MORE GENERALIZATIONS OF PSEUDOCOMPACTNESS

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Abstract. We introduce a covering notion depending on two cardinals, which we call $\mathcal{O}$-$[\mu, \lambda]$-compactness, and which encompasses both pseudocompactness and many other generalizations of pseudocompactness. For Tychonoff spaces, pseudocompactness turns out to be equivalent to $\mathcal{O}$-$[\omega, \omega]$-compactness.

We provide several characterizations of $\mathcal{O}$-$[\mu, \lambda]$-compactness, and we discuss its connection with D-pseudocompactness, for D an ultrafilter. We analyze the behaviour of the above notions with respect to products.

Finally, we show that our results hold in a more general framework, in which compactness properties are defined relative to an arbitrary family of subsets of some topological space X.

1. Introduction

As well-known, there are many equivalent reformulations of pseudocompactness. See, e. g. [St]. Various generalizations and extensions of pseudocompactness have been introduced by many authors; see, among others, [Ar, CoNe, Fr, Ga, GiSa, Gl, Ke, Li4, Re, Sa, SaSt, ScSt, St, StVa]. We introduce here some more pseudocompactness-like properties, focusing mainly on notions related to covering properties and ultrafilter convergence.

The most general form of our notion depends on two cardinals μ and λ; we call it $\mathcal{O}$-$[\mu, \lambda]$-compactness. It generalizes and unifies several pseudocompactness-like notions appeared before. See Remark 2.3.

1991 *Mathematics Subject Classification.* Primary 54D20; Secondary 54A20, 54B10.

Key words and phrases. Pseudocompactness, $\mathcal{O}$-$[\mu, \lambda]$-compactness, D-pseudocompactness, D-limits, in products, regular ultrafilter, family of subsets of a topological space.

The author has received support from MPI and GNSAGA. We wish to express our gratitude to X. Caicedo and S. Garcia-Ferreira for stimulating discussions and correspondence.

In a sense, $\mathcal{O}$-$[\mu, \lambda]$-compactness is to pseudocompactness what $[\mu, \lambda]$-compactness is to countable compactness. See Remark 2.2. In particular, for Tychonoff spaces, $\mathcal{O}$-$[\omega, \omega]$-compactness turns out to be equivalent to pseudocompactness.

We find many conditions equivalent to $\mathcal{O}$-$[\mu, \lambda]$-compactness. In particular, a characterization by means of ultrafilters, Theorem 3.2, plays an important role in this paper. It provides a connection between $\mathcal{O}$-$[\mu, \lambda]$-compactness and D-pseudocompactness, for D a (μ, λ)-regular ultrafilter. The notion of a (μ, λ)-regular ultrafilter arose in a model-theoretical setting, and has proved useful also in some areas of set-theory, and even in topology. See [Li3, Li2] for references.

More sophisticated results are involved when we deal with products, since D-pseudocompactness is productive, but $\mathcal{O}$-$[\mu, \lambda]$-compactness is not productive, as well known in the special case $\mu = \lambda = \omega$, that is, pseudocompactness. We show that if D is a (μ, λ)-regular ultrafilter, then every D-pseudocompact topological space X is $\mathcal{O}$-$[\mu, \lambda]$-compact, hence all (Tychonoff) powers of X are $\mathcal{O}$-$[\mu, \lambda]$-compact, too (Corollary 3.7). The situation is in part parallel to the relationship between the more classical notions of D-compactness and $[\mu, \lambda]$-compactness. In this latter case, an equivalence holds: all powers of a topological space X are $[\mu, \lambda]$-compact if and only if there is some (μ, λ)-regular ultrafilter D such that X is D-compact. We show that an analogous result holds for D-pseudocompactness, provided we deal with a notion slightly stronger than $\mathcal{O}$-$[\mu, \lambda]$-compactness. See Definition 4.1 and Theorem 4.6. In particular, we provide a characterization of those spaces which are D-pseudocompact, for some (μ, λ)-regular ultrafilter D.

In the final section of this note we mention that our results generalize to the abstract framework presented in [Li4]. That is, our proofs work essentially unchanged both for pseudocompactness-like notions and for the corresponding compactness notions. In [Li4] each compactness property is defined relative to a family $\mathcal{F}$ of subsets of some topological space X. The pseudocompactness case is obtained when $\mathcal{F} = \mathcal{O}$, the family of all nonempty open sets of X. When $\mathcal{F}$ is the family of all singletons of X, we obtain results related to $[\mu, \lambda]$-compactness.

Our notation is fairly standard. Unless explicitly mentioned, we assume no separation axiom. However, the reader is warned that there are many conditions equivalent to pseudocompactness, but the equivalence holds only assuming some separation axiom (they are all equivalent only for Tychonoff spaces). For Tychonoff spaces, the particular case $\mu = \lambda = \omega$ of our definitions of $\mathcal{O}$-$[\mu, \lambda]$-compactness (Definition 2.1) turns out to be equivalent to pseudocompactness, but this is not necessarily the case for spaces with lower separation properties. See Remark 2.3.

2. A two cardinal generalization of pseudocompactness

The following definition originally appeared in [Li4] in a more general framework. The letter $\mathcal{O}$ is intended to denote the family of all the nonempty open sets of some topological space X. In this sense, the definition of $\mathcal{O}$-$[\mu, \lambda]$-compactness is the particular case $\mathcal{F} = \mathcal{O}$ of the definition of $\mathcal{F}$-$[\mu, \lambda]$-compactness in [Li4, Definition 4.2]. See also Section 5.

Definition 2.1. We say that a topological space X is $\mathcal{O}$-$[\mu, \lambda]$-*compact* if and only if the following holds.

For every sequence $(C_\alpha)_{\alpha\in\lambda}$ of closed sets of X, if, for every $Z \subseteq \lambda$ with $|Z| < \mu$, there exists a nonempty open set O_Z of X such that $\bigcap_{\alpha\in Z} C_\alpha \supseteq O_Z$, then $\bigcap_{\alpha\in\lambda} C_\alpha \neq \emptyset$.

Clearly, in the above definition, we can equivalently let O_Z vary among the (nonempty) elements of some base of X, rather than among all nonempty open sets. Also, by considering complements, we have that $\mathcal{O}$-$[\mu, \lambda]$-compactness is equivalent to the following statement.

For every λ-indexed open cover $(Q_\alpha)_{\alpha\in\lambda}$ of X, there exists $Z \subseteq \lambda$, with $|Z| < \mu$, such that $\bigcup_{\alpha\in Z} Q_\alpha$ is dense in X.

Remark 2.2. The notion of $\mathcal{O}$-$[\mu, \lambda]$-compactness should be compared with the more classical notion of $[\mu, \lambda]$-compactness.

A topological space X is $[\mu, \lambda]$-*compact* if and only if, for every sequence $(C_\alpha)_{\alpha\in\lambda}$ of closed sets of X, if $\bigcap_{\alpha\in Z} C_\alpha \neq \emptyset$, for every $Z \subseteq \lambda$ with $|Z| < \mu$, then $\bigcap_{\alpha\in\lambda} C_\alpha \neq \emptyset$.

Thus, in the definition of $[\mu, \lambda]$-compactness, we require only the weaker assumption that $\bigcap_{\alpha\in Z} C_\alpha$ is nonempty, for every $Z \subseteq \lambda$ with $|Z| < \mu$, rather than requiring that $\bigcap_{\alpha\in Z} C_\alpha$ contains some nonempty open set. In particular, every $[\mu, \lambda]$-compact space is $\mathcal{O}$-$[\mu, \lambda]$-compact.

Thus, $[\omega, \omega]$-compactness is the same as countable compactness, which is the analogue of pseudocompactness for $\mathcal{O}$-$[\mu, \lambda]$-compactness. Many of the results presented here are versions for $\mathcal{O}$-$[\mu, \lambda]$-compactness of known results about $[\mu, \lambda]$-compactness. Indeed, a simultaneous method of proof is available for both cases, and shall be mentioned in Section 5.

Notice that $[\mu, \lambda]$-compactness is a notion which encompasses both Lindelöfness (more generally, κ-final compactness) and countable compactness (more generally, κ-initial compactness). See, e. g., [Ca2, Gá, Li1, Li2, Va] and references there for further information about $[\mu, \lambda]$-compactness.

Remark 2.3. For Tychonoff spaces, $\mathcal{O}$-$[\omega,\omega]$-compactness is equivalent to pseudocompactness. Without assuming X to be Tychonoff, $\mathcal{O}$-$[\omega,\omega]$-compactness turns out to be equivalent to a condition which is usually called *feeble compactness*. See [Li4, Theorem 4.4(1) and Remark 4.5] and [St].

More generally, the particular case $\mu=\omega$ of Definition 2.1, that is, $\mathcal{O}$-$[\omega,\lambda]$-compactness, has been introduced and studied in [Fr], where it is called *almost λ-compactness*. The notion of $\mathcal{O}$-$[\omega,\lambda]$-compactness has also been studied, under different names, in [SaSt], as *weak-λ-$\aleph_0$-compactness*, and in [Re, StVa] as *weak initial λ-compactness*.

Moreover, [Fr] introduced also a notion which corresponds to $\mathcal{O}$-$[\mu,\lambda]$-compactness for all cardinals λ, calling it *almost μ-Lindelöfness*.

Assuming that X is a Tychonoff space, a property equivalent to $\mathcal{O}$-$[\kappa,\kappa]$-compactness, has been introduced in [CoNe] under the name *pseudo-(κ,κ)-compactness*. See [Li4, Theorem 4.4].

