$.*

The proof, being straightforward, is left to the reader.

1.5.2 Examples

1.5.2.a *Translations ($\mathbb{T}^2$)*

Let us consider the flow $(\mathbb{T}^2, \phi_t, m)$ where $\phi_t(x) = x + \omega t \mod 1$, for some $\omega \in \mathbb{R}^2 \setminus \{0\}$. This is a translation on the two dimensional torus. To investigate we will use Theorem 1.5.2. Consider the set $\Sigma := \{(x, y) \in \mathbb{T}^2 \mid x = 0\}$, this is clearly a Poincaré section, unless $\omega_1 = 0$ (in which case one can choose the section $y = 0$). Obviously Σ is a circle and the Poincaré map is given by

$$T(y) = y + \frac{\omega_2}{\omega_1} \mod 1.$$

The ergodicity of the flow is then reduced to the ergodicity of a circle rotation, thus the flow is ergodic only if ω_1 and ω_2 have an irrational ratio.

The properties of the invariant sets of a dynamical systems have very important reflections on the statistics of the system, in particular on its time averages. Before making this precise (see Theorem 1.6.6) we state few very general and far reaching results.

1.6 Some basic Theorems

Theorem 1.6.1 *(Birkhoff) Let (X, T, μ) be a dynamical system, then for each $f \in L^1(X, \mu)$*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x)$$

exists for almost every point $x \in X$. In addition, setting

$$f^+(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x),$$

holds

$$\int_X f^+ d\mu = \int_X f d\mu.$$

Proof

Since the task at hand is mainly didactic, we will consider explicitly only the case of positive bounded functions, the completion of the proof is left to the reader.

Let $f \in L^\infty(X, d\mu)$, $f \geq 0$, and

$$S_n(x) \equiv \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x).$$

For each $x \in X$, there exists

$$\begin{aligned} \overline{f}^+(x) &= \limsup_{n\to\infty} S_n(x) \\ \underline{f}^+(x) &= \liminf_{n\to\infty} S_n(x). \end{aligned}$$

The first remark is that both $\overline{f}^+$ and $\underline{f}^+$ are invariant functions. In fact,

$$S_n(Tx) = S_n(x) + \frac{1}{n} f(T^n x) - \frac{1}{n} f(x)$$

so, tacking the limit the result follows.[31]

[31]Here we have used the boundedness, this is not necessary. If $f \in L^1(X, d\mu)$ and positive, then $S_n(Tx) \geq S_n(x) - f(x)$, so $\underline{\overline{f}}^+(Tx) \geq \underline{\overline{f}}^+(x)$ and it is and easy exercise to check that any such function must be invariant.

Next, for each $n \in \mathbb{N}$ and $k,\, j \in \mathbb{Z}$ we define

$$D_{n,l,j} = \left\{ x \in X \,\middle|\, \overline{f}^{+}(x) \in \left[\frac{l}{n},\, \frac{l+1}{n}\right) ;\ \underline{f}^{+}(x) \in \left[\frac{j}{n},\, \frac{j+1}{n}\right) \right\},$$

by the invariance of the functions follows the invariance of the sets $D_{n,l,j}$. Also, by the boundedness, follows that for each n exists n_0 such as

$$\bigcup_{j,l \in \{-n_0,\, \ldots,\, n_0\}} D_{n,l,j} = X.$$

The key observation is the following.

Lemma 1.6.2 *For each $n \in \mathbb{N}$ and $l,\, j \in \mathbb{Z}$, setting $A = D_{n,l,j}$, holds*

$$\frac{l+1}{n}\mu(A) < \int_A f d\mu + \frac{3}{n}\mu(A)$$
$$\frac{j}{n}\mu(A) > \int_A f d\mu - \frac{3}{n}\mu(A)$$

From the Lemma follows

$$\begin{aligned} 0 \le \int_X (\overline{f}^{+} - \underline{f}^{+}) d\mu &= \sum_{l,\, j=-n_0}^{n_0} \int_{D_{n,l,j}} (\overline{f}^{+} - \underline{f}^{+}) d\mu \\ &\le \sum_{l,\, j=-n_0}^{n_0} \left[\frac{l+1}{n} - \frac{j}{n}\right] \mu(D_{n,l,j}) < \frac{6}{n} \sum_{l,\, j=-n_0}^{n_0} \mu(D_{n,l,j}) = \frac{6}{n}. \end{aligned}$$

Since n is arbitrary we have

$$\int_X (\overline{f}^{+} - \underline{f}^{+}) d\mu = 0$$

which implies $\overline{f}^{+} = \underline{f}^{+}$ almost everywhere (since $\overline{f}^{+} \ge \underline{f}^{+}$ by definition) proving that the limit exists. Analogously, we can prove

$$\int_X (f - f^{+}) d\mu = 0.$$

Proof of the Lemma 1.6.2 We will prove only the first inequality, the second being proven in exactly the same way.

For each $x \in A$ we will call $k(x)$ the first $m \in \mathbb{N}$ such that

$$S_m(x) > \frac{l-1}{n},$$

by construction $k(x)$ must be finite for each $x \in A$. Hence, setting $X_k = \{x \in A \mid k(x) = k\}$, $\cup_k X_k = A$, and for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\mu\left(\bigcup_{k=1}^{N} X_k\right) \geq \mu(A)(1-\varepsilon).$$

Let us call

$$Y = A \backslash \bigcup_{k=1}^{N} X_k.$$

Then $\mu(Y) \leq \mu(A)\varepsilon$, also set $L = \sup_{x\in A} |f(x)|$. The basic idea is to follow, for each point $x \in A$, the trajectory $\{T^i x\}_{i=0}^{M}$, where $M > N$ will be chosen sufficiently large. If the point would never visit the set Y, we could group the sum $S_M(x)$ in pieces all, in average, larger than $\frac{l-1}{n}$, so the same would hold for $S_M(x)$. The difficulties come from the visits to the set Y.

