REPRESENTABLE TOLERANCES IN VARIETIES

PAOLO LIPPARINI

Abstract. We discuss two possible ways of representing tolerances: first, as a homomorphic image of some congruence; second, as the relational composition of some compatible relation with its converse. The second way is independent from the variety under consideration, while the first way is variety-dependent. The relationships between these two kinds of representations are clarified.

As an application, we show that any tolerance on some lattice $\mathfrak{L}$ is the image of some congruence on a subalgebra of $\mathfrak{L} \times \mathfrak{L}$. This is related to recent results by G. Czédli, G. Grätzer and E. W. Kiss.

1. Introduction

A *tolerance* on some algebra is a binary, compatible, symmetric and reflexive relations. Thus a congruence is just a transitive tolerance. It is quite surprising that the study of tolerances, apart from intrinsic interest, has revealed to be essential in the study of congruences. Indeed, present-day research shows that tolerances are becoming increasingly important even in many other at first look seemingly unrelated contexts. We briefly list some examples, with absolutely no claim to exhaustiveness.

1.1. **Some history.** First of all, A. Day's celebrated Maltsev characterization [13] of congruence modularity uses tolerances in its proof.

As another classical example, it follows easily from Werner [30], and it is stated explicitly, e. g., in Chajda [1], that a variety is congruence permutable if and only if all of its tolerances are congruences. A similar characterization of n-permutable varieties follows from Hagemann and Mitschke [18].

The property that in permutable varieties every tolerance is a congruence and, more generally, that in such varieties every compatible

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reflexive relation is a congruence has been exploited to the greatest extent in J. D. H. Smith's commutator theory [28] for congruence permutable varieties. Smith's theory has been extended to congruence modular varieties by J. Hagemann and C. Herrmann, with further significant contributions by, among many others, R. Freese, H.-P. Gumm, E. Hrushovski, E. W. Kiss, H. Lakser, R. McKenzie, W. Taylor, and S. Tschantz. See the survey books [15, 17] for references. In particular, the Shifting Principle, a tolerance identity remnant of the method of proof of A. Day's characterization of congruence modularity, played an essential role in H.-P. Gumm's geometrical development of commutator theory.

Tolerances played only an occasional role in D. Hobby and R. McKenzie's tame congruence theory [20] for locally finite varieties (e. g., [20, Definition 5.23 and Theorem 5.27]), but they have been extensively used as an auxiliary but fundamental tool in K. Kearnes and E. W. Kiss's subsequent deep study [24] of the shape of congruence lattices of arbitrary varieties.

Particularly interesting applications of tolerances have been discovered in a series of works by G. Czédli and E. K. Horváth [7, 8, 9], with further related results by the above authors and K. Kearnes, P. Lipparini and S. Radeleczki [10, 11, 23, 25, 27]. Most of the theory is summed up and perfected in [10], where a tolerance identity is used in a threefold way to get short and simple proofs for classical results, together with significant improvements. First, [9] showed that all congruence lattice identities implying modularity are characterized by a Maltsev condition (a much longer proof of just particular cases appeared in [15, Chapter XIII]); however, in [10] the "best" Maltsev condition is obtained. Second, a result by R. Freese and B. Jónsson [14] is proved in [10] in a simple way, and strengthened to the effect that congruence modular varieties satisfy M. Haiman's [19] higher Arguesian identities. Finally, H.-P. Gumm's intuition [17] (see also Tschantz [29]) that "congruence modularity is permutability composed with distributivity" is given in [10] the up to now strongest known formulation.

1.2. **Aims of the present paper.** Turning to arguments more directly connected with the present paper, a classical way of representing tolerances (e. g., Grätzer and Wenzel [16, Section 2], and Ježek [22, p. 100]) has recently received renewed attention. See [2, 3, 6, 12].

If Ψ is a binary relation, we sometimes shall use the shorthand $a \mathrel{\Psi} b$ to mean that $(a, b) \in \Psi$.