Definition 2.1 generalizes all the above mentioned notions.

See [Ar, CoNe, Fr, Ga, GiSa, Gl, Ke, Li4, Re, Sa, SaSt, ScSt, St, StVa] for the study of further related notions.

For λ, μ infinite cardinals, $S_\mu(\lambda)$ denotes the set of all subsets of λ of cardinality $<\mu$. We put $\lambda^{<\mu}=\sup_{\mu'<\mu}\lambda^{\mu'}$. Thus, $\lambda^{<\mu}$ is the cardinality of $S_\mu(\lambda)$.

In the next proposition we present some useful conditions equivalent to $\mathcal{O}$-$[\mu,\lambda]$-compactness. A further important characterization will be presented in Theorem 3.2.

Proposition 2.4. *For every topological space X and infinite cardinals λ and μ, the following are equivalent.*

1. *X is $\mathcal{O}$-$[\mu,\lambda]$-compact.*
2. *For every sequence $(P_\alpha)_{\alpha\in\lambda}$ of subsets of X, if, for every $Z\subseteq\lambda$ with $|Z|<\mu$, there exists a nonempty open set O_Z of X such that $\bigcap_{\alpha\in Z}P_\alpha\supseteq O_Z$, then $\bigcap_{\alpha\in\lambda}\overline{P}_\alpha\neq\emptyset$.*
3. *For every sequence $(Q_\alpha)_{\alpha\in\lambda}$ of open sets of X, if, for every $Z\subseteq\lambda$ with $|Z|<\mu$, there exists a nonempty open set O_Z of X such that $\bigcap_{\alpha\in Z}Q_\alpha\supseteq O_Z$, then $\bigcap_{\alpha\in\lambda}\overline{Q}_\alpha\neq\emptyset$.*
4. *For every sequence $\{O_Z\mid Z\in S_\mu(\lambda)\}$ of nonempty open sets of X, it happens that $\bigcap_{\alpha\in\lambda}\overline{\bigcup\{O_Z\mid Z\in S_\mu(\lambda),\alpha\in Z\}}\neq\emptyset$.*
5. *For every sequence $\{O_Z\mid Z\in S_\mu(\lambda)\}$ of nonempty open sets of X, the following holds. If, for every finite subset W of λ, we put $Q_W=\bigcup\{O_Z\mid Z\in S_\mu(\lambda)\text{ and }Z\supseteq W\}$, then $\bigcap\{\overline{Q_W}\mid W\text{ is a finite subset of }\lambda\}\neq\emptyset$.*

(6) *For every sequence* $\{C_Z \mid Z \in S_\mu(\lambda)\}$ *of closed sets of* X*, such that each* C_Z *is properly contained in* X*, if we let, for* $\alpha \in \lambda$*,* P_α *be the interior of* $\bigcap\{C_Z \mid Z \in S_\mu(\lambda), \alpha \in Z\}$*, then we have that* $(P_\alpha)_{\alpha\in\lambda}$ *is not a cover of* X*.*

Proof. (1) ⇒ (2) Just take $C_\alpha = \overline{P}_\alpha$, for $\alpha \in \lambda$.

(2) ⇒ (3) is trivial.

(3) ⇒ (5) The sequence $\{Q_W \mid W$ is a finite subset of $\lambda\}$ is a sequence of λ open sets of X, since there are λ finite subsets of λ.

For every $\nu < \mu$, if $(W_\beta)_{\beta\in\nu}$ is a sequence of finite subsets of λ, then $Z = \bigcup_{\beta\in\nu} W_\beta$ has cardinality $\leq \nu$, and thus belongs to $S_\mu(\lambda)$. Moreover, for each $\beta \in \nu$, we have that $Z \supseteq W_\beta$, hence $Q_{W_\beta} \supseteq O_Z$. This implies that $\bigcap_{\beta\in\nu} Q_{W_\beta} \supseteq O_Z$.

We have proved that the sequence $\{Q_W \mid W$ a finite subset of $\lambda\}$ is a sequence of λ open sets of X such that the intersection of $< \mu$ members of the sequence contains some nonempty open set of X. By applying (3) to this sequence, we have that $\bigcap\{\overline{Q_W} \mid W$ is a finite subset of $\lambda\} \neq \emptyset$.

(5) ⇒ (4) is trivial.

(4) ⇒ (1) Suppose that $(C_\alpha)_{\alpha\in\lambda}$ and O_Z, for $Z \subseteq \lambda$ with $|Z| < \mu$, are as in the premise of the definition of $\mathcal{O}$-$[\mu, \lambda]$-compactness.

For $\alpha \in \lambda$, let $C'_\alpha = \overline{\bigcup\{O_Z \mid Z \in S_\mu(\lambda), \alpha \in Z\}}$. Since C_α is closed, and $C_\alpha \supseteq O_Z$ whenever $\alpha \in Z$, we have that $C_\alpha \supseteq C'_\alpha$. By (4), $\bigcap_{\alpha\in\lambda} C'_\alpha \neq \emptyset$, hence also $\bigcap_{\alpha\in\lambda} C_\alpha \neq \emptyset$. Thus we have proved (1).

We shall also give a direct proof of (3) ⇒ (4), since it is very simple. Given the sequence $\{O_Z \mid Z \in S_\mu(\lambda)\}$ then, for every $\alpha \in \lambda$, put $Q_\alpha = \bigcup\{O_Z \mid Z \in S_\mu(\lambda), \alpha \in Z\}$. For every $Z \in S_\mu(\lambda)$, and every $\alpha \in Z$, we have that $Q_\alpha \supseteq O_Z$. Hence, for every $Z \in S_\mu(\lambda)$, we get $\bigcap_{\alpha\in Z} Q_\alpha \supseteq O_Z$, so that we can apply (3).

(4) ⇔ (6) is immediate by taking complements. □

In the particular case when $\mu = \lambda$ is regular, there are many more conditions equivalent to $\mathcal{O}$-$[\lambda, \lambda]$-compactness.

Theorem 2.5. *Suppose that* X *is a topological space, and* λ *is a regular cardinal. Then the following conditions are equivalent.*

(a) X *is* $\mathcal{O}$-$[\lambda, \lambda]$*-compact.*

(b) Suppose that $(C_\alpha)_{\alpha\in\lambda}$ *is a sequence of closed sets of* X *such that* $C_\alpha \supseteq C_\beta$*, whenever* $\alpha \leq \beta < \lambda$*. If, for every* $\alpha \in \lambda$*, there exists a nonempt open set* O *of* X *such that* $C_\alpha \supseteq O$*, then* $\bigcap_{\alpha\in\lambda} C_\alpha \neq \emptyset$*.*

(c) Suppose that $(C_\alpha)_{\alpha\in\lambda}$ *is a sequence of closed sets of* X *such that* $C_\alpha \supseteq C_\beta$*, whenever* $\alpha \leq \beta < \lambda$*. Suppose further that, for every* $\alpha \in \lambda$*,* C_α *is the closure of some open set of* X*. If, for every* $\alpha \in \lambda$*, there exists a nonempt open set* O *of* X *such that* $C_\alpha \supseteq O$*, then* $\bigcap_{\alpha\in\lambda} C_\alpha \neq \emptyset$*.*

(d) For every sequence $(O_\alpha)_{\alpha\in\lambda}$ *of nonempty open sets of* X*, there exists* $x \in X$ *such that* $|\{\alpha \in \lambda \mid U \cap O_\alpha \neq \emptyset\}| = \lambda$*, for every neighborhood* U *of* x *in* X*.*

(e) For every sequence $(O_\alpha)_{\alpha\in\lambda}$ *of nonempty open sets of* X*, there exists some ultrafilter* D *uniform over* λ *such that* $(O_\alpha)_{\alpha\in\lambda}$ *has a* D*-limit point (see Definition 3.1).*

(f) For every λ*-indexed open cover* $(O_\alpha)_{\alpha\in\lambda}$ *of* X*, such that* $O_\alpha \subseteq O_\beta$ *whenever* $\alpha \leq \beta < \lambda$*, there exists* $\alpha \in \lambda$ *such that* O_α *is dense in* X*.*

In all the above statements we can equivalently require that the elements of the sequence $(C_\alpha)_{\alpha\in\lambda}$*, respectively,* $(O_\alpha)_{\alpha\in\lambda}$*, are all distinct.*

Proof. By [Li4, Theorem 4.4], taking $\mathcal{F}$ there to be the family $\mathcal{O}$ of all the nonempty open sets of X.

Since λ is regular, the last statement is trivial, as far as conditions (b), (c) and (f) are concerned. It follows from [Li4, Proposition 3.3(a)] in case (d). Then apply [Li4, Proposition 4.1] in order to get (e). □

Remark 2.6. At this point, we should mention a significant difference between $\mathcal{O}$-$[\mu, \lambda]$-compactness and $[\mu, \lambda]$-compactness.

It is true that a topological space is $[\mu, \lambda]$-compact if and only if it is $[\kappa, \kappa]$-compact, for every κ such that $\mu \leq \kappa \leq \lambda$. Though simple, the above equivalence has proved very useful in many circumstances. See, e. g., [Li2].