For each $n \in \{0, ..., M\}$ define

$$\widetilde{f}_n(x) = \begin{cases} f(T^n x) & \text{if } T^n x \notin Y \\ \dfrac{l}{n} & \text{if } T^n x \in Y \end{cases}$$

and

$$\widetilde{S}_M(x) = \frac{1}{M}\sum_{n=0}^{M-1} \widetilde{f}_n(x).$$

By definition $y \in Y$ implies $y \notin X_1$, i.e. $f(y) \leq \frac{l-1}{n}$. Accordingly, $\widetilde{f}(x) \geq f(T^n x)$ for each $x \in A$. Note that for each n we change the function $f \circ T^n$ only at some points belonging to the set Y and $\frac{l}{n}$ can be taken less or equal than L (otherwise $\mu(A) = 0$), consequently

$$\int_A f d\mu = \int_A S_M d\mu \geq \int_A \widetilde{S}_M d\mu - L\mu(Y) \geq \int_A \widetilde{S}_M d\mu - L\mu(A)\varepsilon.$$

We are left with the problem of computing the sum. As already mentioned the strategy consists in dividing the points according to their trajectory with respect to the sets X_n. To be more precise, let $x \in A$, then by definition it must belong to some X_n or to Y. We set $k_1(x)$ equal to j is $x \in X_j$ and $k_1(x) = 1$ if $x \in Y$. Next, $k_2(x)$ will have value j if $T^{k_1(x)} x \in X_j$ or value 1 if $T^{k_1(x)} \in Y$. If $k_1(x) + k_2(x) < M$, then we go on and define similarly $k_3(x)$. In this way, to each $x \in A$ we can associate a number $m(x) \in \{1, ..., M\}$ and indices $\{k_i(x)\}_{i=1}^{m(x)}$, $k_i(x) \in \{1, ..., N\}$, such that $M - N \leq \sum_{i=1}^{m(x)-1} k_i(x) < M$, $\sum_{i=1}^{m(x)} k_i(x) \geq M$. Let us call $K_p(x) = \sum_{j=1}^{p} k_j(x)$. Using such a division of the orbit in segments of length $k_i(x)$ we can easily estimate

$$\begin{aligned}\widetilde{S}_M(x) =& \frac{1}{M}\left\{\sum_{i=1}^{m(x)-1} k_i(x)\left[\frac{1}{k_i(x)}\sum_{j=K_{i-1}(x)}^{K_i(x)-1}\widetilde{f}_j(x)\right] + \sum_{i=K_{m(x)-1}(x)}^{M-1}\widetilde{f}(T^i x)\right\}\\ \geq& \frac{1}{M}\sum_{i=1}^{m(x)-1} k_i(x)\frac{l-1}{n} \geq \frac{M-N}{M}\frac{l-1}{n}.\end{aligned}$$

Putting together the above inequalities we get

$$\begin{aligned}\int_A f d\mu &\geq \left\{\frac{(M-N)(l-1)}{Mn} - L\varepsilon\right\}\mu(A)\\ &\geq \frac{l+1}{n}\mu(A) - \left\{\frac{2}{n} + \frac{N(l-1)}{Mn} + L\varepsilon\right\}\mu(A).\end{aligned}$$

which, by choosing first ε sufficiently small and, after, M sufficiently large, concludes the proof. □

To prove the result for all function in $L^1(X, \mu)$ it is convenient to deal at first only with positive functions (which suffice since any function is the difference of two positive functions) and then use the usual trick to cut off a function (that is, given f define f_L by $f_L(x) = f(x)$ if $f(x) \leq L$, and $f_L(x) = L$ otherwise) and then remove the cut off. The reader can try it as an exercise. □

Birkhoff theorem has some interesting consequences.

Corollary 1.6.3 *For each $f \in L^1(X, \mu)$ the following holds*

1. *$f_+ \in L^1(X, \mu)$;*
2. *$f_+(Tx) = f_+(x)$ almost surely.*

The proof is left to the reader as an easy exercise (see Problem 1.18).

Another interesting fact, that starts to show some connections between averages and invariant sets, emerges by considering a measurable set A and its characteristic function χ_A. A little thought shows that the ergodic average $\chi_A^+(x)$ is simply the average frequency of visit of the set A by the trajectory $\{T^n x\}$ (Problem 1.28).

Birkhoff theorem implies also convergence in L^1 and L^2 (see also Problem 1.26). Yet, it is interesting to note that convergence in L^2 can be proven in a much more direct way.

Theorem 1.6.4 (Von Neumann) *Let (X, T, μ) be a Dynamical System, then for each $f \in L^2(X, \mu)$ the ergodic average converges in $L^2(X, \mu)$.*

PROOF. We have already seen that it can be useful to lift the dynamics at the level of the algebra of function or at the level of measures. This game assumes different guises according to how one plays it, here is another very interesting version.

Let us define $U : L^2(X,\mu) \to L^2(X,\mu)$ as

$$Uf := f \circ T.$$

Then, by the invariance of the measure, it follows $\|Uf\|_2 = \|f\|_2$, so U is an L^2 contraction (actually, and L^2-isometry). If T is invertible, the same argument applied to the inverse shows that U is indeed unitary, otherwise we must content ourselves with

$$\|U^*f\|_2^2 = \langle UU^*f, f\rangle \le \|UU^*f\|_2\|f\|_2 = \|U^*f\|_2\|f\|_2,$$

that is $\|U^*\|_2 \le 1$ (also U^* is and L^2 contraction).

