

EVERY WEAKLY INITIALLY \mathfrak{m} -COMPACT TOPOLOGICAL SPACE IS \mathfrak{m} PCAP

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ABSTRACT. The statement in the title solves a problem raised by T. Retta. We also present a variation of the result in terms of $[\mu, \kappa]$ -compactness.

Let \mathfrak{m} be an infinite cardinal. A topological space is *weakly initially \mathfrak{m} -compact* if and only if every open cover of cardinality $\leq \mathfrak{m}$ has a finite subset with a dense union.

A topological space X is said to be *\mathfrak{m} pcap* [R] if every family of $\leq \mathfrak{m}$ open sets in X has a complete accumulation point, i. e., a point each neighborhood of which meets κ members of the family, where κ is the cardinality of the family. The acronym *\mathfrak{m} pcap* stands for *\mathfrak{m} -pseudocompact in the sense of complete accumulation points*.

The next Theorem solves the last problem in [R].

Theorem 1. *For every infinite cardinal \mathfrak{m} , every weakly initially \mathfrak{m} -compact topological space is \mathfrak{m} pcap.*

Before proving the theorem, we recall some known facts about the notions involved in its statement.

The notion of weak initial \mathfrak{m} -compactness has been introduced by Frolík [F] under the name *almost \mathfrak{m} -compactness*, and has been studied by various authors under various names, such as *weak- \mathfrak{m} - \aleph_0 -compactness*, or *\mathcal{O} - $[\omega, \mathfrak{m}]$ -compactness*. See [L2] for references. By taking complements, it is trivial to see that a topological space X is weakly initially \mathfrak{m} -compact if and only if the following holds. For every sequence $(C_\alpha)_{\alpha \in \mathfrak{m}}$ of closed sets of X , if, for every finite $F \subseteq \mathfrak{m}$, there exists a nonempty open set O_F of X such that $\bigcap_{\alpha \in F} C_\alpha \supseteq O_F$, then $\bigcap_{\alpha \in \mathfrak{m}} C_\alpha \neq \emptyset$.

A topological space is said to be *pseudo- (κ, λ) -compact* [CN] if and only if for every λ -indexed sequence $(O_\alpha)_{\alpha \in \lambda}$ of nonempty open sets of X , there is $x \in X$ such that, for every neighborhood U of x , $|\{\alpha \in \lambda \mid U \cap O_\alpha \neq \emptyset\}| \geq \kappa$.

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T. Retta [R, Theorem 3(d)] proved that a space is mpcap if and only if it is pseudo- (κ, κ) -compact for each $\kappa \leq \mathfrak{m}$.

Proof of the theorem. If $\kappa \leq \mathfrak{m}$, then trivially every weakly initially \mathfrak{m} -compact topological space is weakly initially κ -compact. Thus if we prove that, for every infinite cardinal κ , every weakly initially κ -compact topological space is pseudo- (κ, κ) -compact, then we have that every weakly initially \mathfrak{m} -compact topological space is pseudo- (κ, κ) -compact, for every $\kappa \leq \mathfrak{m}$, and we are done by the mentioned result from [R, Theorem 3(d)].

Hence let X be a weakly initially κ -compact topological space, and let $(O_\alpha)_{\alpha \in \kappa}$ be a sequence of nonempty open sets of X . Let $S_\omega(\kappa)$ be the set of all finite subsets of κ . Since $|S_\omega(\kappa)| = \kappa$, we can reindex the sequence $(O_\alpha)_{\alpha \in \kappa}$ as $(O_F)_{F \in S_\omega(\kappa)}$. For every $\alpha \in \kappa$, let $C_\alpha = \bigcup \{O_F \mid F \in S_\omega(\kappa), \alpha \in F\}$. For every finite subset F of κ , we have that $\bigcap_{\alpha \in F} C_\alpha$ contains the nonempty open set O_F . By weak initial κ -compactness, $\bigcap_{\alpha \in \kappa} C_\alpha \neq \emptyset$.

Let $x \in \bigcap_{\alpha \in \kappa} C_\alpha$. We are going to show that, for every neighborhood U of x , we have that $|\{F \in S_\omega(\kappa) \mid U \cap O_F \neq \emptyset\}| = \kappa$, thus X is pseudo- (κ, κ) -compact, and the theorem is proved.

So, let U be a neighborhood of x , and suppose by contradiction that the cardinality of $H = \{F \in S_\omega(\kappa) \mid U \cap O_F \neq \emptyset\}$ is $< \kappa$. Then $|\bigcup H| < \kappa$. Choose $\alpha \in \kappa$ such that $\alpha \notin \bigcup H$. Thus if $F \in S_\omega(\kappa)$ and $\alpha \in F$, then $F \notin H$, hence $U \cap O_F = \emptyset$. Then we also get $U \cap \bigcup \{O_F \mid F \in S_\omega(\kappa), \alpha \in F\} = \emptyset$, hence $U \cap C_\alpha = \emptyset$, since $C_\alpha = \bigcup \{O_F \mid F \in S_\omega(\kappa), \alpha \in F\}$. In particular, $x \notin C_\alpha$. We have reached a contradiction, and the theorem is proved. \square

In fact, our argument gives something more. Let us say that a topological space is *weakly* $[\lambda, \kappa]$ -compact if and only if every open cover of cardinality $\leq \kappa$ has a subset of cardinality $< \lambda$ with a dense union. This notion has been studied in [L1, L2], sometimes under the name \mathcal{O} - $[\lambda, \kappa]$ -compactness.