It is trivial that every $\mathcal{O}$-$[\mu, \lambda]$-compact space is $\mathcal{O}$-$[\mu', \lambda']$-compact, whenever $\mu \leq \mu' \leq \lambda' \leq \lambda$. In particular, every $\mathcal{O}$-$[\mu, \lambda]$-compact space is $\mathcal{O}$-$[\kappa, \kappa]$-compact, for every κ such that $\mu \leq \kappa \leq \lambda$.

On the contrary, the condition of being $\mathcal{O}$-$[\kappa, \kappa]$-compact, for every κ such that $\mu \leq \kappa \leq \lambda$, is not always a sufficient condition in order to get $\mathcal{O}$-$[\mu, \lambda]$-compactness. See Remark 4.13. This fact limits the usefulness of Theorem 2.5 in the present context.

3. A characterization by means of ultrafilters

The first theorem in this section, Theorem 3.2, furnishes a characterization of $\mathcal{O}$-$[\mu, \lambda]$-compactness by means of the existence of D-limit points of ultrafilters. This characterization is the key for the study of the connections between $\mathcal{O}$-$[\mu, \lambda]$-compactness and D-pseudocompactness, for D a (μ, λ)-regular ultrafilter and shall be used in the next section in connection with properties of products.

Definition 3.1. Suppose that D is an ultrafilter over some set I, and X is a topological space. If $(Y_i)_{i\in I}$ is a sequence of subsets of X, then $x \in X$ is called a *D-limit point* of $(Y_i)_{i\in I}$ if and only if $\{i \in I \mid Y_i \cap U \neq \emptyset\} \in D$, for every neighborhood U of x in X. The notion of a D-limit point is due to [GiSa, Definition 4.1] for non-principal ultrafilters over ω, and appears in [Ga] for uniform ultrafilters over arbitrary cardinals.

We say that an ultrafilter D over $S_\mu(\lambda)$ *covers* λ if and only if, for every $\alpha \in \lambda$, it happens that $\{Z \in S_\mu(\lambda) \mid \alpha \in Z\} \in D$. This notion is connected with (μ, λ)-regularity, as we shall see in Definition 3.5.

Theorem 3.2. *For every topological space X and infinite cardinals λ and μ, the following are equivalent.*

1. *X is $\mathcal{O}$-$[\mu, \lambda]$-compact.*
2. *For every sequence $\{O_Z \mid Z \in S_\mu(\lambda)\}$ of nonempty open sets of X, there exists an ultrafilter D over $S_\mu(\lambda)$ which covers λ and such that $\{O_Z \mid Z \in S_\mu(\lambda)\}$ has a D-limit point.*

Proof. (1) $\Rightarrow$ (2) Suppose that $\{O_Z \mid Z \in S_\mu(\lambda)\}$ is a sequence of nonempty open sets of X. For every finite subset W of λ, let $Q_W = \bigcup\{O_Z \mid Z \in S_\mu(\lambda) \text{ and } Z \supseteq W\}$. By $\mathcal{O}$-$[\mu, \lambda]$-compactness, and Condition (5) in Proposition 2.4, we have that $\bigcap\{\overline{Q_W} \mid W \text{ a finite subset of } \lambda\} \neq \emptyset$. Suppose that $x \in \bigcap\{\overline{Q_W} \mid W \text{ a finite subset of } \lambda\}$.

For every neighborhood U of x in X, let $A_U = \{Z \in S_\mu(\lambda) \mid U \cap O_Z \neq \emptyset\}$. For every $\alpha \in \lambda$, let $[\alpha) = \{Z \in S_\mu(\lambda) \mid \alpha \in Z\}$. We are going to show that the family $\mathcal{A} = \{[\alpha) \mid \alpha \in \lambda\} \cup \{A_U \mid \text{U a neighborhood of } x \text{ in } X\}$ has the finite intersection property.

Indeed, let $U_1, \dots, U_n$ be neighborhoods of x, and $\alpha_1, \dots, \alpha_m$ be elements of λ. Let $U = U_1 \cap \dots \cap U_n$, $W = \{\alpha_1, \dots, \alpha_m\}$, and $[W) = [\alpha_1) \cap \dots \cap [\alpha_m) = \{Z \in S_\mu(\lambda) \mid Z \supseteq W\}$. Since $x \in \overline{Q_W}$, we get that $U \cap Q_W \neq \emptyset$, that is, $U \cap O_Z \neq \emptyset$, for some $Z \in S_\mu(\lambda)$ with $Z \supseteq W$. Hence $Z \in A_U$, and also $Z \in A_{U_1}$, ..., $Z \in A_{U_n}$, since $U_1 \supseteq U$, ..., $U_n \supseteq U$. In conclusion, $Z \in A_{U_1} \cap \dots \cap A_{U_n} \cap [\alpha_1) \cap \dots \cap [\alpha_m)$, hence the above intersection is not empty.

We have showed that $\mathcal{A}$ has the finite intersection property. Hence $\mathcal{A}$ can be extended to some ultrafilter D over $S_\mu(\lambda)$. By construction, $[\alpha) \in D$, for every $\alpha \in \lambda$, hence D covers λ. Again by construction, $A_U \in D$, for every neighborhood U of x in X, and this means exactly that x is a D-limit point of $\{O_Z \mid Z \in S_\mu(\lambda)\}$. Thus, (2) is proved.

In order to prove (2) $\Rightarrow$ (1), it is sufficient to prove that (2) implies Condition (4) in Proposition 2.4. Let $\{O_Z \mid Z \in S_\mu(\lambda)\}$ be a sequence of nonempty open sets of X. Letting $C_\alpha = \overline{\bigcup\{O_Z \mid Z \in S_\mu(\lambda), \alpha \in Z\}}$, for $\alpha \in \lambda$, we need too show that $\bigcap_{\alpha \in \lambda} C_\alpha \neq \emptyset$. Let D be an ultrafilter as given by (2), and suppose that x is a D-limit point of $\{O_Z \mid Z \in S_\mu(\lambda)\}$. We are going to show that $x \in \bigcap_{\alpha \in \lambda} C_\alpha$. Suppose by contradiction that, for some $\alpha \in \lambda$, it happens that $x \notin C_\alpha$. Since C_α is closed, x has some neighborhood U disjoint from C_α. Notice that, if $Z \in S_\mu(\lambda)$

and $\alpha \in Z$, then $C_\alpha \supseteq O_Z$. Hence $\{Z \in S_\mu(\lambda) \mid U \cap O_Z \neq \emptyset\} \cap [\alpha) = \emptyset$, hence $\{Z \in S_\mu(\lambda) \mid U \cap O_Z \neq \emptyset\} \notin D$, since D is an ultrafilter, and $[\alpha) \in D$ by assumption, since D is supposed to cover λ. But $\{Z \in S_\mu(\lambda) \mid U \cap O_Z \neq \emptyset\} \notin D$ contradicts the assumption that x is a D-limit point of $\{O_Z \mid Z \in S_\mu(\lambda)\}$. Hence $x \in \bigcap_{\alpha\in\lambda} C_\alpha$, thus $\bigcap_{\alpha\in\lambda} C_\alpha \neq \emptyset$, and the proof is complete. □

Remark 3.3. Theorem 3.2 is inspired by results by X. Caicedo from his seminal paper [Ca2]. See also [Ca1]. Caicedo proved results similar to Theorem 3.2 for $[\mu,\lambda]$-compactness. The result analogous to the implication (1) ⇒ (2) in Theorem 3.2 is Lemma 3.3 (i) in [Ca2]. A common generalization and strengthening of both Theorem 3.2 and [Ca2, Lemmata 3.1 and 3.2] holds. See Theorem 5.2 (1) ⇒ (7) below.

Notice that, because of the well known result about $[\mu,\lambda]$-compactness mentioned in Remark 2.6, essentially all applications of results in [Ca2] can be obtained using only the particular case $\lambda = \mu$ of [Ca2, Lemmata 3.1 and 3.2]. However, such a reduction is not possible in the case of $\mathcal{O}$-$[\mu,\lambda]$-compactness, by Remark 4.13. Hence it is necessary to deal with the more general case in which $\lambda \neq \mu$ is allowed. The idea from [Ca1, Ca2] of treating the full general case is thus well-justified

Definition 3.4. If D is an ultrafilter over I, then a topological space X is said to be *D-pseudocompact* ([GiSa, Ga]) if and only if every sequence $(O_i)_{i\in I}$ of nonempty open subsets of X has some D-limit point in X.

Definition 3.5. An ultrafilter D over some set I is said to be (μ,λ)-*regular* if and only if there is a function $f : I \to S_\mu(\lambda)$ such that $\{i \in I | \alpha \in f(i)\} \in D$, for every $\alpha \in \lambda$. See, e. g., [Li3] for equivalent definitions and for a survey of results on (μ,λ)-regular ultrafilters.

If D is an ultrafilter over I, and $f : I \to J$ is a function, the ultrafilter $f(D)$ over J is defined by the following clause: $Z \in f(D)$ if and only if $f^{-1}(Z) \in D$.

With the above notation, it is trivial to see that D over I is (μ,λ)-regular if and only if there exists some function $f : I \to S_\mu(\lambda)$ such that $f(D)$ covers λ.

In passing, let us mention that the above definitions involve the so-called Rudin-Keisler order. If D and E are two ultrafilters, respectively over I and J, then E is said to be less than or equal to D in the *Rudin-Keisler* (pre-) order, $E \leq_{\mathrm{RK}} D$ for short, if and only if there exists some function $f : I \to J$ such that $E = f(D)$. If both $E \leq_{\mathrm{RK}} D$ and $D \leq_{\mathrm{RK}} E$, then E and D are said to be (Rudin-Keisler) *equivalent.*

The next fact is trivial, but very useful.