Next, consider $V_1 = \{f \in L^2 \mid Uf = f\}$ and $V_2 = \text{Rank}(\mathbb{1} - U)$. First of all, note that if $f \in V_1$, then

$$\|U^*f - f\|_2^2 = \|U^*f\|_2^2 - \langle f, U^*f\rangle - \langle U^*f, f\rangle + \|f\|_2^2 \le 0.$$

Thus, $f \in V_1^* := \{f \in L^2 \mid U^*f = f\}$. The same argument applied to $f \in V_1^*$ shows that $V_1 = V_1^*$. To continue, consider $f \in V_1$ and $h \in L^2$, then

$$\langle f, h - Uh\rangle = \langle f - U^*f, h\rangle = 0.$$

This implies that $V_1^\perp = \overline{V_2}$, hence $V_1 \oplus \overline{V_2} = L^2$. Finally, if $g \in V_2$, then there exists $h \in L^2$ such that $g = h - Uh$ and

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{\infty} U^i g = \lim_{n\to\infty} \frac{1}{n}(h - U^n h) = 0.$$

On the other hand if $f \in V_1$ then $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{\infty} U^i f = f$. The only function on which we do not still have control are the g belonging to the closure of V_2 but not in V_2. In such a case there exists $\{g_k\} \subset V_2$ with $\lim_{k\to\infty} g_k = g$. Thus,

$$\|\frac{1}{n}\sum_{i=0}^{\infty} U^i g\|_2 \le \|\frac{1}{n}\sum_{i=0}^{\infty} U^i g_k\|_2 + \|g - g_k\|_2 \le \|\frac{1}{n}\sum_{i=0}^{\infty} U^i g_k\|_2 + \frac{\varepsilon}{2},$$

provided we choose k large enough. Then, by choosing n sufficiently large we obtain

$$\|\frac{1}{n}\sum_{i=0}^{\infty} U^i g\|_2 \le \varepsilon.$$

We have just proven that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}U^i = P$$

where P is the orthogonal projection on V_1. □

Another very general result, of a somewhat disturbing nature, is Poincaré return theorem.

Theorem 1.6.5 (Poincaré) *Given a dynamical systems (X, T, μ) and a measurable set A, with $\mu(A) > 0$, there exists infinitely many $n \in \mathbb{N}$ such that*

$$\mu(T^{-n}A \cap A) \neq 0.$$

The proof is rather simple (by contradiction) and the reader can certainly find it out by herself (see Problem 1.19).[32]

Let us go back to the relation between ergodicity and averages. From an intuitive point of view a function from X to $\mathbb{R}$ can be thought as an "observable," since to each configuration it associates a value that can represent some relevant property of the configuration (the property that we observe). So, if we observe the system for a long time via the function f, what we see should be well represented by the function f^+. Furthermore, notice that there is a simple relations between invariant functions and invariant sets. More precisely, if a measurable set A is invariant, then its characteristic function χ_A is a measurable invariant function; if f is an invariant function then for each measurable set $I \in \mathbb{R}$ the set $f^{-1}(I)$ is a measurable invariant set (if the implications of the above discussions are not clear to you, see Problem 1.27).

As a byproduct of the previous discussion it follows that if a system is ergodic then for each function $f \in L^1(X, \mu)$ the function f_+ is almost everywhere constant and equal to $\int_X f$. We have just proven an interesting characterization of the ergodic systems:

Theorem 1.6.6 *A Dynamical System (X, T, μ) is ergodic if and only if for each $f \in L^1(X, \mu)$ the ergodic average f^+ is constant; in fact, $f^+ = \mu(f)$ a.e..*

[32] An unsettling aspect of the theorem is due to the following possibility. Consider a room full of air, the motion of the molecules can be thought to happen accordingly to Newton equations, i.e. it is an Hamiltonian systems, hence a dynamical system to which Poincaré theorem applies. Let A be the set of configurations in which all the air is in the left side of the room. Since we ignore, in general, the past history of the room, it could very well be that at some point in the past the systems was in a configuration belonging to A–maybe some silly experiment was performed. So there is a positive probability for the system to return in the same state. Therefore the disturbing possibility of sudden death by decompression.

In other words, if we observe the time average of some observable for a sufficiently long time then we obtain a value close to its space average. The previous observation is very important especially because the space average of a function does not depend on the dynamics. This is exactly what we where mentioning previously: the fact that the dynamics is sufficiently 'complex' allows us to ignore it completely, provided we are interested only in knowing some average behavior. The relevance of ergodic theory for physical systems is largely connected to this fact.

1.7 Mixing

We have argued the importance of ergodicity, yet from a physical point of view ergodicity may be relevant only if it takes places at a sufficiently fast rate (i.e., if the time average converges to the space average on a physically meaningful time scale). This has prompted the study of stronger statistical properties of which we will give a brief, and by no mean complete, account in the following.

Definition 1.7.1 *A Dynamical System (X, T, μ) is called* mixing *if for every pairs of measurable sets A, B we have*

$$\lim_{n\to\infty} \mu(T^{-n}(A) \cap B) = \mu(A)\mu(B).$$

Obviously, if a system is mixing, then it is ergodic. In fact, if A is an invariant set for T, then $T^{-n}A \subset A$, so, calling A^c the complement of A, we have

$$\mu(A)\mu(A^c) = \lim_{n\to\infty} \mu(T^{-n}A \cap A^c) = 0,$$

and the measure of A is either one or zero.

An equivalent characterization of mixing is the following:

Proposition 1.7.2 *A Dynamical System (X, T, μ) is mixing if and only if*

$$\lim_{n\to\infty} \int_X f \circ T^n g d\mu = \int_X f d\mu \int_X g d\mu$$

for every $f, g \in L^2(X, \mu)$ or for every $f \in L^\infty(X,\mu)$ and $g \in L^1(X,\mu)$.[33]

The proof is rather straightforward and it is left as an exercise to the reader (see Problem 1.29) together with the proof of the next statement.

[33] The quantity $\int_X f \circ T g - \int_X f \int_X g$ is called "correlation," and its tending to zero–which takes places always in mixing systems–it is called "decay of correlation."

Proposition 1.7.3 *A Dynamical System (X, T, μ), with X a compact metric space, T continuous and μ Borel, is mixing if and only if for each probability measure λ absolutely continuous with respect to μ*

$$\lim_{n\to\infty} \lambda(f \circ T^n) = \mu(f)$$

for each $f \in \mathcal{C}^{(0)}(\mathbb{T}^2)$.

This last characterization is interesting from a mathematical point of view. Define, as usual, the evolution of a measure via the equation

$$(T_*\lambda)(f) \equiv \lambda(f \circ T)$$

for each continuous function f. If for each measure, absolutely continuous with respect to the invariant one, the evolved measure converges weakly to the invariant measure, then the system is mixing (and thus the evolved measures converge strongly). This has also a very important physical meaning: if the initial configuration is known only in probability, the probability distribution is absolutely continuous with respect to the invariant measure, and the system is mixing, then, after some time, the configurations are distributed according to the invariant measure. Again the details of the evolution are not important to describe relevant properties of the system.