For $\kappa \geq \lambda \geq \mu$, let $\text{COV}(\kappa, \lambda, \mu)$ denote the minimal cardinality of a family of subsets of κ , each of cardinality $< \lambda$, such that every subset of κ of cardinality $< \mu$ is contained in at least one set of the family. Highly non trivial results about $\text{COV}(\kappa, \lambda, \mu)$ are proved in [S] under the terminology $\text{cov}(\kappa, \lambda, \mu, 2)$. See [S, II, Definition 5.1]. Notice that, trivially, $\text{COV}(\kappa, \lambda, \mu) \leq |S_\mu(\kappa)| = \sup_{\mu' < \mu} \kappa^{\mu'}$. In particular, $\text{COV}(\kappa, \lambda, \omega) = \kappa$, hence the next Proposition is stronger than Theorem 1, via [R, Theorem 3(d)].

Proposition 2. *Suppose that $\kappa \geq \lambda \geq \mu$ are infinite cardinals, and either $\kappa > \lambda$, or κ is regular. Then every weakly $[\mu, \kappa]$ -compact topological space is pseudo- $(\kappa, \text{COV}(\kappa, \lambda, \mu))$ -compact.*

Proof. The proof is essentially the same as the proof of Theorem 1. We shall only point out the differences. Let K be a subset of $S_\lambda(\kappa)$ witnessing $|K| = \text{COV}(\kappa, \lambda, \mu)$. Suppose that X is a weakly $[\mu, \kappa]$ -compact topological space and let $(O_Z)_{Z \in K}$ be a sequence of nonempty open sets of X . For $\alpha \in \kappa$, put $C_\alpha = \overline{\bigcup\{O_Z \mid Z \in K, \alpha \in Z\}}$. If $W \subseteq \kappa$, and $|W| < \mu$, then there is $Z \in K$ such that $Z \supseteq W$, so that $\bigcap_{\alpha \in W} C_\alpha \supseteq \bigcap_{\alpha \in Z} C_\alpha$ contains the nonempty open set O_Z , hence, by weak $[\mu, \kappa]$ -compactness, $\bigcap_{\alpha \in \kappa} C_\alpha \neq \emptyset$.

Now notice that the union of $< \kappa$ sets, each of cardinality $< \lambda$, has cardinality $< \kappa$, and this is the only thing that is used in the final part of the proof of Theorem 1. \square

For κ a regular cardinal, weak $[\kappa, \kappa]$ -compactness is equivalent to pseudo- (κ, κ) -compactness, as proved in [L1] under different terminology.

By replacing everywhere nonempty open sets by points in Proposition 2, we get the following result which, in the present generality, might be new.

Proposition 3. *Suppose that $\kappa \geq \lambda \geq \mu$ are infinite cardinals, and either $\kappa > \lambda$, or κ is regular, and let $\nu = \text{COV}(\kappa, \lambda, \mu)$. If X is a $[\mu, \kappa]$ -compact topological space, then, for every ν -indexed family $(x_\beta)_{\beta \in \nu}$ of elements of X , there is some element $x \in X$ such that, for every neighborhood U of x , the set $\{\beta \in \nu \mid x_\beta \in U\}$ has cardinality $\geq \kappa$.*

A common generalization of both Propositions 2 and 3 can be given along the abstract framework presented in [L1, L2]. If X is a topological space, and \mathcal{F} is a family of subsets of X , we say that X is \mathcal{F} - $[\mu, \kappa]$ -compact if and only if the following holds. For every sequence $(C_\alpha)_{\alpha \in \kappa}$ of closed sets of X , if, for every $Z \subseteq \kappa$ with $|Z| < \mu$, there exists a set $F_Z \in \mathcal{F}$ such that $\bigcap_{\alpha \in Z} C_\alpha \supseteq F_Z$, then $\bigcap_{\alpha \in \kappa} C_\alpha \neq \emptyset$.

Proposition 4. *Suppose that $\kappa \geq \lambda \geq \mu$ are infinite cardinals, and either $\kappa > \lambda$, or κ is regular, and let $\nu = \text{COV}(\kappa, \lambda, \mu)$. Suppose that X is a topological space, and \mathcal{F} is a family of subsets of X . If X is \mathcal{F} - $[\mu, \kappa]$ -compact, then, for every ν -indexed family $(F_\beta)_{\beta \in \nu}$ of elements of \mathcal{F} , there is some element $x \in X$ such that, for every neighborhood U of x , the set $\{\beta \in \nu \mid F_\beta \cap U \neq \emptyset\}$ has cardinality $\geq \kappa$.*

Proposition 2 is the particular case of Proposition 4 when we take \mathcal{F} to be the family of all nonempty subsets of X . Proposition 3 is the

particular case of Proposition 4 when we take \mathcal{F} to be the family of all singletons of X .

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