Definition 1.1. Suppose that $\mathfrak{A}$, $\mathfrak{B}$ are algebras, and $\varphi : \mathfrak{B} \to \mathfrak{A}$ is a surjective homomorphism. It is easy to see that if Ψ is a tolerance on

$\mathfrak{B}$, then $\varphi(\Psi) = \{(\varphi(a), \varphi(b)) \mid a \, \Psi \, b \in B\}$ is a tolerance Θ on $\mathfrak{A}$. In the above situation, we say that Θ *is an image of* Ψ. In case we need to specify $\mathfrak{B}$ explicitly, we shall say that Θ *is the image of a tolerance on* $\mathfrak{B}$.

We shall be mainly concerned with the case when Ψ is a congruence on $\mathfrak{B}$, in which case we shall say that Θ *is the image of a congruence on* $\mathfrak{B}$. Notice that the image of a congruence is a tolerance, but, in general, it is not a congruence.

It has been proved in [3, 12] that every tolerance Θ on some algebra $\mathfrak{A}$ can be represented as the image of some congruence on some algebra $\mathfrak{B}$. However, in most cases, $\mathfrak{A}$ belongs to some specified variety $\mathcal{V}$, and it is a natural request to ask that $\mathfrak{B}$, too, belongs to $\mathcal{V}$. This observation justifies the next definition.

Definition 1.2. Following Chajda, Czédli and Halaš [2] and Czédli and Kiss [12], the property "*the tolerances of* $\mathcal{V}$ *are the homomorphic images of its congruences*" (TImC, for short) states that every tolerance on some algebra in $\mathcal{V}$ is the image of a congruence on some algebra in $\mathcal{V}$.

Czédli and Grätzer [6] were the first to drew attention to such a kind of properties; they showed that, in the above terminology, TImC holds for the variety of all lattices. Some properties implying TImC have been studied in [2] and, by methods inspired by both [6] and [2], it is proved in [3] that TImC holds for every variety definable by linear equations (an equation is *linear* if each variable appears at most once on each side).

The general case of an arbitrary variety is extensively studied in [12], where a characterization of those varieties satisfying TImC is given by means of a Maltsev-like condition, and where it is shown that TImC holds, among others, for all varieties of lattices, for all unary varieties, and for the variety of semilattices. In [12] it is also shown that, on the contrary, there is a variety with a majority term in which TImC fails, and that, for an n-permutable variety $\mathcal{V}$, TImC holds for $\mathcal{V}$ if and only if $\mathcal{V}$ is permutable.

Our approach here is more "local", in the sense that we study conditions which guarantee that certain tolerances in $\mathcal{V}$ (but not necessarily *all* tolerances in $\mathcal{V}$) are the images of congruences in $\mathcal{V}$. Our study is connected with another way of representing tolerances, a kind of representation we have introduced for completely different purposes in [26].

Definition 1.3. [26, Definition 2] Suppose that $\mathfrak{A}$ is an algebra, and R is a compatible reflexive relation on $\mathfrak{A}$. Let R^- denote the *converse* of R (that is, $a \mathrel{\mathrm{R}^-} b$ if and only if $b \mathrel{\mathrm{R}} a$). It is immediate to see that $R \circ R^-$ is a tolerance on $\mathfrak{A}$. We call a tolerance *representable* if it can be expressed in the form $R \circ R^-$ as above.

Not every tolerance is representable, as shown in [26, Section 6]. See also Proposition 7.1 here. The main application of representability in [26] is the theorem that, under some weak assumptions on an identity ε, a variety $\mathcal{V}$ satisfies ε for congruences (that is, all algebras in $\mathcal{V}$ satisfy ε when the variables of ε are interpreted as congruences) if and only if $\mathcal{V}$ satisfies ε for representable tolerances.