Fact 3.6. *If D is an ultrafilter over I, X is a D-pseudocompact topological space, and $f : I \to J$ is a function, then X is $f(D)$-pseudocompact.*

Corollary 3.7. *Suppose that D is a (μ, λ)-regular ultrafilter.*

If X is a D-pseudocompact topological space, then X is $\mathcal{O}$-$[\mu, \lambda]$-compact.

More generally, if $(X_j)_{j \in J}$ is a sequence of D-pseudocompact topological spaces, then the Tychonoff product $\prod_{j \in J} X_j$ is $\mathcal{O}$-$[\mu, \lambda]$-compact.

Proof. By (μ, λ)-regularity, there is $f : I \to S_\mu(\lambda)$ such that $f(D)$ covers λ. By Fact 3.6, X is $f(D)$-pseudocompact, hence $\mathcal{O}$-$[\mu, \lambda]$-compactness of X follows from Theorem 3.2 with $f(D)$ in place of D. Notice that here $f(D)$ works "uniformly" for every sequence , while, in the statement of Theorem 3.2(2), the ultrafilter, in general, depends on the sequence.

The last statement follows from the known fact ([GiSa, Theorem 4.3]) that D-pseudocompactness is preserved under taking products. □

A result analogous to Corollary 3.7 for $[\mu, \lambda]$-compactness is proved in [Ca2, Lemma 3.1].

We now present a nice characterization of D-pseudocompactness.

Theorem 3.8. *Suppose that D is an ultrafilter over some set I, and X is a topological space. Then the following are equivalent.*

1. *X is D-pseudocompact.*
2. *For every sequence $\{O_i \mid i \in I\}$ of nonempty open sets of X, if, for $Z \in D$, we put $C_Z = \overline{\bigcup_{i \in Z} O_i}$, then we have that $\bigcap_{Z \in D} C_Z \neq \emptyset$.*
3. *Whenever $(C_Z)_{Z \in D}$ is a sequence of closed sets of X with the property that, for every $i \in I$, $\bigcap_{i \in Z} C_Z$ contains some nonempty open set of X, then $\bigcap_{Z \in D} C_Z \neq \emptyset$.*
4. *For every open cover $(Q_Z)_{Z \in D}$ of X, there is some $i \in I$ such that $\bigcup_{i \in Z} Q_Z$ is dense in X.*
5. *For every sequence $\{C_i \mid i \in I\}$ of closed sets of X, such that each C_i is properly contained in X, if, for $Z \in D$, we let Q_Z be the interior of $\bigcap_{i \in Z} C_i$, then we have that $(Q_Z)_{Z \in D}$ is not a cover of X.*

Proof. (1) ⇒ (2) By D-pseudocompactness, the sequence $\{O_i \mid i \in I\}$ has some D-limit point x in X, that is, $\{i \in I \mid U \cap O_i \neq \emptyset\} \in D$, for every neighborhood U of x in X.

We are going to show that $x \in \bigcap_{Z \in D} C_Z$. Indeed, let Z be any set in D. If U is a neighborhood of x, then $Z' = Z \cap \{i \in I \mid U \cap O_i \neq \emptyset\}$ is still in D, thus is nonempty. Let $i \in Z'$. Then $U \cap O_i \neq \emptyset$, and

$C_Z \supseteq O_i$, since $i \in Z$. Hence $U \cap C_Z \neq \emptyset$. Since the above argument works for every neighborhood U of x, we have that $x \in C_Z$, since C_Z is a closed set.

We have showed that $x \in C_Z$, for every $Z \in D$, hence $x \in \bigcap_{Z \in D} C_Z$.

(2) ⇒ (3) For every $i \in I$, let O_i be some nonempty open set of X such that $\bigcap_{i \in Z} C_Z \supseteq O_i$. For every $Z \in D$, put $C'_Z = \overline{\bigcup_{i \in Z} O_i}$. By Clause (2), we have that $\bigcap_{Z \in D} C'_Z \neq \emptyset$. Since, for every $i \in Z$, $C_Z \supseteq O_i$, we have that $C_Z \supseteq C'_Z$, for every $Z \in D$. Hence, $\bigcap_{Z \in D} C_Z \supseteq \bigcap_{Z \in D} C'_Z \neq \emptyset$.

(3) ⇒ (1) Suppose that $(O_i)_{i \in I}$ is a sequence of nonempty open sets of X. For $Z \in D$, let $C_Z = \overline{\bigcup_{i \in Z} O_i}$. Hence, for every $i \in Z$, $C_Z \supseteq O_i$, and, for every $i \in I$, $\bigcap_{i \in Z} C_Z$ contains the nonempty open set O_i.

By (3), there is some $x \in X$ such that $x \in \bigcap_{Z \in D} C_Z$. It is enough to show that x is a D-limit point of $(O_i)_{i \in I}$. If not, x has some neighborhood U such that $\{i \in I \mid U \cap O_i \neq \emptyset\} \notin D$, that is, $\{i \in I \mid U \cap O_i = \emptyset\} \in D$. Letting $Z = \{i \in I \mid U \cap O_i = \emptyset\}$, we have that $U \cap \bigcup_{i \in Z} O_i = \emptyset$, but this contradicts $x \in C_Z = \overline{\bigcup_{i \in Z} O_i}$.

(3) ⇔ (4) and (2) ⇔ (5) are obtained by considering complements. □

4. Theorems about products

In this section we consider, for a product space $\prod_{j \in J} X_j$, a variant of $\mathcal{O}$-$[\mu, \lambda]$-compactness, a variant which takes into account all the open sets in the box topology on the set $\prod_{j \in J} X_j$. This notion shall be used in order to provide a characterization of all those spaces X which are D-pseudocompact, for some (μ, λ)-regular ultrafilter D (Theorem 4.6).

We shall need to consider the *set* $\prod_{j \in J} X_j$ endowed both with the Tychonoff topology and with the *box* topology. A base for the latter topology is given by *all* the products $\prod_{j \in J} O_j$, each O_j being an open set of X_j. When we write $\prod_{j \in J} X_j$, we shall always assume that the product is endowed with the Tychonoff topology, while $\square_{j \in J} X_j$ shall denote the product endowed with the box topology.

Definition 4.1. Suppose that $(X_j)_{j \in J}$ is a sequence of topological spaces. We say that the topological space $\prod_{j \in J} X_j$ is $\mathcal{O}^{\square}$-$[\mu, \lambda]$-*compact* if and only if the following holds.

For every sequence $(C_\alpha)_{\alpha \in \lambda}$ of closed sets of $\prod_{j \in J} X_j$, if, for every $Z \subseteq \lambda$ with $|Z| < \mu$, there exists a nonempty open set O_Z of $\square_{j \in J} X_j$ such that $\bigcap_{\alpha \in Z} C_\alpha \supseteq O_Z$, then $\bigcap_{\alpha \in \lambda} C_\alpha \neq \emptyset$.

Notice that $\mathcal{O}^\Box$-$[\mu, \lambda]$-compactness is a notion stronger than $\mathcal{O}$-$[\mu, \lambda]$-compactness, that is, every $\mathcal{O}^\Box$-$[\mu, \lambda]$-compact product $\prod_{j\in J} X_j$ is $\mathcal{O}$-$[\mu, \lambda]$-compact. The two notions are distinct, in general, as we shall see in Remark 4.8. Notice also that every $[\mu, \lambda]$-compact product is $\mathcal{O}^\Box$-$[\mu, \lambda]$-compact.

Remark 4.2. Notice that $\mathcal{O}^\Box$-$[\mu, \lambda]$-compactness is not an intrinsic property of the topological space $Y = \prod_{j\in J} X_j$. That is, $\mathcal{O}^\Box$-$[\mu, \lambda]$-compactness does not only depend on the topology on Y, but depends also on the way Y is realized as a product. There might be two homeomorphic spaces, say, $Y = \prod_{j\in J} X_j$ and $Z = \prod_{h\in H} Y_h$ such that Y, *as a product* $\prod_{j\in J} X_j$, is $\mathcal{O}^\Box$-$[\mu, \lambda]$-compact, while Z, *as a product* $\prod_{h\in H} Y_h$, is not. Just to consider a simple case, if $Y = \prod_{j\in J} X_j$, and Z is a homeomorphic copy of Y, and we consider Z "as itself", that is, as the product of just a single factor, then Z is $\mathcal{O}^\Box$-$[\mu, \lambda]$-compact if and only if it is $\mathcal{O}$-$[\mu, \lambda]$-compact. On the contrary, as we shall see, $\mathcal{O}^\Box$-$[\mu, \lambda]$-compactness and $\mathcal{O}$-$[\mu, \lambda]$-compactness are distinct notions, in general.

The above remark will cause no problem here, since we will always be dealing with a space $Y = \prod_{j\in J} X_j$ together with just one single realization of Y as $\prod_{j\in J} X_j$. In other words, we shall never deal with the homeomorphism equivalence class of Y, but we shall always deal with $Y = \prod_{j\in J} X_j$ just in its concrete realization.

Of course, $\mathcal{O}^\Box$-$[\mu, \lambda]$-compactness can be characterized in a way similar to the characterizations of $\mathcal{O}$-$[\mu, \lambda]$-compactness given in Proposition 2.4. Clause (7) in the next proposition is proved as the last statement of Definition 2.1.

