ADDENDUM TO
"EXISTENTIALLY COMPLETE CLOSURE ALGEBRAS"

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SOMMARIO. Si mostra, usando i metodi di [2], che la teoria universale delle algebre di chiusura è decidibile.

A Closure algebra is a Boolean algebra together with an operator K satisfying the formal properties of closure in a topological space. McKinsey and Tarski [1] announced that the (first-order) theory of closure algebras is undecidable. In contrast, the proof of [2, Theorem 1] shows:

Theorem. The set of universal consequences of the theory of closure algebras is decidable.

We now show how the arguments in [2] can be used to prove the theorem.

A universal sentence $\psi$ is a consequence of the theory of closure algebras iff there is no closure algebra which is a model of the existential sentence $\neg \psi$, that is, iff no extension of the two-element closure algebra is a model of $\neg \psi$.

In [2] we dealt with a more general problem, that is, we considered an arbitrary closure algebra $A$ and an existential formula $\varphi$ with constants from $A$, and we found necessary and sufficient conditions for the existence of an extension of $A$ satisfying $\varphi$ (we supposed disjunctions were factored out, but this is no loss of generality). First, we showed how to reduce the existential formula $\varphi$ to an equivalent form which involved only a polynomial $P$ with certain coefficients $a_i$ (defined in terms of the constants from $A$); then we constructed increasing sequences $w_{sd}(m,\hat{a})$, such that $w_{sd}(0,\hat{a})=a_i$, and such that an extension as above exists iff for all $m$ the $w_{sd}(m,\hat{a})'s$ satisfy a certain property.

By our first remark here, we need to consider the particular case when $A$ is the two-element closure algebra; in this case each $a_i$ and each $w_{sd}(m,\hat{a})$ is either 0 or 1; since each $w_{sd}(m,\hat{a})$ is increasing, as a function of $m$, it is eventually constant; moreover, the indexes $s$ and $t$ vary on a finite set, so that it is enough to consider a finite number of $m$'s (actually, just one). The procedure described in [2] is effective, and, by the preceding period, in this particular case, terminates in a finite number of steps, so that we have an effective way to decide whether a universal sentence is a consequence of the theory of closure algebras.

REMARKS. (i) A different proof can be found in [1]. We suspect that the algorithm furnished by the present proof is more efficient, but we have not checked this.

(ii) The constructions in [2] show that if a universal sentence fails in some closure algebra then it fails in a closure algebra whose underlying Boolean algebra is atomic and complete; that is, the sentence fails in a topological space (an atomic and complete Boolean algebra is isomorphic to the set of subsets of a set; we get a topological space by taking, as closed subsets, those $x's$ satisfying $Kx=x$).

(iii) The same arguments show that set of universal consequences of the theory $V_a$ is decidable, too (cf. [2, Remark 2]).

We discovered the above arguments shortly after [2] had been printed, but only now we have written down the details.

References.


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