Here we show that there is a deep connection between representability of tolerances in the sense of [26], and the property of being the image of some congruence in the sense of Definition 1.1. In particular, we prove (Theorem 3.1) that every representable tolerance on some algebra $\mathfrak{A}$ is the image of a congruence on $\mathfrak{A} \times \mathfrak{A}$, and that if some tolerance Θ on $\mathfrak{A}$ can be expressed as the intersection of a family of representable tolerances, then Θ is the image of some congruence in an algebra belonging to $\mathcal{V}(\mathfrak{A})$, the variety generated by $\mathfrak{A}$.

In certain cases, e. g. a 3-permutable variety $\mathcal{V}$, the properties of being representable and of being the image of some congruence in $\mathcal{V}$ are equivalent (Proposition 5.1); on the other hand, for every set A, and every reflexive, symmetric and not transitive relation Θ over A, it is possible (Proposition 7.1) to give A an algebraic structure $\mathfrak{A}$ in such a way that Θ is a tolerance on $\mathfrak{A}$, Θ is the image of some congruence on $\mathcal{V}(\mathfrak{A})$, but Θ is not representable, not even expressible as the intersection of representable tolerances.

The particular case of varieties of lattices, which might be of independent interest, is dealt in a particularly simple way in Section 2. A broad generalization of the case of lattices is presented in Section 6.

2. A first example

We first exemplify our methods in the particular case of lattices.

Theorem 2.1. *If $\mathfrak{L}$ is a lattice and Θ is a tolerance on $\mathfrak{L}$, then Θ is the image of some congruence on some subalgebra of $\mathfrak{L} \times \mathfrak{L}$.*

Proof. The partial order $\leq$ induced by the lattice operations is a compatible relation on $\mathfrak{L}$, thus also $\leq \cap\ \Theta$ is compatible. Hence the binary relation $\leq \cap\ \Theta$ can be considered as a subalgebra $\mathfrak{B}$ of $\mathfrak{L} \times \mathfrak{L}$. Let $\varphi : \mathfrak{B} \to \mathfrak{A}$ be the first projection, and β on $\mathfrak{B}$ be the kernel of the second projection.

We shall show that $\varphi(\beta) = \Theta$. Indeed, if $a \,\Theta\, b$, then $a = a \vee a \,\Theta\, a \vee b$, thus $(a, a \vee b) \in B$, since $a \leq a \vee b$. Similarly, $(b, a \vee b) \in B$. Trivially, $(a, a \vee b)\, \beta\, (b, a \vee b)$, $\varphi(a, a \vee b) = a$, $\varphi(b, a \vee b) = b$, thus $\Theta \subseteq \varphi(\beta)$.

Conversely, if $(a, b) \in \varphi(\beta)$, then there is $c \in L$ such that $(a, c) \in B$, $(b, c) \in B$, hence $a \,\Theta\, c$, $a \leq c$, $c \,\Theta\, b$, $b \leq c$, thus $a = a \wedge c \,\Theta\, c \wedge b = b$. Hence $\varphi(\beta) \subseteq \theta$, and the theorem is proved. □

Notice that we have not used all the properties of a lattice, thus Theorem 2.1 allows some strengthening, see Proposition 6.1 below. Moreover, the proof of Theorem 2.1 applies not only to lattices, but also to lattices with additional operations, provided the additional operations respect the lattice order, that is, the order remains a compatible relation with respect to the additional operations.

Notice also that if $\mathfrak{L}$ belongs to some variety of lattices $\mathcal{V}$, then every subalgebra of $\mathfrak{L} \times \mathfrak{L}$ belongs to $\mathcal{V}$. In fact, we do not need the full assumption that $\mathcal{V}$ is a variety: we get that if $\mathcal{V}$ is a class of lattices, and $\mathcal{V}$ is closed under subalgebras and finite products, then every tolerance on some lattice in $\mathcal{V}$ is the image of some congruence on some lattice in $\mathcal{V}$.

