# INITIAL $\lambda$-COMPACTNESS IN LINEARLY ORDERED SPACES

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Abstract. We show that a linearly ordered topological space is initially $\lambda$-compact if and only if it is $\lambda$-bounded, that is, every set of cardinality $\leq \lambda$ has compact closure. As a consequence, every product of initially $\lambda$-compact linearly ordered topological spaces is initially $\lambda$-compact.

A topological space is *initially $\lambda$-compact* if every open cover by at most $\lambda$ sets has a finite subcover. According to a celebrated theorem, Stephenson and Vaughan [SV, Theorem 1.1], if $\lambda$ is a strong limit singular cardinal, then every product of initially $\lambda$-compact topological spaces is still initially $\lambda$-compact. We prove a much stronger result for products whose factors are linearly ordered topological spaces: for such spaces, the above theorem holds for *every* infinite cardinal $\lambda$. In fact, our proof works for *generalized ordered spaces*, for short, *GO spaces*, that is, Hausdorff spaces equipped with a linear order and with a base of order-convex sets. See, e. g., Bennet and Lutzer [BL] for more information about GO spaces.

We shall prove a chain of equivalences which involve several notions, such as $\lambda$-boundedness, $D$-compactness, $D$-pseudocompactness, conditions asking for the existence of "complete accumulation points" of sequences of open sets, and a condition simply asking that strictly increasing or decreasing sequences indexed by a regular cardinal converge. To state our theorem in such a full generality we need to recall some definitions. If $D$ is an ultrafilter over some set $I$, then a topological space $X$ is said to be *$D$-compact* if every $I$-indexed sequence $(x_i)_{i\in I}$ of elements of $X$ *$D$-converges* to some $x \in X$, that is, $\{i \in I \mid x_i \in U\} \in D$, for every open neighborhood $U$ of $x$. The space $X$ is said to be *$D$-pseudocompact* if every $I$-indexed sequence $(O_i)_{i\in I}$ of nonempty open subsets of $X$ has some *$D$-limit point* in $X$, that is, there is some $x \in X$ such that $\{i \in I \mid U \cap O_i \neq \emptyset\} \in D$, for every open neighborhood $U$ of $x$. If $\beta$ is a limit ordinal, we say that a sequence $(x_\gamma)_{\gamma<\beta}$ of elements of a topological space *converges* to some point $x$ if, for every neighborhood $U$ of $x$, there is $\gamma < \beta$ such that $x_{\gamma'} \in U$, for every $\gamma' > \gamma$.

**Theorem.** *For every infinite cardinal $\lambda$, and every GO space $X$, the following conditions are equivalent.*

(1) *$X$ is initially $\lambda$-compact.*
(2) *$X$ is* weakly initially $\lambda$-compact, *that is, every open cover of $X$ by at most $\lambda$ sets has a finite subcollection with dense union.*
(3) *For every infinite (equivalently, every infinite regular) cardinal $\nu \leq \lambda$, and every family $(O_\gamma)_{\gamma<\nu}$ of $\nu$ open nonempty sets of $X$, there is $x \in X$ such that $|\{\gamma < \nu \mid O_\gamma \cap U \neq \emptyset\}| = \nu$, for every neighborhood $U$ of $x$.*

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2010 *Mathematics Subject Classification.* 54F05, 54A20, 54D20, 54B10.
*Key words and phrases.* Linearly ordered, generalized ordered, initially $\lambda$-compact, $D$-(pseudo)-compact, $\lambda$-bounded topological space; product; ultrafilter.