1.7.1 Examples

1.7.1.a *Rotations*

We have seen that the translations by an irrational angle are ergodic. They are not mixing. The reader can easily see why.

1.7.1.b *Bernoulli shift*

The key observation is that, given a measurable set A, for each $\varepsilon > 0$ there exists a set $A_\varepsilon \in \mathcal{A}$, thus depending only on a finite subset of indices,[34] with the property[35]

$$\mu(A_\varepsilon \backslash A) \leq \varepsilon.$$

Then, given A, B measurable, and for each $\varepsilon > 0$, let A_ε, B_ε be such an approximation, and I_A, I_B the defining sets of indices, then

$$\left|\mu(T^{-m}A \cap B) - \mu(A)\mu(B)\right| \leq 4\varepsilon + \left|\mu(T^{-m}A_\varepsilon \cap B_\varepsilon) - \mu(A_\varepsilon)\mu(B_\varepsilon)\right|.$$

[34] Remember, this means that there exists a finite set $I \subset \mathbb{Z}$ such that it is possible to decide if $\sigma \in \Sigma_n$ belongs or not to A_ε only by looking at $\{\sigma_i\}_{i\in I}$.

[35] This follows from our construction of the σ-algebra and by the definition of outer measure, see Examples 1.1.1–Bernoulli shift.

If we choose m so large that $(I_A+m)\cap I_B=\emptyset$, then by the definition of Bernoulli measure we have

$$\mu(T^{-m}A_\varepsilon\cap B_\varepsilon)=\mu(T^{-m}A_\varepsilon)\mu(B_\varepsilon)=\mu(A_\varepsilon)\mu(B_\varepsilon),$$

which proves

$$\lim_{m\to\infty}\mu(T^{-m}A\cap B)=\mu(A)\mu(B).$$

1.7.1.c *Dilation*

This system is mixing. In fact, let $f,\, g\in\mathcal{C}^{(1)}(\mathbb{T})$, then we can represent them via their Fourier series $f(x)=\sum_{k\in\mathbb{Z}}e^{2\pi ikx}f_k,\ f_{-k}=\overline{f}_k$. It is well known that $\sum_{k\in\mathbb{Z}}|f_k|<\infty$ and $|f_k|\leq\frac{c}{|k|}$, for some constant c depending on f. Therefore,

$$f(T^nx)=\sum_{k\in\mathbb{Z}}e^{2\pi i2^nkx}f_k,$$

which implies that the only Fourier coefficients of $f\circ T^n$ different from zero are the $\{2^nk\}_{k\in\mathbb{Z}}$. Hence,

$$\left|\int_{\mathbb{T}}f\circ T^ng-\int_{\mathbb{T}}f\int_{\mathbb{T}}g\right|=\left|\sum_{k\in\mathbb{Z}}f_kg_{2^nk}-f_0g_0\right|\leq c2^{-n}\sum_{k\in\mathbb{Z}}|f_k|.$$

The previous inequalities imply the exponential decay of correlations for each smooth function. The proof is concluded by a standard approximation argument: given $f,\, g\in L^2(X,\,d\mu)$, for each $\varepsilon>0$ exists $f_\varepsilon,\, g_\varepsilon\in\mathcal{C}^{(1)}(X)$: $\|f-f_\varepsilon\|_2<\varepsilon$ and $\|g-g_\varepsilon\|_2<\varepsilon$. Thus,

$$\left|\int_{\mathbb{T}}f\circ T^ng-\int_{\mathbb{T}}f\int_{\mathbb{T}}g\right|\leq\left|\int_{\mathbb{T}}f_\varepsilon\circ T^ng_\varepsilon-\int_{\mathbb{T}}f_\varepsilon\int_{\mathbb{T}}g_\varepsilon\right|+2(\|f\|_2+\|g\|_2)\varepsilon,$$

which yields the result by choosing first ε small and then n sufficiently large.

1.8 Stronger statistical properties

One very fruitful idea in the realm of measurable dynamical systems is the idea of *entropy*. In some sense the entropy measure the complexity of the motions from a measure theoretical point of view.

To define it one starts by considering a partition of the space into measurable sets $\xi:=\{A_1,\dots A_n\}$ and defines[36]

$$H_\mu(\xi)-\sum_i\mu(A_i)\log\mu(A_i).$$

[36]The case of a countable partition, or even an uncountable partition, can be handled and it is very relevant, but outside the aims of this book, see [73] for a complete treatment of the subject.

Given two partitions $\xi = \{A_i\}, \eta = \{B_j\}$ we define $\xi \vee \eta := \{A_i \cap B_j\}$. Let then be

$$\xi^T_{-n} := \xi \vee T^{-1}(\xi) \vee \cdots \vee T^{-n+1}(\xi).$$

It is then possible to prove that the sequence $H_\mu(\xi^T_{-n}$ is sub-additive, hence the limit

$$h_\mu(T, \xi) := \lim_{n\to\infty} \frac{1}{n} H_\mu(\xi^T_{-n}$$

exists.

Definition 1.8.1 *The entropy of T with respect to μ is defined as*

$$h_\mu(T) := \sup\{h_\mu(T, \xi) \mid H(\xi) < \infty\}$$

Clearly if a system has positive metric entropy this means that the motion has a high complexity and it is very far from regular. One of the main property of entropy is that it is a metric invariant, that is if two systems are metrically conjugate (see the following), then they have the same metric entropy.

Even more extreme form statistical behaviors are possible, to present them we need to introduce the idea of equivalent systems. This is done via the concept of conjugation that we have already seen informally in Example 1.4.1 (logistic map, circle map).

Definition 1.8.2 *Two Dynamical Systems (X_1, T_1, μ_1), (X_2, T_2, μ_2) are (measurably) conjugate if there exists a measurable map $\phi : X_1 \to X_2$ almost everywhere invertible*[37] *such that $\mu_1(A) = \mu(\phi(A))$ and $T_2 \circ \phi = \phi \circ T_1$.*

Clearly, the conjugation is an equivalence relation. Its relevance for the present discussion is that conjugate systems have the same ergodic properties (Problem1.41).[38]

We can now introduce the most extreme form of stochasticity.

