# The non-equivalence of two definitions of selective pseudocompactness

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Abstract. We show that two possible definitions of selective pseudocompactness are not equivalent in the class of $T_1$ topological spaces.

Let $X$ be a topological space. The following conditions on $X$ have been considered in the literature.

(A) For every sequence $(U_n)_{n\in\omega}$ of pairwise disjoint non-empty open subsets of $X$, there is a sequence $(x_n)_{n\in\omega}$ of elements of $X$ such that $x_n \in U_n$, for every $n \in \omega$, and the set $\{x_n \mid n \in \omega\}$ is not closed in $X$.

(B) For every sequence $(U_n)_{n\in\omega}$ of non-empty open subsets of $X$, there is a sequence $(x_n)_{n\in\omega}$ of elements of $X$ such that $x_n \in U_n$, for every $n \in \omega$, and there is $y \in X$ such that, for every open neighborhood $V$ of $y$, the set $\{n \in \omega \mid x_n \in V\}$ is infinite.

The above conditions are, respectively, Conditions (i) and (iii) in Theorem 2.1 in Dorantes-Aldama and Shakhmatov [2], to which we refer for motivations, history, comments on terminology and further references. Other works on the subject include Angoa, Ortiz-Castillo and Tamariz-Mascarúa [1], García-Ferreira and Ortiz-Castillo [3], García-Ferreira and Tomita [4] and possibly others. By the way, let us mention that Conditions (iii) and (iv) in Theorem 2.1 in [2] are equivalent for every topological space (no separation axiom needed), since the proof that (iii) implies (iv) in [2] uses no separation axiom, and (iv) implies (iii) trivially.

It is proved in [2] that (A) and (B) are equivalent in the class of Tychonoff spaces. We provide a counterexample showing that (A) does not imply (B) in the class of $T_1$ spaces.

Let $X = \omega \times \omega$ be the product of two copies of the set of natural numbers and, for every $n \in \omega$, let $X_n = \{n\} \times \omega$. Let $\tau = \{\emptyset\} \cup \{A \subseteq X \mid X_n \setminus A$ is finite, for all but finitely many $n \in \omega\}$. In other words, a nonempty subset $A$ of $X$ belongs to $\tau$ if and only if $A \cap X_n$ is cofinite in $X_n$, for all but finitely many $n \in \omega$.

**Theorem 1.** *$(X, \tau)$ is a $T_1$ topological space which satisfies (A) but does not satisfy (B).*

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*Proof.* It is obvious that $(X,\tau)$ is a $T_1$ topological space. Moreover, $(X,\tau)$ vacuously satisfies (A), since any two nonempty sets in $\tau$ have nonempty intersection.

It remains to show that $(X,\tau)$ does not satisfy (B). For each $n \in \omega$, consider the family $(U_n)_{n\in\omega}$, where $U_n = [n,\infty) \times \omega = \bigcup_{m\geq n} X_m$. Each $U_n$ belongs to $\tau$. Suppose that $(x_n)_{n\in\omega}$ is a sequence of elements of $X$ such that $x_n \in U_n$, for $n \in \omega$. For every $n \in \omega$, $\{m \in \omega \mid x_m \in X_n\}$ is finite, since if $m > n$, then $U_m \cap X_n = \emptyset$, hence $x_m \notin X_n$, so each $X_n$ can contain only finitely many $x_m$'s. For every $n \in \omega$, let $\bar{m}(n) = \sup\{m \in \omega \mid x_m \in X_n\}$, where we let $\sup\emptyset = 0$. Furthermore, put $Y_n = \{n\} \times (\bar{m}(n),\infty)$ and $Y = \bigcup_{n\in\omega} Y_n$. By construction, $Y \cap \{x_n \mid n \in \omega\} = \emptyset$. Hence, for every $y \in X$, $V = \{y\} \cup Y$ is an open neighborhood of $y$ such that $V \cap \{x_n \mid n \in \omega\} \subseteq \{y\}$. Hence (B) fails. □

The ideas from [1, 2, 3, 4] of introducing selective pseudocompactness properties (sometimes in different terminology) can be extended to the general framework presented in [5, 6]. If $X$ is a topological space, $I$ is a set, $(Y_i)_{i\in I}$ is an $I$-indexed sequence of subsets of $X$ and $F$ is a filter over $I$, a point $x \in X$ is said to be an *$F$-limit point* of the sequence $(Y_i)_{i\in I}$ if $\{i \in I \mid Y_i \cap U \neq \emptyset\} \in F$, for every open neighborhood $U$ of $x$. As usual, if $Y_i = \{x_i\}$ is a singleton, for each $i \in I$, we simply write that some $x$ as above is a limit point of $(x_i)_{i\in I}$. If $\mathcal{P}$ is a family of filters over $I$, we defined in [6, Section 6] a space $X$ to be *sequencewise $\mathcal{P}$-pseudocompact* if, for every $I$-indexed sequence $(U_i)_{i\in I}$ of nonempty open subsets of $X$, there is $F \in \mathcal{P}$ such that $(U_i)_{i\in I}$ has an $F$-limit point. The next definition presents the "selective" stronger variant.

**Definition 2.** Let $\mathcal{P}$ be a family of filters over some set $I$. A topological space $X$ is *selectively sequencewise $\mathcal{P}$-pseudocompact* if, for every $I$-indexed sequence $(U_i)_{i\in I}$ of nonempty open subsets of $X$, there is a sequence $(x_i)_{i\in I}$ of elements of $X$ such that $x_i \in U_i$, for every $i \in I$, and there is $F \in \mathcal{P}$ such that $(x_i)_{i\in I}$ has an $F$-limit point.

The arguments from [5, 6] show that property (B), as well as many of the properties discussed in [1, 2, 3, 4] can be obtained as special cases of Definition 2.

We have not yet performed a completely accurate search in order to check whether some of the results presented here are already known. Credits for already known results should go to the original discoverers.

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