Proposition 4.3. *For every sequence $(X_j)_{j\in J}$ of topological spaces, and λ, μ infinite cardinals, the following are equivalent, where, in items (2)-(5), closures are computed in $\prod_{j\in J} X_j$.*

(1) *$\prod_{j\in J} X_j$ is $\mathcal{O}^\Box$-$[\mu, \lambda]$-compact.*
(2) *For every sequence $(P_\alpha)_{\alpha\in\lambda}$ of subsets of $\prod_{j\in J} X_j$, if, for every $Z \subseteq \lambda$ with $|Z| < \mu$, there exists a nonempty open set O_Z of $\Box_{j\in J} X_j$ such that $\bigcap_{\alpha\in Z} P_\alpha \supseteq O_Z$, then $\bigcap_{\alpha\in\lambda} \overline{P_\alpha} \neq \emptyset$.*
(3) *For every sequence $(Q_\alpha)_{\alpha\in\lambda}$ of open sets of $\Box_{j\in J} X_j$, if, for every $Z \subseteq \lambda$ with $|Z| < \mu$, there exists a nonempty open set O_Z of $\Box_{j\in J} X_j$ such that $\bigcap_{\alpha\in Z} Q_\alpha \supseteq O_Z$, then $\bigcap_{\alpha\in\lambda} \overline{Q_\alpha} \neq \emptyset$.*
(4) *For every sequence $\{O_Z \mid Z \in S_\mu(\lambda)\}$ of nonempty open sets of $\Box_{j\in J} X_j$, it happens that $\bigcap_{\alpha\in\lambda} \overline{\bigcup\{O_Z \mid Z \in S_\mu(\lambda), \alpha \in Z\}} \neq \emptyset$.*
(5) *For every sequence $\{O_Z \mid Z \in S_\mu(\lambda)\}$ of nonempty open sets of $\Box_{j\in J} X_j$, the following holds. If, for every finite subset W*

of λ*, we put* $Q_W = \bigcup\{O_Z \mid Z \in S_\mu(\lambda)$ *and* $Z \supseteq W\}$*, then* $\bigcap\{\overline{Q_W} \mid W$ *is a finite subset of* $\lambda\} \neq \emptyset$*.*

(6) *For every sequence* $\{C_Z \mid Z \in S_\mu(\lambda)\}$ *of closed sets of* $\Box_{j\in J}X_j$*, such that each* C_Z *is properly contained in* X*, if we let, for* $\alpha \in \lambda$*,* P_α *be the interior (computed in* $\prod_{j\in J} X_j$*) of* $\bigcap\{C_Z \mid Z \in S_\mu(\lambda), \alpha \in Z\}$*, then we have that* $(P_\alpha)_{\alpha\in\lambda}$ *is not a cover of* X*.*

(7) *For every* λ*-indexed open cover* $(Q_\alpha)_{\alpha\in\lambda}$ *of* $\prod_{j\in J} X_j$*, there exists* $Z \subseteq \lambda$*, with* $|Z| < \mu$*, such that* $\bigcup_{\alpha\in Z} Q_\alpha$ *is a dense subset in* $\Box_{j\in J}X_j$*.*

The proof of Theorem 3.2 carries over essentially unchanged in order to get the following useful theorem.

Theorem 4.4. *For every sequence* $(X_j)_{j\in J}$ *of topological spaces, and* λ*,* μ *infinite cardinals, the following are equivalent.*

(1) $\prod_{j\in J} X_j$ *is* $\mathcal{O}^\Box$*-*$[\mu, \lambda]$*-compact.*

(2) *For every sequence* $\{O_Z \mid Z \in S_\mu(\lambda)\}$ *of nonempty open sets of* $\Box_{j\in J}X_j$*, there exists an ultrafilter* D *over* $S_\mu(\lambda)$ *which covers* λ *and such that* $\{O_Z \mid Z \in S_\mu(\lambda)\}$ *has a* D*-limit point in* $\prod_{j\in J} X_j$*.*

Theorem 4.4 can be used to improve the last statement in Corollary 3.7.

Corollary 4.5. *Suppose that* D *is a* (μ, λ)*-regular ultrafilter.*

If $(X_j)_{j\in J}$ *is a sequence of* D*-pseudocompact topological spaces, then* $\prod_{j\in J} X_j$ *is* $\mathcal{O}^\Box$*-*$[\mu, \lambda]$*-compact.*

We are now going to show that a topological space X is D-pseudocompact for some (μ, λ)-regular ultrafilter D if and only if all (Tychonoff) powers of X are $\mathcal{O}^\Box$-$[\mu, \lambda]$-compact. We shall denote by X^δ the Tychonoff product of δ-many copies of X.

Theorem 4.6. *For every topological space* X*, and* λ*,* μ *infinite cardinals, the following are equivalent.*

(1) *There exists some ultrafilter* D *over* $S_\mu(\lambda)$ *which covers* λ*, and such that* X *is* D*-pseudocompact.*

(2) *There exists some* (μ, λ)*-regular ultrafilter* D *(over any set) such that* X *is* D*-pseudocompact.*

(3) *There exists some* (μ, λ)*-regular ultrafilter* D *such that, for every cardinal* δ*, the space* X^δ *is* D*-pseudocompact.*

(4) *The power* X^δ *is* $\mathcal{O}^\Box$*-*$[\mu, \lambda]$*-compact, for every cardinal* δ*.*

(5) *The power* X^δ *is* $\mathcal{O}^\Box$*-*$[\mu, \lambda]$*-compact, for* $\delta = \min\{2^{2^\kappa}, (w(X))^\kappa\}$*, where* $\kappa = \lambda^{<\mu}$*.*

Proof. (1) $\Rightarrow$ (2) is trivial, since if D is over $S_\mu(\lambda)$ and covers λ, then D is (μ, λ)-regular.

(2) $\Rightarrow$ (3) follows from the mentioned result from [GiSa, Theorem 4.3], asserting that a product of D-pseudocompact spaces is still D-pseudocompact.

(3) $\Rightarrow$ (4) follows from Corollary 4.5.

(4) $\Rightarrow$ (5) is trivial.

(5) $\Rightarrow$ (1) We first consider the case $\delta = (w(X))^\kappa$.

Let $\mathcal{B}$ be a base of X of cardinality $w(X)$. Thus, there are δ-many $S_\mu(\lambda)$-indexed sequences of elements of $\mathcal{B}$, since $|S_\mu(\lambda)| = \kappa$. Let us enumerate them as $\{Q_{\beta,Z} \mid Z \in S_\mu(\lambda)\}$, β varying in δ. In X^δ consider the sequence $\{\prod_{\beta\in\delta} Q_{\beta,Z} \mid Z \in S_\mu(\lambda)\}$. For every $Z \in S_\mu(\lambda)$, the set $\prod_{\beta\in\delta} Q_{\beta,Z}$ is open in the box topology on X^δ. By the $\mathcal{O}^\Box$-$[\mu, \lambda]$-compactness of X^δ, and by Theorem 4.4(1) $\Rightarrow$ (2), there exists an ultrafilter D over $S_\mu(\lambda)$ which covers λ and such that $\{\prod_{\beta\in\delta} Q_{\beta,Z} \mid Z \in S_\mu(\lambda)\}$ has some D-limit point x in X^δ.

We are going to show that X is D-pseudocompact. So, let $\{O_Z \mid Z \in S_\mu(\lambda)\}$ be a sequence of nonempty open sets of X. Since $\mathcal{B}$ is a base for X, then, for every $Z \in S_\mu(\lambda)$, there is a nonempty B_Z in $\mathcal{B}$ such that $O_Z \supseteq B_Z$. Choose one such B_Z for each $Z \in S_\mu(\lambda)$. The sequence $\{B_Z \mid Z \in S_\mu(\lambda)\}$ is an $S_\mu(\lambda)$-indexed sequences of elements of $\mathcal{B}$. Since, by construction, all such sequences are enumerated by $\{Q_{\beta,Z} \mid Z \in S_\mu(\lambda)\}$, there is some $\beta_0 \in \delta$ such that $B_Z = Q_{\beta_0,Z}$, for every $Z \in S_\mu(\lambda)$.

By what we have proved before, the sequence $\{\prod_{\beta\in\delta} Q_{\beta,Z} \mid Z \in S_\mu(\lambda)\}$ has some D-limit point x in X^δ, say $x = (x_\beta)_{\beta\in\delta}$. A trivial property of D-limits implies that, for every $\beta \in \delta$, we have that x_β is a D-limit of $\{Q_{\beta,Z} \mid Z \in S_\mu(\lambda)\}$. In particular, by taking $\beta = \beta_0$, we get that x_{β_0} is a D-limit point of $\{B_Z \mid Z \in S_\mu(\lambda)\}$.

Since $O_Z \supseteq B_Z$, for every $Z \in S_\mu(\lambda)$, we get that x_{β_0} is also a D-limit point of $\{O_Z \mid Z \in S_\mu(\lambda)\}$. We have proved that every sequence $\{O_Z \mid Z \in S_\mu(\lambda)\}$ of nonempty open sets of X has some D-limit point in X, that is, X is D-pseudocompact.