*In the above condition we can equivalently ask either that the $O_\gamma$'s are pairwise disjoint, or that $O_\gamma \subseteq O_{\gamma'}$, for $\gamma > \gamma'$.*

(4) *For every infinite regular cardinal $\nu \leq \lambda$, and every strictly increasing (resp., strictly decreasing) $\nu$-indexed sequence of elements of $X$, the sequence has a supremum (resp., an infimum) to which it converges.*

(5) *$X$ is $D$-compact, for every ultrafilter $D$ over any set of cardinality $\leq \lambda$.*

(6) *$X$ is $D$-pseudocompact, for every ultrafilter $D$ over any set of cardinality $\leq \lambda$.*

(7) *$X$ is* $\lambda$-bounded, *that is, every subset of cardinality $\leq \lambda$ has compact closure.*

*Proof.* We shall first prove the following chain of equivalences: (1) $\Rightarrow$ (2) $\Rightarrow$ $(3)_{\text{reg}}$ $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1), where by $(3)_{\text{reg}}$ we denote the condition (3) restricted to regular $\nu$'s. By the way, notice that the implication (1) $\Rightarrow$ (4) is trivial, hence the reader interested only in the proof of the equivalence of (1), (4), (5) and (7) could skip the next three passages.

(1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3) is known, and true for every topological space. First, we prove here (2) $\Rightarrow$ $(3)_{\text{reg}}$. Suppose that (2) holds, and that the conclusion of $(3)_{\text{reg}}$ fails. Then, since $\nu$ is regular, for every $x \in X$ we can choose an open neighborhood $U_x$ of $x$ and some $\delta_x < \nu$ such that $O_\gamma \cap U_x = \emptyset$, for every $\gamma > \delta_x$. For every $\delta < \nu$, let $V_\delta = \bigcup_{\delta_x = \delta} U_x$. Thus $(V_\delta)_{\delta < \nu}$ is an open cover of $X$ by $\leq \lambda$ sets, hence, by (2), it has a finite subcollection with dense union, say, $V_{\delta_1}, \dots, V_{\delta_n}$. If $\gamma = \sup\{\delta_1, \dots, \delta_n\} + 1$, then $O_\gamma \cap (V_{\delta_1} \cup \dots \cup V_{\delta_n}) = \emptyset$, a contradiction, since $O_\gamma$ is nonempty.

$(3)_{\text{reg}}$ $\Rightarrow$ (4) is easy. Suppose that $\nu$ is a regular cardinal, and that $(x_\gamma)_{\gamma < \nu}$ is, say, a strictly increasing sequence. For $\gamma < \nu$, define $O_\gamma = (x_\gamma, x_{\gamma+2}) = \{x \in X \mid x_\gamma < x < x_{\gamma+2}\}$. The $O_\gamma$'s are open and nonempty, since $x_{\gamma+1} \in O_\gamma$. It is immediate to see that the $x$ given by $(3)_{\text{reg}}$ is a supremum of $(x_\gamma)_{\gamma < \nu}$ to which the sequence converges.

If the open sets in $(3)_{\text{reg}}$ are required to be disjoint, simply take only the "even" above sets, namely, for $\gamma = \alpha + n$, with $\alpha = 0$ or $\alpha$ limit, let $O_\gamma = (x_{\alpha+2n}, x_{\alpha+2n+2})$.

If the sequence of open sets in $(3)_{\text{reg}}$ is required to be $\subseteq$-decreasing, take $O_\gamma = \bigcup_{\gamma' > \gamma} (x_\gamma, x_{\gamma'})$. Thus the proof of $(3)_{\text{reg}}$ $\Rightarrow$ (4) is complete in each case.

Next, we concentrate on the proof of (4) $\Rightarrow$ (5). Suppose that (4) holds, and that, without loss of generality, $D$ is an ultrafilter over some cardinal $\kappa \leq \lambda$. Let $f : \kappa \to X$ be a function: we have to show that $(f(\gamma))_{\gamma < \kappa}$ $D$-converges in $X$. For some ordinal $\kappa' \leq \kappa + 1$, we shall construct inductively two sequences $(l_\alpha)_{\alpha < \kappa'}$ and $(r_\alpha)_{\alpha < \kappa'}$ of elements of $X \cup \{-\infty, \infty\}$, where, as usual, $-\infty$ and $\infty$ are two new elements intended to satisfy $-\infty < x < \infty$, for every $x \in X$. At a certain point the construction ends, giving a point to which $(f(\gamma))_{\gamma < \kappa}$ $D$-converges.

