IMPLICIT FUNCTION THEOREM
(A QUANTITATIVE VERSION)

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1. The theorem

Let $n, m \in \mathbb{N}$ and $F \in \mathcal{C}^1(\mathbb{R}^{m+n}, \mathbb{R}^m)$ and let $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^m$ such that $F(x_0, y_0) = 0$. For each $\delta > 0$ let $V_\delta = \{(x, y) \in \mathbb{R}^{n+m} \;:\; \|x - x_0\| \leq \delta, \|y - y_0\| \leq \delta\}$.

thm:implicit-func **Theorem 1.1.** *Assume that $\partial_x F(x_0, y_0)$ is invertible and choose $\delta > 0$ such that $\sup_{(x,y)\in V_\delta} \|\mathbb{1} - [\partial_x F(x_0, y_0)]^{-1}\partial_x F(x, y)\| \leq \frac{1}{2}\}$. Let $B_\delta = \sup_{(x,y)\in V_\delta} \|\partial_y F(x, y)\|$ and $M = \|\partial_x F(x_0, y_0)^{-1}\|$. Set $\delta_1 = (2MB_\delta)^{-1}\delta$ and $\Lambda_{\delta_1} := \{y \in \mathbb{R}^m \;:\; \|y - y\| < \delta_1\}$. Then there exists $g \in \mathcal{C}^1(\Lambda_{\delta_1}, \mathbb{R}^m)$ such that all the solutions of the equation $F(x, y) = 0$ in the set $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \;:\; \|y - y_0\| < \delta_1, \|x - x_0\| < \delta\}$ are given by $(g(y), y)$. In addition,*

$$\partial_y g(y) = -(\partial_x F(g(y), y))^{-1}\partial_y F(g(y), y).$$

We will do the proof in several steps.

1.1. Existence of the solution.

Let $A(x, y) = \partial_x F(x, y)$, $M = \|A(x_0, y_0)^{-1}\|$.

We want to solve the equation $F(x, y) = 0$, various approaches are possible. Here we will use a simplification of Newton method, made possible by the fact that we already know a good approximation of the zero we are looking for. Let y be such that $\|y - y_0\| < \delta_1 \leq \delta$. Consider $U_\delta = \{x \in \mathbb{R}^n \;:\; \|x - x_0\| \leq \delta\}$ and the function $\Theta_y : U_\delta \to \mathbb{R}^n$ defined by[1]

eq:Newton-imp
$$\Theta_y(x) = x - A(x_0, y_0)^{-1}F(x, y). \tag{1.1}$$

Exercise 1. *Prove that, for $x \in U(y)$, $F(x, y) = 0$ is equivalent to $x = \Theta_y(x)$.*

Next,

$$\|\Theta_y(x_0) - \Theta_{y_0}(x_0)\| \leq M\|F(x_0, y)\| \leq MB_\delta\delta_1.$$

In addition, $\|\partial_x\Theta_y\| = \|\mathbb{1} - A(x_0, y_0)^{-1}A(x, y)\| \leq \frac{1}{2}$. Thus,

$$\|\Theta_y(x) - x_0\| \leq \frac{1}{2}\|x - x_0\| + \|\Theta_y(x_0) - x_0\| \leq \frac{1}{2}\|x - x_0\| + MB_\delta\delta_1 \leq \delta.$$

The idea is then to define recursively the sequence of functions $g_n \in \mathcal{C}^1(\Lambda_{\delta_1}, \mathbb{R}^n)$ such that $g_0(y) = x_* \in U_\delta$ and

$$g_{n+1}(x) = \Theta_y(g_n(y)).$$

[1] The Newton method would consist in finding a fixed point for the function $x - A(x, y)^{-1}F(x, y)$. This gives a much faster convergence and hence is preferable in applications, yet here it would make the estimates a bit more complicated.