Now we consider the case $\delta = 2^{2^\kappa}$. We shall prove that if $\delta = 2^{2^\kappa}$ and (1) fails, then (5) fails. If (1) fails, then, for every ultrafilter D over $S_\mu(\lambda)$ which covers λ, there is a sequence $\{O_Z \mid Z \in S_\mu(\lambda)\}$ of nonempty open sets of X which has no D-limit point. Since there are δ-many ultrafilters over $S_\mu(\lambda)$, we can enumerate the above sequences as $\{O_{\beta,Z} \mid Z \in S_\mu(\lambda)\}$, β varying in δ. Now, given any ultrafilter D over $S_\mu(\lambda)$ and covering λ, it is not the case that the sequence $\{\prod_{\beta\in\delta} O_{\beta,Z} \mid Z \in S_\mu(\lambda)\}$ has some D-limit point. Indeed, were $x =$

$(x_\beta)_{\beta\in\delta}$ a D-limit point of $\{\prod_{\beta\in\delta} O_{\beta,Z} \mid Z \in S_\mu(\lambda)\}$, then, by a trivial property of D-limits, for every $\beta \in \delta$, x_β would be a D-limit point of $\{O_{\beta,Z} \mid Z \in S_\mu(\lambda)\}$. This is a contradiction since, by construction, for every ultrafilter D over $S_\mu(\lambda)$ covering λ, there exists some $\beta \in \delta$ such that $\{O_{\beta,Z} \mid Z \in S_\mu(\lambda)\}$ has no D-limit point.

We have showed that for no ultrafilter D over $S_\mu(\lambda)$ and covering λ the sequence $\{\prod_{\beta\in\delta} O_{\beta,Z} \mid Z \in S_\mu(\lambda)\}$ has some D-limit point. Since, for every $Z \in S_\mu(\lambda)$, $\prod_{\beta\in\delta} O_{\beta,Z}$ is an open set of the box topology on X^δ, we get that, by Theorem 4.4 (1) $\Rightarrow$ (2), X^δ is not $\mathcal{O}^\Box$-$[\mu,\lambda]$-compact, that is, (5) fails. □

Remark 4.7. Condition (5) in Theorem 4.6 can be improved to the effect that we can take κ there to be equal to the cofinality of the partial order $S_\mu(\lambda)$. A subset H of $S_\mu(\lambda)$ is said to be *cofinal* in $S_\mu(\lambda)$ if and only if, for every $Z \in S_\mu(\lambda)$, there is $Z' \in H$ such that $Z \subseteq Z'$. The *cofinality* $\operatorname{cf} S_\mu(\lambda)$ of $S_\mu(\lambda)$ is the minimal cardinality of some subset H cofinal in $S_\mu(\lambda)$. Notice that if λ is regular, then $\operatorname{cf} S_\lambda(\lambda) = \lambda$ and, more generally, $\operatorname{cf} S_\lambda(\lambda^+) = \lambda^+$. Highly non trivial results about $\operatorname{cf} S_\mu(\lambda)$ are consequences of Shelah's pcf-theory [Sh].

For the rest of this remark, let us fix some subset H cofinal in $S_\mu(\lambda)$.

All the definitions and results involving $S_\mu(\lambda)$ can be modified in order to apply to H, too. In particular, in the definitions of $\mathcal{O}$-$[\mu,\lambda]$-compactness and of $\mathcal{O}^\Box$-$[\mu,\lambda]$-compactness, we get an equivalent notion if we consider only those $Z \in H$. Similarly, in Propositions 2.4 and 4.3 we can equivalently consider H-indexed sequences, rather than $S_\mu(\lambda)$-indexed sequences, that is, we can replace everywhere $Z \in S_\mu(\lambda)$ by $Z \in H$, still obtaining the results.

Moreover, we can say that an ultrafilter D over H covers λ if and only if, for every $\alpha \in \lambda$, it happens that $[\alpha)_H = \{Z \in H \mid \alpha \in Z\} \in D$. With this definition, we have that Theorems 3.2 and 4.4, too, hold, if $Z \in S_\mu(\lambda)$ is everywhere replaced by $Z \in H$.

Moreover, let $f : S_\mu(\lambda) \to H$ be defined in such a way that $Z \subseteq f(Z)$. If D is over $S_\mu(\lambda)$ and covers λ, then $f(D)$ is over (a subset of) H, and $f(D)$, too, covers λ. The above observations give us the possibility of proving Theorem 4.6 with the improved value $\kappa = \operatorname{cf} S_\mu(\lambda)$ in Condition (5).

Remark 4.8. In order to get results like Theorem 4.6, it is actually necessary to deal with $\mathcal{O}^\Box$-$[\mu,\lambda]$-compactness, rather than with $\mathcal{O}$-$[\mu,\lambda]$-compactness. Indeed, [GiSa, Example 4.4] constructed a Tychonoff space X such that all powers of X are pseudocompact but for no ultrafilter D uniform over ω, X is D-pseudocompact. By Remark 2.3, all powers of X are $\mathcal{O}$-$[\omega, \omega]$-compact. The condition that, for no ultrafilter D uniform over ω, X is D-pseudocompact is easily seen to be equivalent to the property that for no ultrafilter D over $S_\omega(\omega)$ and covering ω, X is D-pseudocompact. The equivalence can be proved directly; otherwise, notice that, for $\mu = \lambda$ a regular cardinal, Condition (4) in Theorem 4.6 coincides with Condition (5) in [Li4, Corollary 5.5], hence the respective Conditions (1) are equivalent.

Since, for no ultrafilter D over $S_\omega(\omega)$ and covering ω, X is D-pseudocompact, we get, by Theorem 4.6, that not every power of X is $\mathcal{O}^\Box$-$[\omega, \omega]$-compact, but, as we remarked, every power of X is $\mathcal{O}$-$[\omega, \omega]$-compact, thus the two notions are distinct, in general. Indeed, by Remark 4.7, we have that X^δ is not $\mathcal{O}^\Box$-$[\omega, \omega]$-compact for $\delta = 2^{2^\omega}$.

In particular, Conditions (4) and (5) in Theorem 4.6 are in general not equivalent to the other conditions, if we replace $\mathcal{O}^\Box$-$[\mu, \lambda]$-compactness with $\mathcal{O}$-$[\mu, \lambda]$-compactness.

Indeed, as is the case for pseudocompactness, we can show that the $\mathcal{O}$-$[\mu, \lambda]$-compactness of a product depends only on the $\mathcal{O}$-$[\mu, \lambda]$-compactness of all subproducts of some small number of factors. Thus, we have an analogue for $\mathcal{O}$-$[\mu, \lambda]$-compactness of the equivalence (4) $\Leftrightarrow$ (5) in Theorem 4.6.

Lemma 4.9. *If X and Y are topological spaces, $f : X \to Y$ is a continuous and surjective function, and X is $\mathcal{O}$-$[\mu, \lambda]$-compact then also Y is $\mathcal{O}$-$[\mu, \lambda]$-compact.*

Theorem 4.10. *Suppose that $(X_j)_{j \in J}$ is a sequence of topological spaces. Then the product $\prod_{j \in J} X_j$ is $\mathcal{O}$-$[\mu, \lambda]$-compact if and only if any subproduct of $\leq \kappa$ factors is $\mathcal{O}$-$[\mu, \lambda]$-compact, where $\kappa = \lambda^{<\mu}$. Indeed, the result can be improved to $\kappa = \operatorname{cf} S_\mu(\lambda)$.*

Proof. The only-if part is immediate from Lemma 4.9.

To prove the converse, given $(C_\alpha)_{\alpha \in \lambda}$ as in the definition of $\mathcal{O}$-$[\mu, \lambda]$-compactness, we might assume, without loss of generality, that the O_Z's are members of the canonical base of $\prod_{j \in J} X_j$, that is, each O_Z has the form $\prod_{j \in J} Q_j$, where each Q_j is an open set of X_j, and $Q_j = X_j$, for all $j \in J \setminus J_Z$, for some finite $J_Z \subseteq J$.

If $J' = \bigcup_{Z \in S_\mu(\lambda)} J_Z$, and $\pi : \prod_{j \in J} X_j \to \prod_{j \in J'} X_j$ is the canonical projection, then, by assumption, $\prod_{j \in J'} X_j$ is $\mathcal{O}$-$[\mu, \lambda]$-compact, since $|J'| \leq \kappa$, hence $\bigcap_{\alpha \in \lambda} \pi(C_\alpha) \neq \emptyset$, and this clearly implies $\bigcap_{\alpha \in \lambda} C_\alpha \neq \emptyset$.

By arguments similar to those in Remark 4.7, we can improve the value of κ to $\operatorname{cf} S_\mu(\lambda)$. $\Box$

For sake of simplicity, in the statement of Theorem 4.6 we have dealt with a single topological space X. However, a version of the theorem holds for families of topological spaces.

Theorem 4.11. *For every family T of topological spaces, and λ, μ infinite cardinals, the following are equivalent.*

(1) *There exists some (μ, λ)-regular ultrafilter D (which can be taken over $S_\mu(\lambda)$) such that, for every $X \in T$, we have that X is D-pseudocompact.*
(2) *Every product of any number of members of T (allowing repetitions) is $\mathcal{O}^\square$-$[\mu, \lambda]$-compact.*
(3) *Every product of members of T (allowing repetitions) with at most δ factors is $\mathcal{O}^\square$-$[\mu, \lambda]$-compact, where $\delta = \min\{2^{2^\kappa}, \sup\{|T|, \nu\}\}$, for $\nu = \sup_{X \in T}(w(X))^\kappa$ and $\kappa = \lambda^{<\mu}$ (indeed, this can be improved to $\kappa = \operatorname{cf} S_\mu(\lambda)$).*

Corollary 4.12. *For μ, λ, μ' and λ' infinite cardinals, the following are equivalent.*

(a) Every (μ, λ)-regular ultrafilter is (μ', λ')-regular.