Czédli and Grätzer [6] proved that the variety of all lattices satisfies TImC, and G. Czédli and E. W. Kiss [12] extended the result to an arbitrary variety of lattices. Theorem 2.1 furnishes an alternative simple proof of the above results, actually, apparently, a slight strengthening.

As we shall see in the next section, the reason why Theorem 2.1 holds is that tolerances in lattices are representable in the sense of Definition 1.3.

3. Representable tolerances

Theorem 3.1. *If Θ is a tolerance on $\mathfrak{A}$, and Θ can be expressed as the intersection of λ-many representable tolerances, then Θ is the image of a congruence on some subalgebra of $\mathfrak{A} \times \mathfrak{A}^\lambda$.*

In particular, if Θ is representable, then Θ is the image of a congruence on some subalgebra of $\mathfrak{A} \times \mathfrak{A}$.

Proof. Suppose that $\Theta = \bigcap_{i \in \lambda} \Theta_i$, where each Θ_i has the form $R_i \circ {R_i}^-$, for certain reflexive compatible relations R_i. Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A} \times \mathfrak{A}^\lambda$ whose base set is $B = \{(a, (a_i)_{i \in \lambda}) \mid a, a_i \in A, \text{ and } a \,\mathrm{R}_i\, a_i, \text{ for each } i \in \lambda\}$. The assumption that each R_i is compatible implies that $\mathfrak{B}$ is indeed a subalgebra of $\mathfrak{A} \times \mathfrak{A}^\lambda$.

Let $\varphi : \mathfrak{B} \to \mathfrak{A}$ be the first projection. Since each R_i is reflexive, we have that φ is surjective. Let β be the kernel of the second projection $\pi : \mathfrak{B} \to \mathfrak{A}^\lambda$. We shall show that $\varphi(\beta) = \Theta$.

Indeed, for every $a, c \in A$, since $\Theta = \bigcap_{i \in \lambda}(R_i \circ {R_i}^{-})$, the following is a chain of equivalent conditions.

(1) $a \, \Theta \, c$;
(2) for every $i \in \lambda$, there is $b_i \in A$ such that $(a, b_i) \in R_i$ and $(b_i, c) \in {R_i}^{-}$;
(3) for every $i \in \lambda$, there is $b_i \in A$ such that $(a, b_i), (c, b_i) \in R_i$.
(4) there is a sequence $\bar{b} = (b_i)_{i \in \lambda}$ of elements from A such that $(a, \bar{b}), (c, \bar{b}) \in B$ (thus, $(a, \bar{b}) \beta (c, \bar{b})$).
(5) $(a, c) \in \varphi(\beta)$.

We have shown that $\Theta = \varphi(\beta)$, thus the theorem is proved. $\square$

G. Czédli observed that every tolerance on a lattice is representable, as a consequence of Lemma 2 in [5]. Cf. also Chajda and Zelinka [4]. See [26, Proposition 11] and Section 6 below, for some slightly more general results. Hence Theorem 2.1 is actually a particular case of Theorem 3.1. We have given a direct proof of Theorem 2.1 since it is relatively short and simple, and moreover it is a good introduction to the methods used in this paper.

4. Expressing tolerances as images in varieties

First, a note on terminology. We shall say that a tolerance Θ is in a variety $\mathcal{V}$ to mean that Θ is a tolerance on some algebra $\mathfrak{A} \in \mathcal{V}$. Technically, this is justified since a tolerance on $\mathfrak{A}$ can be seen as a subalgebra of $\mathfrak{A} \times \mathfrak{A}$ (and Θ and $\mathfrak{A}$ generate the same variety, since, in the above sense, $\mathfrak{A}$ is isomorphic to a substructure of Θ). A similar remark applies to congruences in place of tolerances.

Definition 4.1. If Θ is a tolerance on $\mathfrak{A} \in \mathcal{V}$, we say that Θ *is the image of a congruence in* $\mathcal{V}$ if it is possible to chose $\mathfrak{B} \in \mathcal{V}$, β a congruence on $\mathfrak{B}$, and $\varphi : \mathfrak{B} \to \mathfrak{A}$ a surjective homomorphism such that $\Theta = \varphi(\beta)$.