Definition 1.8.3 *A dynamical system (X, T, μ) is called Bernoulli if there exists a Bernoulli shift (M, ν, σ) and a measurable isomorphism $\phi : X \to M$ (i.e., a measurable map one one and onto apart from a set of zero measure and with measurable inverse) such that, for each $A \in X$,*

$$\nu(\phi(A)) = \mu(A)$$

and

$$T = \phi^{-1} \circ \sigma \circ \phi.$$

[37] This means that there exists a measurable function $\phi^{-1} : X_2 \to X_1$ such that $\phi \circ \phi^{-1} =$ id μ_2-a.e. and $\phi^{-1} \circ \phi =$ id μ_1-a.e..

[38] Of course the reader can easily imagine other forms of conjugacy, e.g. topological or differential conjugation.

That is a system is Bernoulli if it is isomorphic to a Bernoulli shift. Since we have seen that Bernoulli systems are very stochastic (remind that they can be seen as describing a random event like coin tossing) this is certainly a very strong condition on the systems. In particular it is immediate to see that Bernoulli systems are mixing (Problem1.41).

1.8.1 Examples

1.8.1.a *Dilation*

We will show that such a system is indeed Bernoulli. The map ϕ is obtained by dividing $[0, 1)$ in $[0, \frac{1}{2})$ and $[\frac{1}{2}, 1)$. Then, given $x \in \mathbb{T}$, we define $\phi : \mathbb{T} \to \Sigma_2^+$ by

$$\phi(x)_i = \begin{cases} 1 & \text{if } T^i x \in [0, \frac{1}{2}) \\ 2 & \text{if } T^i x \in [\frac{1}{2}, 1) \end{cases}$$

the reader can check that the map is measurable and that it satisfy the required properties. Note that the above shows that the Bernoulli measure with $p_1 = p_2 = \frac{1}{2}$ is nothing else than Lebesgue measure viewed on the numbers written in basis two. This may explain why we had to be so careful in the construction of the Bernoulli measure.

1.8.1.b *Baker*

Let us define ϕ^{-1}; for each $\sigma \in \Sigma_2$

$$x = \sum_{i=0}^{\infty} \frac{\sigma_{-i}}{2^{i+1}},$$
$$y = \sum_{i=1}^{\infty} \frac{\sigma_i}{2^i}.$$

Again the rest is left to the reader.

1.8.1.c *Forced Pendulum*

In the introduction we have seen that there exists a square Q with stable and unstable sides such that, calling T the map introduced by the flow at a proper time, $TQ \cap Q \supset Q_0^u \cup Q_1^u$. Where Q_i^u are rectangles that go from one stable side of Q to the other and, in analogy, $T^{-1}Q \cap Q \supset Q_0^s \cup Q_1^s$.

We can use this fact to code the dynamics similarly to what we have done for the Backer map. Namely, given the set $\Lambda = \bigcap_{n\in\mathbb{Z}} T^n Q$ (this set it is non empty–see Example 1.4.1–Horseshoe) and $\phi : \Lambda \to \Sigma_2$ define by

$$[\phi(x)]_k = \begin{cases} i \in \{0,1\} & \text{if } k \geq 0 \text{ and } T^k x \in Q_i^u \\ i \in \{0,1\} & \text{if } k < 0 \text{ and } T^k x \in Q_i^s. \end{cases}$$

It is easy to verify that ϕ is onto and that it is a.e. invertible. It remains to specify the measure on the Horseshoe, we can just pull back any invariant measure on the shift and we will get an invariant measure on the set Λ.

Let us conclude with a final remark on the physical relevance of the concept just introduced. As we mentioned, if f is an observable, then its ergodic average represents the result of an observation over a very long time (the time scale being determined by the mixing properties of the system). Yet, in reality, it may happen that we look for too short a time or, after studying a certain quantity, we can get a grant to buy the needed apparatus to perform more precise measurements. What would we see in such a case? Clearly, we would not see a constant, even for an ergodic system, and we would interpret the non constant part as fluctuations. In many cases it may happen that this fluctuations have a very special nature: they are Gaussian. In such a case we say that the system satisfies the Central Limit Theorem (CLT). Let us be more precise: define $S_n f := \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} f \circ T^i$.

Definition 1.8.4 *Given a Dynamical System (X, T, μ) and a class of observables $\mathcal{A} \subset L^2(X, \mu)$ we say that the class $\mathcal{A}$ satisfies the CLT if $\forall f \in \mathcal{A}$, $\mu(f) = 0$,*

$$\lim_{n\to\infty} \mu(\{x \mid S_n f \geq t\}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{x^2}{2\sigma^2}} dx,$$

where (the variance) σ is defined by $\sigma^2 = \mu(f) + 2\sum_{i=1}^{\infty} \mu(f \circ T^i f)$.[39]

The relevance of the above theorem is the following: if the system is ergodic and satisfies the CLT, then $\frac{1}{n}\sum_{i=0}^{n-1} f \circ T^i - \mu(f) = \mathcal{O}(\frac{1}{\sqrt{n}})$, we have thus the precise scale on which the fluctuations should appear.

In this book we will be mainly interested in the question of how to establish if a given system is ergodic or not.

Unfortunately, neither ergodicity is a typical property of dynamical systems, nor is regular motion. It is a frustrating fact of life that generically dynamical systems present some kind of mixed behavior. Nevertheless, there are some class of systems that are known to be ergodic and among them the hyperbolic systems are probably the most relevant. We will discuss them in the next chapters.

[39] This definition is a bit stricter than usual because, in general, there may be cases in which the fluctuations are Gaussian but the formula for the variance does not hold as written.

Problems

1.1 Given a measurable Dynamical Systems (X, T, μ) verify that, for each measurable set A, if $T(A)$ is measurable, then $\mu(TA) \geq \mu(A)$.