(b) For every family T of topological spaces, if every product of any number of members of T (allowing repetitions) is $\mathcal{O}^\square$-$[\mu, \lambda]$-compact, then every product of any number of members of T (allowing repetitions) is $\mathcal{O}^\square$-$[\mu', \lambda']$-compact.

(c) For every topological space X, if every power of X is $\mathcal{O}^\square$-$[\mu, \lambda]$-compact, then every power of X is $\mathcal{O}^\square$-$[\mu', \lambda']$-compact.

(d) Same as (c), restricted to Tychonoff spaces.

Proof. (a) $\Rightarrow$ (b) Suppose that the assumption in (b) holds. By Theorem 4.11 (2) $\Rightarrow$ (1), there exists some (μ, λ)-regular ultrafilter D such that, for every $X \in T$, we have that X is D-pseudocompact. By (a), D is (μ', λ')-regular. Hence, by Theorem 4.11 (1) $\Rightarrow$ (2), every product of any number of members of T is $\mathcal{O}^\square$-$[\mu', \lambda']$-compact.

(b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (d) are trivial.

(d) $\Rightarrow$ (a) Garcia-Ferreira [Ga] constructs, for every ultrafilter D, say over I, a Tychonoff space $P_{RK}(D)$ such that, for every ultrafilter E, say over J, the space $P_{RK}(D)$ is E-pseudocompact if and only if $E = f(D)$ for some function $f : I \to J$, that is if and only if $E \leq_{\mathrm{RK}} D$ in the Rudin-Keisler order.

Let D be a (μ, λ)-regular ultrafilter, say over I. By above, $X = P_{RK}(D)$ is D-pseudocompact, hence, by Theorem 4.6 (2) $\Rightarrow$ (4), every power of X is $\mathcal{O}^\square$-$[\mu, \lambda]$-compact.

By (d), every power of X is $\mathcal{O}^\square$-$[\mu', \lambda']$-compact and, by Theorem 4.6 (2) $\Rightarrow$ (4), X is E-pseudocompact, for some (μ', λ')-regular ultrafilter E

over some set J. By the above-mentioned result from [Ga], $E = f(D)$, for some function $f : I \to J$. By a trivial property of the Rudin-Keisler order, D is (μ', λ')-regular, thus (a) is proved. □

Many results are known about cardinals for which Condition (a) in Corollary 4.12 holds. See [Li3] for a survey. Corollary 4.12 can be applied in each of these cases.

Remark 4.13. As we mentioned in Remark 2.6, $[\mu, \lambda]$-compactness is equivalent to $[\kappa, \kappa]$-compactness for every κ such that $\mu \leq \kappa \leq \lambda$. We now show that the analogous result fails, in general, for $\mathcal{O}$-$[\mu, \lambda]$-compactness.

Under some set-theoretical assumption, [Ka] constructed an ultrafilter D uniform over ω_1 and an ultrafilter D' over ω such that, for every ultrafilter E, it happens that $E \leq_{\mathrm{RK}} D$ if and only if E is Rudin-Keisler equivalent either to D or to D'. By the results from [Ga] mentioned in the proof of Corollary 4.12, the space $P_{RK}(D)$ is both D-pseudocompact and D'-pseudocompact, hence both $\mathcal{O}$-$[\omega, \omega]$-compact and $\mathcal{O}$-$[\omega_1, \omega_1]$-compact, since every uniform ultrafilter over some cardinal λ is (λ, λ)-regular (see, e. g., [Li3]). Indeed, by Corollary 4.5, all powers of $P_{RK}(D)$ are even both $\mathcal{O}^{\Box}$-$[\omega, \omega]$-compact and $\mathcal{O}^{\Box}$-$[\omega_1, \omega_1]$-compact.

However, [Ga] proved that $P_{RK}(D)$ is not even ω_1-pseudocompact. Since, by [Re, Theorem 2(c)], every $\mathcal{O}$-$[\omega, \lambda]$-compact Tychonoff space is λ-pseudocompact, we have that $P_{RK}(D)$ is not $\mathcal{O}$-$[\omega, \omega_1]$-compact ($\mathcal{O}$-$[\omega, \lambda]$-compact spaces are called weakly-initially compact in [Re]).

5. The abstract framework

In this final section we mention that our results actually hold in the general framework introduced in [Li4]. In [Li4] each compactness property is defined relative to some family $\mathcal{F}$ of subsets of a topological space X. By taking $\mathcal{F}$ to be either the set of all singletons of X, or the set of all nonempty open sets of X, this generalized approach provides a unified treatment of definitions and results about $[\mu, \lambda]$-compactness and related compactness notions, on one side, and about $\mathcal{O}$-$[\mu, \lambda]$-compactness and related pseudocompactness-like notions, on the other side.

In the case of $[\mu, \lambda]$-compactness, as we shall point after each single result, most of the theorems we get are known; in the case when $\mathcal{F} = \mathcal{O}$ we usually get back the results obtained in the previous sections.

Definition 5.1. The definitions of $\mathcal{F}$-$[\mu,\lambda]$-compactness and of $\mathcal{F}$-D-compactness can be obtained, respectively, from the definitions of $\mathcal{O}$-$[\mu,\lambda]$-compactness (Definition 2.1) and of D-pseudocompactness (Definition 3.4), by replacing the family $\mathcal{O}$ off all nonempty open sets with another specified family $\mathcal{F}$ of subsets of X.

In more detail, let X be a topological space, and let $\mathcal{F}$ be any family of subsets of X.

Let λ and μ be infinite cardinals. We say that X is $\mathcal{F}$-$[\mu,\lambda]$-*compact* if and only if, for every sequence $(C_\alpha)_{\alpha\in\lambda}$ of closed sets of X, if, for every $Z \subseteq \lambda$ with $|Z| < \mu$, there exists $F \in \mathcal{F}$ such that $\bigcap_{\alpha\in Z} C_\alpha \supseteq F$, then $\bigcap_{\alpha\in\lambda} C_\alpha \neq \emptyset$.

Let D be an ultrafilter over some set Z. We say that X is $\mathcal{F}$-D-*compact* if and only if every sequence $(F_z)_{z\in Z}$ of members of $\mathcal{F}$ has some D-limit point in X.

When, in the preceding definitions, $\mathcal{F} = \mathcal{O}$, the family of all the nonempty open sets of X, we get back Definitions 2.1 and 3.4. When $\mathcal{F}$ is taken to be the family of all singletons of X, we get back the more familiar notions of, respectively, $[\mu,\lambda]$-compactness and of D-compactness. See [Li4] for more information. In particular, notice that, for $\mu = \lambda$ a regular cardinal, [Li4] provides a very refined theory of $\mathcal{F}$-$[\lambda,\lambda]$-compactness. In the particular case $\mu = \lambda$ regular, the results presented in [Li4] are usually stronger than the results presented here for $\mathcal{F}$-$[\mu,\lambda]$-compactness. Notice also that, by Remark 4.13, the theory of $\mathcal{F}$-$[\mu,\lambda]$-compactness, in general, cannot be "reduced" to the theory of $\mathcal{F}$-$[\kappa,\kappa]$-compactness. On the contrary, it is a very useful fact that $[\mu,\lambda]$-compactness can be studied in terms of $[\kappa,\kappa]$-compactness, for $\mu \leq \kappa \leq \lambda$ (Remark 2.6).

Notice that if X is realized as a Tychonoff product $\prod_{j\in J} X_j$, then $\mathcal{O}^\Box$-$[\mu,\lambda]$-compactness, as introduced in Definition 4.1, is the same as $\mathcal{F}$-$[\mu,\lambda]$-compactness of $\prod_{j\in J} X_j$, when we take $\mathcal{F}$ to be the family of all open sets in $\Box_{j\in J} X_j$, that is, the open sets in the box topology.

If $\mathcal{F}$ is a family of subsets of some topological space, we denote by $\bigvee \mathcal{F}$ (resp., $\bigvee_{\leq\kappa} \mathcal{F}$), the family of all subsets of X which can be obtained as the union of the members of some subfamily of $\mathcal{F}$ (resp., some subfamily of cardinality $\leq \kappa$).

Theorem 5.2. *Suppose that X is a topological space, $\mathcal{F}$ is a family of subsets of X, and λ and μ are infinite cardinals. Then the following are equivalent.*

(1) *X is $\mathcal{F}$-$[\mu,\lambda]$-compact.*

(2) *For every sequence* $(P_\alpha)_{\alpha\in\lambda}$ *of subsets of* X*, if, for every* $Z \subseteq \lambda$ *with* $|Z| < \mu$*, there exists some* $F_Z \in \mathcal{F}$ *such that* $\bigcap_{\alpha\in Z} P_\alpha \supseteq F_Z$*, then* $\bigcap_{\alpha\in\lambda} \overline{P}_\alpha \neq \emptyset$.

(3) *For every sequence* $(Q_\alpha)_{\alpha\in\lambda}$ *of sets in* $\bigvee \mathcal{F}$ *(equivalently, in* $\bigvee \mathcal{F}_{\leq\kappa}$*, where* $\kappa = \lambda^{<\mu}$*), if, for every* $Z \subseteq \lambda$ *with* $|Z| < \mu$*, there exists some* $F_Z \in \mathcal{F}$ *such that* $\bigcap_{\alpha\in Z} Q_\alpha \supseteq F_Z$*, then* $\bigcap_{\alpha\in\lambda} \overline{Q}_\alpha \neq \emptyset$*. The value of* κ *can be improved to* $\operatorname{cf} S_\mu(\lambda)$.