The above definition is a local version of TImC, as introduced in Definition 1.2.

Though in the above definitions $\mathcal{V}$ is intended to stand for a variety, our results generally hold for an arbitrary class $\mathcal{V}$ which is closed under taking subalgebras and products, in particular, for quasivarieties. Actually, in most cases, it is enough to assume that $\mathcal{V}$ is closed under taking subalgebras and finite products.

Theorem 3.1 suggests the next definition.

Definition 4.2. A tolerance is *weakly representable* (*finitely representable*, resp.) if it can be expressed as an intersection of representable tolerances (of a finite number of representable tolerances, resp.). See again [26, Section 6] for more informations about weakly representable tolerances.

With the above terminology, as an immediate consequence of Theorem 3.1, we get:

Corollary 4.3. *Let $\mathcal{V}$ be a class of algebras closed under subalgebras and finite products (arbitrary products, resp.).*

Every finitely representable (weakly representable, resp.) tolerance in $\mathcal{V}$ is the image of some congruence in $\mathcal{V}$.

In particular, if every tolerance in $\mathcal{V}$ is finitely representable (weakly representable, resp.), then the tolerances of $\mathcal{V}$ are the images of its congruences.

The converse of Corollary 4.3 is not true. As a particular case of [3, 12], in every variety defined by the empty set of equations the tolerances of $\mathcal{V}$ are the images of its congruences, but, by [26, Proposition 10] (see also Proposition 7.1 below), there exists a non representable tolerance on some algebra (which trivially belongs to a variety defined by an empty set of equations).

However, it is possible to show that, within a given variety, a tolerance is the image of some congruence if and only if it is the image of some representable tolerance (see Corollary 4.5 below). This can be useful, since if we want to show that some variety $\mathcal{V}$ satisfies TImC, it is enough to show that the tolerances of $\mathcal{V}$ are the images of its representable tolerances.

Lemma 4.4. *If Θ, Ψ and Φ are tolerances, Θ is an image of Ψ, and Ψ is an image of Φ, then Θ is an image of Φ.*

Proof. Let Θ be on $\mathfrak{A}$, Ψ be on $\mathfrak{C}$, and Φ be on $\mathfrak{D}$, and let the assumption of the lemma be witnessed by surjective homomorphisms $\psi : \mathfrak{C} \to \mathfrak{A}$ and $\varphi : \mathfrak{D} \to \mathfrak{C}$. Then $\psi \circ \varphi : \mathfrak{D} \to \mathfrak{A}$ witnesses that Θ is an image of Φ. $\square$

Corollary 4.5. *Let $\mathcal{V}$ be a class of algebras closed under subalgebras and products (in particular, a variety). For every tolerance Θ in $\mathcal{V}$, the following conditions are equivalent.*

(1) *Θ is the image of a congruence in $\mathcal{V}$.*
(2) *Θ is the image of a representable tolerance in $\mathcal{V}$.*
(3) *Θ is the image of a weakly representable tolerance in $\mathcal{V}$.*

In particular, for every $\mathcal{V}$ as above, the tolerances of $\mathcal{V}$ are the images of its congruences if and only if the tolerances of $\mathcal{V}$ are the images of its (weakly) representable tolerances.

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are trivial, since every congruence β is representable (as $\beta = \beta \circ \beta$), and since every representable tolerance is weakly representable.

(3) $\Rightarrow$ (1) If Θ is the image of a weakly representable tolerance Ψ in $\mathcal{V}$, then, by Theorem 3.1, Ψ is the image of some congruence β in $\mathcal{V}$, hence, by Lemma 4.4, Θ is the image of β. $\square$

Corollary 4.6. *Suppose that Θ is a tolerance on the algebra $\mathfrak{A}$. Then the following conditions are equivalent.*

(1) *Θ is the image of a congruence on some subalgebra of some power $\mathfrak{A}^I$, for some set I.*
(2) *Θ is the image of a congruence in $\mathcal{V}(\mathfrak{A})$, the variety generated by $\mathfrak{A}$.*

In all the preceding conditions we can equivalently replace the word "congruence" with either "representable tolerance" or "weakly representable tolerance".