1.2 Set $\mathcal{M}^1(X) = \{\mu \in \mathcal{M} \mid \mu(X) = 1\}$ and $\mathcal{M}^1_T(X) = \mathcal{M}^1(X) \cap \mathcal{M}_T(X)$. Prove that $\mathcal{M}^1_T(X)$ and $\mathcal{M}^1(X)$ are convex sets in $\mathcal{M}(x)$.

1.3 Call $\mathcal{M}^e(X) \subset \mathcal{M}^1(X)$ the set of ergodic probability measures. Show that $\mathcal{M}^e(X)$ consists of the extremal points of $\mathcal{M}_T(X)$. (Hint: Krein-Milman Theorem [37]).

1.4 Prove that the Lebesgue measure is invariant for the rotations on $\mathbb{T}$.

1.5 Consider a rotation by $\omega \in \mathbb{Q}$, find invariant measures different from Lebesgue.

1.6 Prove that the measure μ_h defined in Examples 1.1.1 (Hamiltonian systems) is invariant for the Hamiltonian flow. (Hint: Use the properties of H to deduce $\langle \nabla_{\phi^t x} H,\, d_x\phi^t \nabla_x H\rangle = \|\nabla_x H\|^2$, and thus $d_x\phi^t\nabla_x H = \frac{\|\nabla_x H\|^2}{\|\nabla_{\phi^t x} H\|^2}\nabla_{\phi^t x}H + v$ where $\langle \nabla_{\phi^t x} H,\, v\rangle = 0$. Then study the evolution of an arbitrarily small parallelepiped with one side parallel to $\nabla_x H$–or look at the volume form if you are more mathematically incline–remembering the invariance of the volume with respect to the flow.)

1.7 Given a Poincaré section prove that there exists $c > 0$ such that $\inf \tau_\Sigma \geq c > 0$.

1.8 Show that ν_Σ, defined in (1.2.1) is well defined.(Hint: use the invariance of μ and the fact that, by Problem 1.7, if $A \subset \Sigma$ then $\mu(\phi^{[0,\delta]}(A) \cap \phi^{[n\delta,\,(n+1)\delta]}A) = 0$ provided $(n+1)\delta \leq c$.)

1.9 Show that the return time τ_Σ is finite ν_Σ-a.e. .(Hint: let $\delta < c$ and $\Sigma_\delta := \phi^{[0,\delta]}\Sigma$, apply Poincaré return theorem to Σ_δ.)

1.10 Show that ν_Σ is T_Σ invariant. Verify that, collecting the results of the last exercises, $(\Sigma,\, T_\Sigma,\, \nu_\Sigma)$ is a Dynamical System.

1.11 something about holomorphic dynamics?

1.12 Prove that the Bernoulli measure is invariant with respect to the shift. (Hint: check it on the algebra $\mathcal{A}$ first.)

1.13 Let Σ_p be the set of periodic configurations of Σ. If μ is the Bernoulli measure prove that $\mu(\Sigma_p) = 0$ (Hint: Σ_p is the countable union of zero measure sets.)

1.14 Consider the Bernoulli shift on $\mathbb{Z}$ and define the following equivalence relation: $\sigma \sim \sigma'$ iff there exists $n \in \mathbb{Z}$ such that $T^n\sigma = \sigma'$ (this means that two sequences are equivalent if they belong to the same orbit). Consider now the equivalence classes (the space of orbits) and choose[40] a representative from each class, call the set so obtained K. Show that K cannot be a measurable set. (Hint: show that $K \cap T^n K \subset \Sigma_p$, then by using Problem 1.13 show that if K is measurable $\sum_{i=-\infty}^{\infty} \mu(T^n K) = 1$ which, by the invariance of μ, is impossible).

1.15 Compute the transfer operator for maps of $\mathbb{T}$. (Hint: Use the equivalent definition $\int g\mathcal{L}f dm = \int fg \circ T dm$.) Prove that $\|\mathcal{L}h\|_1 \leq \|h\|_1$.

1.16 Prove the Lasota-York inequality (1.4.4).

1.17 Prove that for each sequence $\{h_n\} \subset \mathcal{C}^{(1)}(\mathbb{T})$, with the property $\sup_{n\in\mathbb{N}} \|h_n'\|_1 + \|h_n\|_1 < \infty$, it is possible to extract a subsequence converging in L^1. (Hint: Consider partitions $\mathcal{P}_n$ of $\mathbb{T}$ in intervals of size $\frac{1}{n}$. Define the conditional expectation $\mathbb{E}(h|\mathcal{P}_n)(x) = \frac{1}{m(I(x))}\int_{I(x)} hdm$, where $x \in I(x) \in \mathcal{P}_n$. Prove that $\|\mathbb{E}(h|\mathcal{P}_n) - h\|_1 \leq \frac{1}{n}\|h'\|_1$. Notice that the functions $\mathbb{E}(h_n|\mathcal{P}_m)$ have only m distinct values and, by using the standard diagonal trick, construct an subsequence h_{n_j} such that all the $\mathbb{E}(h_{n_j}|\mathcal{P}_m)$ are converging. Prove that h_{n_j} converges in L^1.)

1.18 Prove Corollary 1.6.3.(Hint: ??)

1.19 Prove Theorem 1.6.5 (Hint: Note that $\mu(T^{-n}A \cap T^{-m}A) \neq 0$ then, supposing without loss of generality $n < m$, $\mu(A \cap T^{-m+n}A) \neq 0$. Then prove the theorem by absurd remembering that $\mu(X) < \infty$.)

1.20 Let $U \subset X$ of positive measure, consider

$$f_U(x) = \lim \frac{1}{n}\sum_{i=0}^{n-1} \chi_U(T^i x).$$

Show that the limit exists and that the set $A_0 := \{x \in U \mid f_U(x) = 0\}$ has zero measure. (Hint: The existence follows from Birkhoff theorem, it also follows that A_0 is an invariant set, then

$$0 = \int_{A_0} f_U = \int_{A_0} \chi_U = \mu(A_0).$$

)

[40]Attention !!!: here we are using the *Axiom of choice*.

