The non-equivalence of two definitions of selective pseudocompactness

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Abstract. We show that two possible definitions of selective pseudocompactness are not equivalent in the class of $T_1$ topological spaces.

Let $X$ be a topological space. The following conditions on $X$ have been considered in the literature.

(A) For every sequence $(U_n)_{n \in \omega}$ of pairwise disjoint non-empty open subsets of $X$, there is a sequence $(x_n)_{n \in \omega}$ of elements of $X$ such that $x_n \in U_n$, for every $n \in \omega$, and the set $\{x_n \mid n \in \omega\}$ is not closed in $X$.

(B) For every sequence $(U_n)_{n \in \omega}$ of non-empty open subsets of $X$, there is a sequence $(x_n)_{n \in \omega}$ of elements of $X$ such that $x_n \in U_n$, for every $n \in \omega$, and there is $y \in X$ such that, for every open neighborhood $V$ of $y$, the set $\{n \in \omega \mid x_n \in V\}$ is infinite.

The above conditions are, respectively, Conditions (i) and (iii) in Theorem 2.1 in Dorantes-Aldama and Shakhmatov [2], to which we refer for motivations, history, comments on terminology and further references. Other works on the subject include Angoa, Ortiz-Castillo and Tamariz-Mascarúa [1], García-Ferreira and Ortiz-Castillo [3], García-Ferreira and Tomita [4] and possibly others. By the way, let us mention that Conditions (iii) and (iv) in Theorem 2.1 in [2] are equivalent for every topological space (no separation axiom needed), since the proof that (iii) implies (iv) in [2] uses no separation axiom, and (iv) implies (iii) trivially.

It is proved in [2] that (A) and (B) are equivalent in the class of Tychonoff spaces. We provide a counterexample showing that (A) does not imply (B) in the class of $T_1$ spaces.

Let $X = \omega \times \omega$ be the product of two copies of the set of natural numbers and, for every $n \in \omega$, let $X_n = \{n\} \times \omega$. Let $\tau = \{\emptyset\} \cup \{A \subseteq X \mid X_n \setminus A \text{ is finite}, \text{for all but finitely many } n \in \omega\}$. In other words, a nonempty subset $A$ of $X$ belongs to $\tau$ if and only if $A \cap X_n$ is cofinite in $X_n$, for all but finitely many $n \in \omega$.

Theorem 1. $(X, \tau)$ is a $T_1$ topological space which satisfies (A) but does not satisfy (B).
Proof. It is obvious that \((X, \tau)\) is a \(T_1\) topological space. Moreover, \((X, \tau)\) vacuously satisfies (A), since any two nonempty sets in \(\tau\) have nonempty intersection.

It remains to show that \((X, \tau)\) does not satisfy (B). For each \(n \in \omega\), consider the family \((U_n)_{n \in \omega}\), where \(U_n = [n, \infty) \times \omega = \bigcup_{m \geq n} X_m\). Each \(U_n\) belongs to \(\tau\). Suppose that \((x_n)_{n \in \omega}\) is a sequence of elements of \(X\) such that \(x_n \in U_n\), for \(n \in \omega\). For every \(n \in \omega\), \(\{m \in \omega \mid x_m \in X_n\}\) is finite, since if \(m > n\), then \(U_m \cap X_n = \emptyset\), hence \(x_m \notin X_n\), so each \(X_n\) can contain only finitely many \(x_m\)'s. For every \(n \in \omega\), let \(m(n) = \sup\{m \in \omega \mid x_m \in X_n\}\), where we let \(\sup \emptyset = 0\). Furthermore, put \(Y_n = \{n\} \times (m(n), \infty)\) and \(Y = \bigcup_{n \in \omega} Y_n\). By construction, \(Y \cap \{x_n \mid n \in \omega\} = \emptyset\). Hence, for every \(y \in X\), \(V = \{y\} \cup Y\) is an open neighborhood of \(y\) such that \(V \cap \{x_n \mid n \in \omega\} \subseteq \{y\}\). Hence (B) fails.

\[\Box\]

The ideas from [1, 2, 3, 4] of introducing selective pseudocompactness properties (sometimes in different terminology) can be extended to the general framework presented in [5, 6]. If \(X\) is a topological space, \(I\) is a set, \((Y_i)_{i \in I}\) is an \(I\)-indexed sequence of subsets of \(X\) and \(F\) is a filter over \(I\), a point \(x \in X\) is said to be an \(F\)-limit point of the sequence \((Y_i)_{i \in I}\) if \(\{i \in I \mid Y_i \cap U \neq \emptyset\} \in F\), for every open neighborhood \(U\) of \(x\). As usual, if \(Y_i = \{x_i\}\) is a singleton, for each \(i \in I\), we simply write that some \(x\) as above is a limit point of \((x_i)_{i \in I}\).

If \(\mathcal{P}\) is a family of filters over \(I\), we defined in [6, Section 6] a space \(X\) to be sequencewise \(\mathcal{P}\)-pseudocompact if, for every \(I\)-indexed sequence \((U_i)_{i \in I}\) of nonempty open subsets of \(X\), there is \(F \in \mathcal{P}\) such that \((U_i)_{i \in I}\) has an \(F\)-limit point. The next definition presents the “selective” stronger variant.

**Definition 2.** Let \(\mathcal{P}\) be a family of filters over some set \(I\). A topological space \(X\) is selectively sequencewise \(\mathcal{P}\)-pseudocompact if, for every \(I\)-indexed sequence \((x_i)_{i \in I}\) of elements of \(X\) such that \(x_i \in U_i\), for every \(i \in I\), and there is \(F \in \mathcal{P}\) such that \((x_i)_{i \in I}\) has an \(F\)-limit point.

The arguments from [5, 6] show that property (B), as well as many of the properties discussed in [1, 2, 3, 4] can be obtained as special cases of Definition 2.

We have not yet performed a completely accurate search in order to check whether some of the results presented here are already known. Credits for already known results should go to the original discoverers.

**References**


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