The sequences will satisfy the following properties.

(a) $l_\gamma \leq l_\alpha$, whenever $\gamma < \alpha < \kappa'$, that is, $(l_\alpha)_{\alpha < \kappa'}$ is increasing;
(b) $r_\alpha \leq r_\gamma$, whenever $\gamma < \alpha < \kappa'$, that is, $(r_\alpha)_{\alpha < \kappa'}$ is decreasing;
(c) $f^{-1}((l_\alpha, r_\alpha)) \in D$, for every $\alpha < \kappa'$ (hence, in particular, $l_\alpha \leq r_\alpha$, for every $\alpha < \kappa'$);
(d) $f(\gamma) \notin (l_\alpha, r_\alpha)$, whenever $\gamma < \alpha < \kappa'$, and $\gamma < \kappa$.

For $\alpha = 0$, take $l_0 = -\infty$ and $r_0 = \infty$; Clause (c) is trivially satisfied, and the other conditions are vacuously true.

Suppose that $\beta > 0$, and that $(l_\alpha)_{\alpha<\beta}$ and $(r_\alpha)_{\alpha<\beta}$ have been constructed satisfying the above properties.

If $\beta = \alpha + 1$ is a successor ordinal, and $f(\alpha) \notin (l_\alpha, r_\alpha)$, simply take $l_\beta = l_\alpha$ and $r_\beta = r_\alpha$; (a)-(d) are trivially satisfied.

If $\beta = \alpha + 1$, and, instead, $f(\alpha) \in (l_\alpha, r_\alpha)$, let $x = f(\alpha)$. Observe that $(l_\alpha, r_\alpha) = (l_\alpha, x) \cup \{x\} \cup (x, r_\alpha)$, hence, since $D$ is an ultrafilter and, by (c), $f^{-1}((l_\alpha, r_\alpha)) \in D$, then either (i) $f^{-1}(\{x\}) \in D$, or (ii) $f^{-1}((l_\alpha, x)) \in D$, or (iii) $f^{-1}((x, r_\alpha)) \in D$. In the first eventuality, $(f(\gamma))_{\gamma<\kappa}$ trivially $D$-converges to $x$, the construction ends, and we have obtained the desired conclusion. In the second case, put $l_\beta = l_\alpha$ and $r_\beta = x$. Since $x = f(\alpha) < r_\alpha$, then Clause (b) is satisfied, and all the other conditions are satisfied by construction and by the inductive hypothesis. Symmetrically, in the third case, put $l_\beta = x$, and $r_\beta = r_\alpha$; as above, (a)-(d) are satisfied.