1.21 A topological Dynamical System (X, T) is called *Topologically transitive*, if it has a dense orbit. Show that if $(\mathbb{T}^d, T, m)$ is ergodic and T is continuous, then the system is topologically transitive. (Hint: For each $n \in \mathbb{N}$, $x \in \mathbb{T}^d$ consider $B_{\frac{1}{m}}(x)$–the ball of radius $\frac{1}{m}$ centered at x. By compactness, there are $\{x_i\}$ such that $\cup_i B_{\frac{1}{m}}(x_i) = \mathbb{T}^d$. Let

$$A_{m,i} = \{y \in \mathbb{T}^d \mid T^k y \cap B_{\frac{1}{M}}(X_I) = \emptyset \ \ \forall k \in \mathbb{N}\},$$

clearly $A_{m,i} = \cap_{k\in\mathbb{N}} T^{-k} B_{\frac{1}{m}}(x_i)^c$ has the property $T^{-1}A_{m,i} \supset A_{m,i}$. It follows that $\tilde{A}_{m,i} = \cup_{n\in\mathbb{N}} T^{-n} A_{m,i} \supset A_{m,i}$ is an invariant set and it holds $\mu(\tilde{A}_{m,i} \setminus A_{m,i}) = 0$. Since $A_{m,i}$ it is not of full measure, $\tilde{A}_{m,i}$, and thus $A_{m,i}$, must have zero measure. Hence, $\bar{A}_m = \cap_i A_{m,i}$ has zero measure. This means that $\cup_{m\in\mathbb{N}} \bar{A}_m$ has zero measure. Prove now that, for each $y \in \mathbb{T}^d$, the trajectories that never get closer than $\frac{2}{m}$ to y are contained in $\bar{A}_m$, and thus have measure zero. Hence, almost every point has a dense orbit.)
Extend the result to the case in which X is a compact metric space and μ charges the open sets (that is: if $U \subset X$ is open, then $\mu(U) > 0$).

1.22 Give an example of a system with a dense orbit which it is not ergodic. (Hint: A system with two periodic orbits, and the measure supported on them. Along such lines more complex examples can be readily constructed)

1.23 Give an example of an ergodic system with no dense orbit. (Hint: A non transitive system with a measure supported on a periodic orbit.)

1.24 Give an example of a Dynamical Systems which does not have any invariant probability measure. (Hint: $X = \mathbb{R}^d$, $Tx = x + v$, $v \neq 0$.)

1.25 Show that a Dynamical Systems (X, T, μ) is ergodic if and only if there does not exists any invariant probability measure absolutely continuous with respect to μ, beside μ itself.

1.26 Prove that Birkhoff theorem implies Von Neumann theorem. (Hint: Note that the ergodic average is a contraction in L^∞, an isometry in L^2 and that $L^1 \subset L^2$ (since the measure is finite). Use Lebesgue dominate convergence theorem to prove convergence in L^2 for bounded functions. Use Fatou to show that if $f \in L^2$ then $f^+ \in L^2$ and a $3 - \varepsilon$ argument to conclude).

1.27 Prove that if (X, T, μ) is ergodic, then all $f \in L^1(X, \mu)$ such that $f \circ T = f$ are a.e. constant. Prove also the converse.

1.28 For each measurable set A, let

$$F_{A,n}(x) = \frac{1}{n}\sum_{i=0}^{n-1}\chi_A(T^i x).$$

be the average number of times x visits A in the time n. Show that there exists $F_A = \lim_{n\to\infty} F_{A,n}$ a.e. and prove that, if the system is ergodic, $F_A = \mu(A)$. (Hint: Birkhoff theorem and Theorem 1.6.6).

1.29 Prove Proposition 1.7.2 and Proposition 1.7.3. (Hint: Note that for each measurable set A and $\varepsilon > 0$ there exists $f \in \mathcal{C}^{(0)}(X)$ such that $\mu(|f - \chi_A|) < \varepsilon$ –by Uryshon Lemma and by the regularity of Borel measures. To prove that $\mu(T^{-n}A \cap B) \to \mu(A)\mu(B)$ choose $d\lambda = \mu(B)^{-1}\chi_B d\mu$ and use the invariance of μ to obtain the uniform estimate $\lambda(|f \circ T^n - \chi_A \circ T^n|) \leq \mu(B)^{-1}\mu(|f - \chi_A|)$.)

1.30 Show that the irrational rotations are not mixing.

1.31 Prove that if $f \in \mathcal{C}^{(2)}(\mathbb{T})$, then its Fourier series converges uniformly.[41] (Hint: Remember that $f_n = \frac{1}{2\pi}\int_{\mathbb{T}} e^{2\pi inx} f(x)dx$. Thus

$$f_n = \frac{1}{(2\pi in)^2 2\pi}\int_{\mathbb{T}} e^{2\pi inx} f^{(2)}(x)dx.$$

)

1.32 Let ν be a Borel measure on $Q = [0,1]^2$ such that $\nu(\partial_x f) = 0$ for all $f \in \mathcal{C}^{(1)}_{\text{per}}(Q) = \{f \in \mathcal{C}^{(1)}(Q) \mid f(0,y) = f(1,y)\ \forall\, y \in [0,1]\}$. Prove that there exists a Borel measure ν_1 on $[0,1]$ such that $\nu = m \times \nu_1$. (Hint: The measure ν_1 is nothing else then the marginal with respect to x, that is: for each continuous function $f : [0,1] \to \mathbb{R}$ define $\tilde{f} : Q \to \mathbb{R}$ by $\tilde{f}(x,y) = f(y)$, then $\nu_1(f) = \nu(\tilde{f})$. To prove the statement use Fourier series. If f is smooth enough $f(x,y) = \sum_{k\in\mathbb{Z}} \hat{f}_k(y)e^{2\pi ikx}$ where the Fourier series for f and $\partial_x f$ converge uniformly. Then notice that $0 = \nu(\partial_x e^{2\pi ik\cdot}) = 2\pi ik\nu(e^{2\pi ik\cdot})$ implies $\nu(f) = \nu(\hat{f}_0) = m \times \nu_1(f)$.)

1.33 Prove that is a flow is ergodic (mixing) so is each Poincarè section. Prove that is a map is ergodic so is any suspension on the map. Give an example of a mixing map with a non-mixing suspension (constant ceiling).