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1) Let Θ be an image of γ, a congruence on $\mathfrak{C} \in \mathcal{V}(\mathfrak{A})$. By the HSP characterization of $V(\mathfrak{A})$, there are a set I, an algebra $\mathfrak{B} \subseteq \mathfrak{A}^I$, and a surjective homomorphism $\varphi : \mathfrak{B} \to \mathfrak{C}$. Then $\beta = \varphi^{-1}(\gamma) = \{(b, b') \mid b, b' \in B \text{ and } (\varphi(b), \varphi(b')) \in \gamma\}$ is a congruence on $\mathfrak{B}$, and $\varphi(\beta) = \gamma$, in the sense of Definition 1.1.

Thus Θ is an image of γ, which is an image of β, hence, by Lemma 4.4, Θ is an image of β, a congruence on $\mathfrak{A}^I$, and (1) is proved.

The last statement is immediate from Corollary 4.5. $\square$

Corollary 4.7. *A variety $\mathcal{V}$ satisfies TImc if and only if every tolerance Θ on any algebra $\mathfrak{A} \in \mathcal{V}$ is the image of a congruence (equivalently, of a weakly representable tolerance) on some subalgebra of some power $\mathfrak{A}^I$.*

5. In n-permutable varieties

Under certain conditions, the converse of Theorem 3.1 does hold.

Proposition 5.1. *Suppose that $\mathfrak{A}$ is an algebra in a 3-permutable variety $\mathcal{V}$, and Θ is a tolerance on $\mathfrak{A}$. Then the following conditions are equivalent.*

(1) *Θ is representable.*
(2) *Θ is weakly representable.*
(3) *Θ is the image of some congruence on some subalgebra of $\mathfrak{A} \times \mathfrak{A}$.*
(4) *Θ is the image of some congruence in $\mathcal{V}(\mathfrak{A})$.*
(5) *Θ is a congruence of $\mathfrak{A}$.*

If we only assume that every subalgebra of $\mathfrak{A} \times \mathfrak{A}$ *has 3-permutable congruences, then Conditions (1), (3) and (5) above are still equivalent.*

Proof. (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are trivial.

(1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4) follow from Theorem 3.1.

(4) $\Rightarrow$ (5) Clearly $\mathcal{V}(\mathfrak{A})$, being a subvariety of $\mathcal{V}$, is 3-permutable, too, hence Θ is the image of some congruence on some algebra with 3 -permuting congruences. But it is well-known that this implies that Θ is a congruence, see [22, Chapter 7, Theorem 1.10].

(5) $\Rightarrow$ (1) is trivial, since if Θ is a congruence, then $\Theta = \Theta \circ \Theta^-$.

Under the assumption that every subalgebra of $\mathfrak{A} \times \mathfrak{A}$ has 3-permutable congruences, (3) $\Rightarrow$ (5) holds, again by Theorem 1.10 in [22, Chapter 7]. The implications (1) $\Rightarrow$ (3) and (5) $\Rightarrow$ (1) do not use 3 -permutability at all. □

We expect that parts of Proposition 5.1 hold under assumptions weaker than 3-permutability. However, globally (that is, if we ask that the conditions hold *for every* tolerance in a 3-permutable variety - even, in an n-permutable variety), Proposition 5.1 is essentially an empty result, in the sense that the conditions hold only in permutable varieties (in which they are trivially true).