To complete the induction, we have to consider the case when $\beta$ is a limit ordinal; say, $\operatorname{cf} \beta = \nu$; notice that $\nu$ is a regular cardinal, and that $\nu \leq \lambda$, since $\beta < \kappa' \leq \kappa+1$, hence $\beta \leq \kappa \leq \lambda$. The sequence $(l_\alpha)_{\alpha<\beta}$ is either eventually constant, or, since $\nu$ is regular, has a strictly increasing subsequence of length $\nu$ and unbounded in $\{l_\alpha\}_{\alpha<\beta}$. In this latter case, Condition (4) asserts the existence of a supremum to which the subsequence converges, and hence also the sequence $(l_\alpha)_{\alpha<\beta}$ converges to it, since $(l_\alpha)_{\alpha<\beta}$ is increasing, by Clause (a). If this is the case, let $l_\beta$ be such a supremum. If, instead, $(l_\alpha)_{\alpha<\beta}$ is eventually constant, let $l_\beta$ be the value on which $(l_\alpha)_{\alpha<\beta}$ becomes constant. Notice that, in both cases, $(l_\alpha)_{\alpha<\beta}$ converges to $l_\beta$ (unless $l_\alpha = -\infty$ constantly, for $\alpha \leq \beta$). Symmetrically, $(r_\alpha)_{\alpha<\beta}$ is either eventually constant, or has a strictly decreasing subsequence of length $\nu$ unbounded in $\{r_\alpha\}_{\alpha<\beta}$. Let $r_\beta$ be either the above constant value, or the infimum given by Condition (4). If $(f(\gamma))_{\gamma<\kappa}$ $D$-converges either to $l_\beta$ or to $r_\beta$, we are done, and we can stop the construction. Otherwise, unless either $l_\beta = -\infty$ or $r_\beta = \infty$, there are a convex neighborhood $L$ of $l_\beta$, and a convex neighborhood $R$ of $r_\beta$ such that $f^{-1}(L) \notin D$, and $f^{-1}(R) \notin D$. Since $(l_\alpha)_{\alpha<\beta}$ converges to $l_\beta$, there is $\alpha < \beta$ such that $l_\alpha \in L$. We have that $f^{-1}((-\infty, l_\beta]) \notin D$, since, $L$ being convex, $(-\infty, l_\beta] \subseteq (-\infty, l_\alpha] \cup L$, and, moreover, $f^{-1}((-\infty, l_\alpha]) \notin D$, by (c), and $f^{-1}(L) \notin D$. Symmetrically, $f^{-1}([r_\beta, \infty)) \notin D$, hence necessarily $f^{-1}((l_\beta, r_\beta)) \in D$, since $D$ is an ultrafilter, and $X = (-\infty, l_\beta] \cup (l_\beta, r_\beta) \cup [r_\beta, \infty)$. Thus Clause (c) is satisfied for $\beta$. All the other conditions are trivial. The cases in which either $l_\beta = -\infty$, or $r_\beta = \infty$ are treated in a similar way, and, in fact, are simpler (we just need only the "right part", or the "left part" of the above arguments).

Now observe that the construction cannot be completed up to stage $\kappa' = \kappa + 1$. Indeed, by (d), at stage $\kappa + 1$ we would have $f^{-1}((l_\kappa, r_\kappa)) = \emptyset$, contradicting (c). Thus the construction ends at stage $\kappa + 1$, or before, and we showed that when the construction ends we get some element to which $(f(\gamma))_{\gamma<\kappa}$ $D$-converges.

The proof of the implication (4) $\Rightarrow$ (5) is thus complete.

(5) $\Rightarrow$ (1) is nowadays a well-known standard argument, and, in fact, the implication holds for every topological space. See, e. g., [St, implication (7) in Diagram 3.6].

We have proved the equivalence of (1), (2), $(3)_{\text{reg}}$, (4) and (5). Now notice that (5) $\Rightarrow$ (6) $\Rightarrow$ (3) $\Rightarrow$ $(3)_{\text{reg}}$ are trivial: to show (6) $\Rightarrow$ (3), just consider, for every $\nu \leq \lambda$, some uniform ultrafilter over $\nu$.

Finally, the equivalence of (5) and (7) is well-known, and holds for every Hausdorff regular space, [Sa, Theorems 5.3 and 5.4] (recall that it can be proved that every GO space is regular). □

**Corollary.** *Suppose that $X$ is a product of topological spaces, and that all factors but at most one are GO spaces. Then $X$ is initially $\lambda$-compact if and only if each factor is initially $\lambda$-compact.*

*Proof.* One implication is trivial. For the other direction, by the equivalence of (1) and (7) in the Theorem, all but at most one factor are $\lambda$-bounded. It is well-known that a product of regular $\lambda$-bounded spaces is still $\lambda$-bounded [St, Theorem 5.7 and implications (1), (1′) in Diagram 3.6], and that a product of a $\lambda$-bounded space with an initially $\lambda$-compact space is initially $\lambda$-compact [Sa, Theorem 5.2 and implications (1), (2) in Diagram 3.6]. Hence the corollary follows by first grouping together the GO spaces, and then, in case, multiplying their product with the possibly non GO factor. □

The particular cases of the above Theorem and Corollary when $\lambda = \omega$ appeared in Sanchis and Tamariz-Mascarúa [STM, Section 2], or are immediate consequences of the statements there.

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