[41] This result is far from optimal, see [1] if you want to get deeper in the theory of Fourier series.

1.34 Consider $([0,1],T)$ where

$$T(x) = \frac{1}{x} - \left[\frac{1}{x}\right]$$

($[a]$ is the integer part of a), and

$$\mu(f) = \frac{1}{\ln 2}\int_0^1 f(x)\frac{1}{1+x}dx.$$

Prove that $([0,1],T,\mu)$ is a Dynamical System.[42] (Hint: write $\mu(f \circ T) = \sum_{i=1}^{\infty}\int_{\frac{1}{i+1}}^{\frac{1}{i}} f \circ T(x)\mu(dx)$, change variable and use the identity $\frac{1}{a^2+a} = \frac{1}{a} - \frac{1}{a+1}$ to obtain a series with alternating signs.)

1.35 Prove that for each $x \in \mathbb{Q} \cap [0,1]$ holds $\lim_{n\to\infty} T^n(x) = 0$. (Hint: if $x = \frac{p_0}{q_0}$, $p_0 \le q_0$, then $q_0 = k_1 p_0 + p_1$, with $p_1 < p_0$, and $T(x) = \frac{p_1}{p_0}$. Let $q_1 = p_0$ and go on noticing that $p_{i+1} < p_i$.)[43]

1.36 In view of the two previous exercises explain why it is problematic to study the statistical properties of the Gauss map on a computer.(Hint: The computer uses only rational numbers. It is quite amazing that these type of pathologies arises rather rarely in the numerical studies carried out by so many theoretical physicist.)

1.37 Prove that any infinite continuous fraction of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

with $a_i \in \mathbb{N}$ defines a real number. (Hint: Note that if you fix the first n $\{a_i\}$, this corresponds to specifying which elements of the partition

[42] The above map is often called *Gauss map* since to him is due the discovery of the above invariant measure.

[43] This is nothing else that the *Euclidean algorithm* to find the greatest common divisor of two integers [38] Elements, Book VII, Proposition 1 and 2. The greatest common divisor is clearly the last non-zero p_i. This provides also a remarkable way of writing rational numbers: *continuous fractions*

$$\frac{p_0}{q_0} = \cfrac{1}{k_1 + \cfrac{1}{k_2 + \ddots + \cfrac{1}{k_n}}}.$$

$\{[\frac{1}{i+1}, \frac{1}{i}]\}$ are visited by the trajectory of $\{T^i x\}$. By the expansivity of the map readily follows that x must belong to an interval of size λ^{-n} for some $\lambda > 1$.)

1.38 Prove that, for each $a \in \mathbb{N}$,

$$x = \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \ddots}}} = \frac{-a + \sqrt{a^2 + 4}}{2}.$$

(Hint: Note that $T(x) = x$.) Study periodic continuous fractions of period two.

1.39 Choose a number in $[0, 1]$ at random according to Lebesgue distribution. Assuming that the Gauss map is mixing (which it is, see ???) compute the average percentage of numbers larger than n in the associated continuous fraction. (Hint: Define $f(x) = [x^{-1}]$, then the entries of the continuous fraction of x are $\{f \circ T^i\}$. The quantity one must compute is then $m(\lim_{k\to\infty} \frac{i}{k} \sum_{i=0}^{k-1} \chi_{[n,\infty)} \circ f \circ T^i) = \mu([n, \infty))$.)

1.40 Let (X_0, T_0, μ_0) be a Dynamical System and $\phi : X_0 \to X_1$ an homeomorphism. Define $T_1 := \phi \circ T_0 \circ \phi^{-1}$ and $\mu_1(f) = \mu_0(f \circ \phi^{-1})$. Prove that (X_1, T_1, μ_1) is a Dynamical System.

1.41 Let (X_0, T_0, μ_0) be measurably conjugate to (X_1, T_1, μ_1), then show that one of the two is ergodic if and only if the other is ergodic. Prove the same for mixing.

1.42 Show that the systems described in Examples ??–strange attractor and horseshoe, are Bernoulli.

1.43 Prove Lebesgue density theorem: for each measurable set A, $m(A) > 0$, there exists $x \in A$ such that for each $\varepsilon > 0$ exists $\delta > 0$ such that $m(A \cap [x - \delta, x + \delta]) > (1 - \varepsilon)2\delta$. (Hint: we have seen in Examples 1.8.1-Dilations that Lebesgue measure is equivalent to Bernoulli measure and that the cylinder correspond to intervals. It then suffices to prove the theorem for the latter. Let $A \subset \Sigma^+$ such that $\mu(A) > 0$, then, for each $\varepsilon > 0$,there exists $A_\varepsilon \in \mathcal{A}$ such that $A_\varepsilon \supset A$ and $\mu(A_\varepsilon) - \mu(A) < \varepsilon\mu(A)$. Since $A_\varepsilon \in \mathcal{A}$, it exists $n_\varepsilon \in \mathbb{N}$ such that it is possible to decide if $\sigma \in A_\varepsilon$ only by looking at $\{\sigma_1, \ldots, \sigma_{n_\varepsilon}\}$. Consider all the cylinders $\mathcal{I}\{A(0; k_1, \ldots, k_{n_\varepsilon})\}$, clearly if $I \in \mathcal{I}$ then $I \cap A_\varepsilon$ is either I or $\emptyset$. Let $\mathcal{I}_+ = \{I \in \mathcal{I} \mid I \cap A_\varepsilon = I\}$ and $\mathcal{I}_+ = \{I \in \mathcal{I} \mid I \cap A_\varepsilon = \emptyset\}$. Now suppose that for each $I \in \mathcal{I}_+$ holds $\mu(I \cap A) \leq (1-\varepsilon)\mu(I)$ then

$$\mu(A) = \sum_{I \in \mathcal{I}_+} \mu(A \cap I) \leq (1-\varepsilon)\mu(A_\varepsilon) < \mu(A),$$

which is absurd. Thus there must exists $I \in \mathcal{I}_+$: $\mu(A \cap I) > (1-\varepsilon)\mu(I)$.)

Notes

Give references for SRB and Gibbs, mention entropy, K-systems. diffeo with holes, strange attractors, history of the field

. . . .