Corollary 5.2. *Suppose that* $\mathcal{V}$ *is an* n*-permutable variety, for some* n*. Then the following conditions are equivalent.*

(1) *Every tolerance in* $\mathcal{V}$ *is representable.*
(2) *Every tolerance in* $\mathcal{V}$ *is weakly representable.*
(3) *Every tolerance in* $\mathcal{V}$ *is the image of a congruence in* $\mathcal{V}$*.*
(4) $\mathcal{V}$ *is permutable.*
(5) *Every tolerance in* $\mathcal{V}$ *is a congruence.*

Proof. (1) $\Rightarrow$ (2) and (5) $\Rightarrow$ (1) are trivial.

(2) $\Rightarrow$ (3) follows from Corollary 4.3.

(3) $\Rightarrow$ (4) is [12, Theorem 5.3].

(4) $\Rightarrow$ (5) is immediate from a classical result from [30], parts of which are due independently to G. Hutchinson [21]. Actually, Conditions (4) and (5) are equivalent for every variety, as follows easily from the above papers, and explicitly stated, e. g., in [1]. □

6. Beyond lattices

We now provide a generalization of Theorem 2.1. Its proof exploits exactly the only properties of lattices which were used in the proof of 2.1.

Proposition 6.1. *Suppose that $\mathfrak{A}$ is an algebra with two binary operations $\vee$ and $\wedge$ (among possibly other operations), and with a compatible binary relation M, which satisfy the following conditions:*

(1) $a \vee a = a$, *for every* $a \in A$.
(2) $a \,\mathrm{M} (a \vee b)$, *and* $b \,\mathrm{M} (a \vee b)$, *for every* $a, b \in A$.
(3) $a = a \wedge c = c \wedge a$, *for every* $a, c \in A$ *such that* $a \,\mathrm{M}\, c$.

Then every tolerance Θ of $\mathfrak{A}$ is representable, and is an image of some congruence on some subalgebra of $\mathfrak{A} \times \mathfrak{A}$.

Proof. Same as the proof of Theorem 2.1, using M in place of $\leq$: Θ is representable as $R \circ R^{-}$, with $R = \mathrm{M} \cap \Theta$. $\square$

Remark 6.2. Condition (2) in Proposition 6.1 is satisfied in case M is defined by

$$a \,\mathrm{M}\, b \text{ if and only if } a \vee b = b,$$

and $\mathfrak{A}$ satisfies $a \vee (a \vee b) = a \vee b$ and $b \vee (a \vee b) = a \vee b$, for every $a, b \in A$.

By Proposition 6.1, and writing explicitly the condition that the M given by Remark 6.2 is compatible, we get:

Proposition 6.3. *Suppose that $\mathfrak{A}$ is an algebra with (exactly) two binary operations $\vee$ and $\wedge$ satisfying the following conditions:*

(0) *For every* $a, a', b, b' \in A$, *if* $a \vee b = b$ *and* $a' \vee b' = b'$, *then* $(a \vee a') \vee (b \vee b') = b \vee b'$ *and* $(a \wedge a') \vee (b \wedge b') = b \wedge b'$.
(1) $a \vee a = a$, *for every* $a \in A$.
(2) $a \vee (a \vee b) = a \vee b$, *and* $b \vee (a \vee b) = a \vee b$, *for every* $a, b \in A$.
(3) *For every* $a, c \in A$, *if* $a \vee c = c$, *then* $a = a \wedge c = c \wedge a$.

Then every tolerance Θ of $\mathfrak{A}$ is representable, and is an image of some congruence on some subalgebra of $\mathfrak{A} \times \mathfrak{A}$.

Problems 6.4. Notice that, again by [4, 5], tolerances in lattices satisfy a property stronger than representability. Indeed, if Θ is a tolerance on a lattice $\mathfrak{L}$, then

(1) there is a compatible and reflexive relation R such that $\Theta = R \circ R^{-} = (R \circ R^{-}) \cap (R^{-} \circ R)$,

or even

(2) there is a compatible and reflexive relation R such that $a \,\Theta\, b$ if and only if there are c and d such that $a \,\mathrm{R}\, c \,\mathrm{R}^{-} b$, $a \,\mathrm{R}^{-} d \,\mathrm{R}\, b$, and $d \,\mathrm{R}\, c$ (just take $R = \Theta \cap \leq$, $c = a \vee b$ and $d = a \wedge b$).