(4) *For every sequence* $\{F_Z \mid Z \in S_\mu(\lambda)\}$ *of members of* $\mathcal{F}$*, it happens that* $\bigcap_{\alpha\in\lambda} \overline{\bigcup\{F_Z \mid Z \in S_\mu(\lambda), \alpha \in Z\}} \neq \emptyset$.

(5) *For every sequence* $\{F_Z \mid Z \in S_\mu(\lambda)\}$ *of members of* $\mathcal{F}$*, the following holds. If, for every finite subset* W *of* λ*, we put* $Q_W = \bigcup\{F_Z \mid Z \in S_\mu(\lambda)$ *and* $Z \supseteq W\}$*, then* $\bigcap\{\overline{Q_W} \mid W$ *is a finite subset of* $\lambda\} \neq \emptyset$.

(6) *For every* λ*-indexed open cover* $(Q_\alpha)_{\alpha\in\lambda}$ *of* X*, there exists* $Z \subseteq \lambda$*, with* $|Z| < \mu$*, such that* $F \cap \bigcup_{\alpha\in Z} Q_\alpha \neq \emptyset$*, for every* $F \in \mathcal{F}$.

(7) *For every sequence* $\{F_Z \mid Z \in S_\mu(\lambda)\}$ *of elements of* $\mathcal{F}$*, there exists an ultrafilter* D *over* $S_\mu(\lambda)$ *which covers* λ *and such that* $\{F_Z \mid Z \in S_\mu(\lambda)\}$ *has some* D*-limit point in* X.

Proof. Same as the proofs of Proposition 2.4, of the last remark in Definition 2.1 and of Theorem 3.2. See also Remark 4.7. □

Proposition 2.4 and Theorem 3.2 can be obtained as the particular case of Theorem 5.2, when $\mathcal{F} = \mathcal{O}$ is the family of the nonempty open sets of X.

Proposition 4.3 and Theorem 4.4 can be obtained as the particular case of Theorem 5.2, when X is the topological space $\prod_{j\in J} X_j$ (with the Tychonoff topology), and $\mathcal{F}$ is the family of the nonempty open sets of $\square_{j\in J} X_j$ (with the box topology).

Thus, Theorem 5.2 provides a generalization of all the above results.

As we mentioned in Remark 3.3, in the particular case when $\mathcal{F}$ is the family $\mathcal{S}$ of all singletons, the implication (1) ⇒ (7) in Theorem 5.2 is proved in [Ca1, Ca2]. Again when $\mathcal{F} = \mathcal{S}$, the equivalence of (1) and (2) in Theorem 5.2 has been proved in [Gá], with different notation. See also [Va, Lemma 5(b)].

Theorem 5.3. *Suppose that* X *is a topological space,* $\mathcal{F}$ *is a family of subsets of* X*, and* D *is an ultrafilter over some set* I*. Then the following are equivalent.*

(1) X *is* $\mathcal{F}$*-*D*-compact.*

(2) *For every sequence* $\{F_i \mid i \in I\}$ *of members of* $\mathcal{F}$*, if, for* $Z \in D$*, we put* $C_Z = \overline{\bigcup_{i\in Z} F_i}$*, then we have that* $\bigcap_{Z\in D} C_Z \neq \emptyset$.

(3) *Whenever* $(C_Z)_{Z \in D}$ *is a sequence of closed sets of* X *with the property that, for every* $i \in I$*, there exists some* $F \in \mathcal{F}$ *such that* $\bigcap_{i \in Z} C_Z \supseteq F$*, then* $\bigcap_{Z \in D} C_Z \neq \emptyset$*.*
(4) *For every open cover* $(O_Z)_{Z \in D}$ *of* X*, there is some* $i \in I$ *such that* $F \cap \bigcup_{i \in Z} O_Z \neq \emptyset$*, for every* $F \in \mathcal{F}$*.*

Proof. Similar to the proof of Theorem 3.8. □

Theorem 3.8 could be obtained as the particular case $\mathcal{F} = \mathcal{O}$ of Theorem 5.3.

The particular case of Theorem 5.3 when $\mathcal{F}$ is the set of all singletons of X might be new, so we state it explicitly.

Corollary 5.4. *Suppose that* X *is a topological space, and* D *is an ultrafilter over some set* I*. Then the following are equivalent.*

(1) X *is* D*-compact.*
(2) *For every sequence* $\{x_i \mid i \in I\}$ *of elements of* X*, if, for* $Z \in D$*, we put* $C_Z = \overline{\{x_i \mid i \in Z\}}$*, then we have that* $\bigcap_{Z \in D} C_Z \neq \emptyset$*.*
(3) *Whenever* $(C_Z)_{Z \in D}$ *is a sequence of closed sets of* X *with the property that, for every* $i \in I$*,* $\bigcap_{i \in Z} C_Z \neq \emptyset$*, then* $\bigcap_{Z \in D} C_Z \neq \emptyset$*.*
(4) *For every open cover* $(O_Z)_{Z \in D}$ *of* X*, there is some* $i \in I$ *such that* $(O_Z)_{i \in Z}$ *is a cover of* X*.*

Theorem 5.5. *Suppose that* λ *and* μ *are infinite cardinals,* T *is a family of topological spaces, and, for every* $X \in T$*,* $\mathcal{F}_X$ *is a family of subsets of* X*.*

To every product $\prod_{j \in J} X_j$*, where each* X_j *belongs to* T*, associate the family* $\mathcal{F} = \{\prod_{j \in J} F_j \mid F_{X_j} \in \mathcal{F}_j$*, for every* $j \in J\}$*.*

Then the following are equivalent.

(1) *There exists some ultrafilter* D *over* $S_\mu(\lambda)$ *which covers* λ*, and such that, for every* $X \in T$*, we have that* X *is* $\mathcal{F}_X$*-*D*-compact.*
(2) *There exists some* (μ, λ)*-regular ultrafilter* D *(over any set) such that, for every* $X \in T$*, we have that* X *is* $\mathcal{F}_X$*-*D*-compact.*
(3) *There exists some* (μ, λ)*-regular ultrafilter* D *such that, for every set* J*, every product* $\prod_{j \in J} X_j$ *of members of* T *(allowing repetitions) is* $\mathcal{F}$*-*D*-compact.*
(4) *For every set* J*, every product* $\prod_{j \in J} X_j$ *of members of* T *(allowing repetitions), is* $\mathcal{F}$*-*$[\mu, \lambda]$*-compact.*
(5) *Let* $\delta = \min\{2^{2^\kappa}, \sup\{|T|, \sup_{X \in T} |\mathcal{F}_X|^\kappa\}$*, where* $\kappa = \lambda^{<\mu}$ *(indeed, this can be improved to* $\kappa = \operatorname{cf} S_\mu(\lambda)$*). For every set* J *with* $|J| \leq \delta$*, every product* $\prod_{j \in J} X_j$ *of members of* T *(allowing repetitions) is* $\mathcal{F}$*-*$[\mu, \lambda]$*-compact.*

Proof. Same as the proofs of Corollary 3.7 and of Theorem 4.6, using [Li4, Fact 6.1 and Proposition 5.1 (b) with $\nu = |J^+|$] and Theorem 5.2 (7) $\Leftrightarrow$ (1). For (5), see also Remark 4.7. □

Theorem 5.5 is more general than Theorems 4.6 and 4.11. In the particular case when $\mathcal{F}$ is the family $\mathcal{S}$ of all singletons, Theorem 5.5 is essentially [Ca2, Theorem 3.4] (in some cases, our evaluation of δ might be slightly sharper). Corollaries 3.7 and 4.5 are immediate consequences of Theorem 5.5 (2) $\Rightarrow$ (4), by taking, for every $j \in J$, $\mathcal{F}_j$ to be the family of all nonempty open sets of X_j.

The following easy proposition, generalizing Lemma 4.9, describes the behavior of $\mathcal{F}$-D-compactness with respect to quotients.

Proposition 5.6. *Suppose that X and Y are topological spaces, and $f : X \to Y$ is a continuous function. Suppose that $\mathcal{F}$ is a family of subsets of X, and suppose that $\mathcal{G}$ is a family of subsets of Y, such that for every $G \in \mathcal{G}$ there is $F \in \mathcal{F}$ such that $F \subseteq f^{-1}(G)$.*

Then the following hold.

(1) *If X is $\mathcal{F}$-$[\mu, \lambda]$-compact then Y is $\mathcal{G}$-$[\mu, \lambda]$-compact.*
(2) *If X is $\mathcal{F}$-D-compact then Y is $\mathcal{G}$-D-compact.*

We end with a trivial but useful property of $\mathcal{F}$-$[\mu, \lambda]$-compactness.

Proposition 5.7. *Every $\mathcal{F}$-$[\mathrm{cf}\,\lambda, \mathrm{cf}\,\lambda]$-compact topological space is $\mathcal{F}$-$[\lambda, \lambda]$-compact.*

In particular, every $\mathcal{O}$-$[\mathrm{cf}\,\lambda, \mathrm{cf}\,\lambda]$-compact topological space is $\mathcal{O}$-$[\lambda, \lambda]$-compact.

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