Note that, if $g \in \mathcal{C}^1(\Lambda_{\delta_1}, B_\delta)$, then the same holds for g_{n+1}. So the sequence has such a property since it holds for g_0. Moreover, given any two functions $\phi, \varphi \in \mathcal{C}^1(\Lambda_{\delta_1}, B_\delta)$ we have

$$\|\Theta_y(\phi(y)) - \Theta_y(\varphi(y))\| \leq \int_0^1 \left\| \frac{d}{dt}\Theta_y((1-t)\phi(y) + t\varphi(y)) \right\| \leq \frac{1}{2}\|\phi - \varphi\|_{\mathcal{C}^0(\Lambda_{\delta_1}, B_\delta)}.$$

Accordingly,

$$\begin{aligned}\|g_{n+m} - g_n\|_{\mathcal{C}^0(\Lambda_{\delta_1}, B_\delta)} &\leq 2^{-1}\|g_{n+m-1} - g_{n-1}\|_{\mathcal{C}^0(\Lambda_{\delta_1}, B_\delta)} \\ &\leq 2^{-n}\|g_m - g_0\|_{\mathcal{C}^0(\Lambda_{\delta_1}, B_\delta)} \leq 2^{-n+1}\delta.\end{aligned}$$

Accordingly, $\{g_n\}$ is a Cauchy sequence in the sup norm. It follows that is has a limit $g = \lim_{n\to\infty} g_n$. As the limit is uniform, it follows that $g \in \mathcal{C}^0(\Lambda_{\delta_1}, B_\delta)$. We have so obtained a function $g : \{y \;:\; \|y - y_0\| \leq \delta_1\} = \Lambda_{\delta_1} \to \mathbb{R}^n$ such that $F(g(y), y) = 0$.

Exercise 2. *Show that the limit does not depend on the initial choice x_*.*

To see that all the solutions of $F(y, x) = 0$ have such a form suppose that $F(y_*, x_*) = 0$, then choose $g_0(y) = x_*$ as initial condition for the sequence. Note that $g_1(y_*) = x_* + A(x_0, y_0)^{-1}F(x_*, y_*) = x_*$ and, by induction, $g_n(y_*) = x_*$. Hence $g(y_*) = x_*$, hence the zero belongs to the set $\{(g(y), y)\}$ as announced.

It remains the question of the regularity.

1.2. Lipschitz continuity and Differentiability.

Let $y, y' \in \Lambda_{\delta_1}$. By (1.1)

$$\|g(y) - g(y')\| \leq \frac{1}{2}\|g(y) - g(y')\| + MB_\delta|y - y'|$$

This yields the Lipschitz continuity of the function g. To obtain the differentiability we note that, by the differentiability of F and the above Lipschitz continuity of g, for $h \in \mathbb{R}^m$ small enough,

$$\|F(g(y+h), y+h) - F(g(y), y) + \partial_x F[g(y+h) - g(y)] + \partial_y F h\| = o(\|h\|).$$

Since $F(g(y+h), y+h) = F(g(y), y) = 0$, we have that

$$\lim_{h\to 0} \|h\|^{-1}\|g(y+h) - g(y) + [\partial_x F]^{-1}\partial_y F h\| = 0$$

which concludes the proof of the Theorem, the continuity of the derivative being obvious by the obtained explicit formula.

2. Generalization

First of all note that the above theorem implies the inverse function theorem. Indeed if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a function such that $\partial_x f$ is invertible at some point x_0, then one can consider the function $F(x, y) = f(x) - y$. Applying the implicit function theorem to the equation $F(x, y) = 0$ it follows that $y = f(x)$ are the only solution, hence the function is locally invertible.

Exercise 3. *Make the above precise and prove that if $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ and $\det[\partial_x f(x_0)] \neq 1$, then there exists $\delta > 0$ such that, setting $B(x_0, \delta) = \{x \in \mathbb{R}^n \;:\; \|x - x_0\| < \delta\}$, we have that f, restricted to $B(x_0, \delta)$ is invertible and*

$$\partial_y f^{-1}(y) = \left[(\partial_x f) \circ f^{-1}(y)\right]^{-1}.$$

The above theorem can be generalized in several ways, e.g.

prob:implicit-cr **Exercise 4.** *Show that if F in Theorem* thm:implicit-func *1.1 is $\mathcal{C}^r$, then also g is $\mathcal{C}^r$.*

As I mentioned the statement of Theorem thm:implicit-func 1.1 is suitable for quantitative applications.

prob:implicit-quant **Exercise 5.** *Suppose that in Theorem* thm:implicit-func *1.1 we have $F \in \mathcal{C}^2$, then show that we can chose*

$$\delta = [2\|D\partial_x F\|_\infty]^{-1}.$$

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