Which parts of the theory of tolerances on lattices follow just from the assumption (1) or (2)?

Notice that we do not need all the axioms for lattices, in order to get (1) above: the properties listed in Proposition 6.1, together with their symmetric duals suffice.

7. Concluding remarks

We now show that the converse of Theorem 3.1 fails in a large class of algebras.

Proposition 7.1. *For every set A, and every reflexive and symmetric relation Θ on A which is not transitive, there is an algebra $\mathfrak{A}$ with base set A and such that Θ is a tolerance on $\mathfrak{A}$ which is not weakly representable, but Θ is the image of some congruence on some subalgebra of some power of $\mathfrak{A}$.*

Proof. For every $a, b \in A$ such that $a \,\Theta\, b$, and for every function $f : A \to \{a, b\}$, add to A a unary function symbol representing f. It is easy to see that Θ is a tolerance on the algebra thus obtained, and that Θ is not weakly representable. Indeed, every nontrivial compatible relation R on $\mathfrak{A}$ contains Θ, and, since Θ is not transitive, then $\Theta \subset \Theta \circ \Theta \subseteq R \circ R^-$ (see [26, Proposition 12] for more details).

Consider $\mathcal{V}(\mathfrak{A})$, the variety generated by $\mathfrak{A}$. Since $\mathcal{V}(\mathfrak{A})$ is unary, then, by [12, Corollary 4.4], tolerances are images of congruences in $\mathcal{V}(\mathfrak{A})$. Then Corollary 4.6(3) $\Rightarrow$ (1) implies that Θ is the image of some congruence on some subalgebra of some power of $\mathfrak{A}$. $\square$

Finally, we show that the conditions exploited in the proof of Theorem 3.1 actually characterize representable tolerances. We shall treat only the case $\lambda = 1$ for sake of simplicity. The case of arbitrary λ can be dealt with in a similar way, and is left to the interested reader.

Proposition 7.2. *Suppose that Θ is a tolerance on the algebra $\mathfrak{A}$. Then Θ is representable if and only if Θ can be realized as the image of a congruence on some subalgebra $\mathfrak{B}$ of $\mathfrak{A} \times \mathfrak{A}$, such that $\mathfrak{B}$ contains $\Delta = \{(a, a) \mid a \in A\}$, and in such a way that φ and $\Psi = \beta$ in Definition 1.1 can be chosen to be, respectively, the first projection and the kernel of the second projection.*

Proof. The construction used in the proof of Theorem 3.1 shows that if Θ is representable, then $\mathfrak{B}$, φ and β can be chosen to satisfy the desired requirements.

Conversely, suppose that we have $\mathfrak{B} \subseteq \mathfrak{A} \times \mathfrak{A}$, φ and β satisfying the conditions in the statement of the proposition. Being a subalgebra of $\mathfrak{A} \times \mathfrak{A}$, $\mathfrak{B}$ can be thought of as a compatible relation on $\mathfrak{A}$. We shall take $R = B$. Since B contains Δ, then R is reflexive. By assumption, $a \,\Theta\, c$

if and only if $(a,c) \in \varphi(\beta)$. Noticing that the equivalence of items (5) and (2) in the proof of Theorem 3.1 holds also in the present situation, we get that $a \,\Theta\, c$ if and only if there is $b \in A$ such that $(a,b) \in R$ and $(b,c) \in R^-$. This means exactly that $\Theta = R \circ R^-$. $\square$

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Dipartimento Novello di Matematica, Viale della Ricerca Scientìfica, II Università di Roma (Tor Vergata), I-00133 ROME ITALY

URL: http://www.mat.uniroma2.it/~lipparin