Decomposable ultrafilters and possible cofinalities

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Abstract  We use Shelah’s theory of possible cofinalities in order to solve some problems about ultrafilters.

Theorem 1  Suppose that $\lambda$ is a singular cardinal, $\lambda' < \lambda$, and the ultrafilter $D$ is $\kappa$-decomposable for all regular cardinals $\kappa$ with $\lambda' < \kappa < \lambda$. Then $D$ is either $\lambda$-decomposable, or $\lambda^+$-decomposable.

Corollary 2  If $\lambda$ is a singular cardinal, then an ultrafilter is $(\lambda, \lambda)$-regular if and only if it is either $\text{cf} \lambda$-decomposable or $\lambda^+$-decomposable.

We give applications to topological spaces and to abstract logics (Corollaries 8, 9 and Theorem 10).

If $F$ is a family of subsets of some set $I$, and $\lambda$ is an infinite cardinal, a $\lambda$-decomposition for $F$ is a function $f : I \to \lambda$ such that whenever $X \subseteq \lambda$ and $|X| < \lambda$ then $\{i \in I | f(i) \in X\} \notin F$. The family $F$ is $\lambda$-decomposable if and only if there is a $\lambda$-decomposition for $F$. If $D$ is an ultrafilter (that is, a maximal proper filter) let us define the decomposability spectrum $K_D$ of $D$ by $K_D = \{\lambda \geq \omega | D$ is $\lambda$-decomposable\}.

The question of the possible values the spectrum $K_D$ may take is particularly intriguing. Even the old problem from [P; Si] of characterizing those cardinals $\mu$ for which there is an ultrafilter $D$ such that $K_D = \{\omega, \mu\}$ is not yet completely solved [Shr, p. 1007].

The case when $K_D$ is infinite is even more involved. [P] studied the situation in which $\lambda$ is limit and $K_D \cap \lambda$ is unbounded in $\lambda$; he found some assumptions which imply that $\lambda \in K_D$. This is not always the case; if $\mu$ is strongly compact and $\text{cf} \lambda < \mu < \lambda$ then there is an ultrafilter $D$ such that $K_D \cap \lambda$ is unbounded in $\lambda$, and $D$ is not $\lambda$-decomposable. If we are in the

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above situation, \( D \) is necessarily \( \lambda^+ \)-decomposable (by [So, Lemma 3] and the proof of [P, Proposition 2]).

The above examples suggest the problem (implicit in [P]) whether \( K_D \cap \lambda \) unbounded in \( \lambda \) implies that either \( \lambda \in K_D \) or \( \lambda^+ \in K_D \). In general, the problem is still open; here we solve it affirmatively in the particular case when there is \( \lambda' < \lambda \) such that \( K_D \) contains all regular cardinals in the interval \([\lambda', \lambda)\). This is sufficient for all applications we know of: see Corollaries 2, 7, 8, 9, and Theorem 10.

We briefly review some known results on \( K_D \). If \( \kappa \) is regular and \( \kappa^+ \in K_D \), then \( \kappa \in K_D \). If \( \kappa \in K_D \) is singular, then \( \kappa \in K_D \). Results from [D] imply that if there is no inner model with a measurable cardinal then \( K_D \) is always an interval with minimum \( \omega \). On the other hand, it is trivial that \( K_D = \{\mu\} \) if and only if \( \mu \) is either \( \omega \) or a measurable cardinal. If a measurable cardinal \( \mu \) is made singular by Prikry forcing, then in the resulting model we have an ultrafilter \( D \) such that \( K_D = \{\omega, \mu\} \). Further comments and constraints on \( K_D \) are given in [L3; L4]. Apparently, the problem of determining which sets of cardinals can be represented as \( K_F = \{\lambda \geq \omega | F \) is \( \lambda \)-decomposable\} for a filter \( F \) has not been studied.

If \( (\lambda_j)_{j \in J} \) are regular cardinals, the cofinality of \( \prod_{j \in J} \lambda_j \) of the product \( \prod_{j \in J} \lambda_j \) is the smallest cardinality of a set \( G \subseteq \prod_{j \in J} \lambda_j \) having the property that for every \( f \in \prod_{j \in J} \lambda_j \) there is \( g \in G \) such that \( f(j) \leq g(j) \) for all \( j \in J \).

We shall state our results in a quite general form, involving arbitrary filters, rather than ultrafilters. In what follows, the reader interested in ultrafilters only can always assume that \( F \) is an ultrafilter.

**Proposition 3** \( (\lambda_j)_{j \in J} \) are infinite regular cardinals, \( \mu = \text{cf} \prod_{j \in J} \lambda_j \) and the filter \( F \) is \( \lambda_j \)-decomposable for all \( j \in J \), then \( F \) is \( \mu' \)-decomposable for some \( \mu' \) with \( \text{sup}_{j \in J} \lambda_j \leq \mu' \leq \mu \).

**Proof** Let \( F \) be over \( I \), and let \((g_\alpha)_{\alpha \in \mu} \) witness \( \mu = \text{cf} \prod_{j \in J} \lambda_j \). For every \( j \in J \) let \( f(j, -) : I \to \lambda_j \) be a \( \lambda_j \)-decomposition for \( F \). For any fixed \( i \in I \), \( f(-, i) \in \prod_{j \in J} \lambda_j \), thus there is \( \alpha(i) \in \mu \) such that \( f(j, i) \leq g_{\alpha(i)}(j) \) for all \( j \in J \).

Let \( X \) be a subset of \( \mu \) with minimal cardinality with respect to the property that \( Y = \{i \in I | \alpha(i) \in X\} \subseteq F \). Let \( \mu' = |X| \). Thus, whenever \( X' \subseteq \mu \) and \( |X'| < \mu' \), we have \( Y' = \{i \in I | \alpha(i) \in X'\} \subseteq F \). Define \( h(i) = \alpha(i) \) for \( i \in Y \), and \( h(i) = 0 \) for \( i \notin Y \). Thus, \( h : I \to X \cup \{0\} \).

If \( |X'| < \mu' \) then \( \{i \in I | h(i) \in X'\} \subseteq Y' \cup (I \setminus Y) \notin F \) (otherwise, since \( F \) is a filter, \( Y' \subseteq Y \cap (Y' \cup (I \setminus Y)) \subseteq F \), contradiction). This shows that, modulo a bijection from \( X \cup \{0\} \) onto \( \mu' \), \( h \) is a \( \mu' \)-decomposition for \( F \). Trivially, \( \mu' \leq \mu \).

Hence, it remains to show that \( \text{sup}_{j \in J} \lambda_j \leq \mu' \). Suppose to the contrary that \( \mu' < \lambda_j \) for some \( j \in J \). Then \( |\{g_{\alpha(i)}(j) | i \in Y\}| \leq |\{\alpha(i) | \alpha(i) \in X\}| \leq |X| = \mu' < \lambda_j \). Since \( \lambda_j \) is regular, we have that \( \beta = \text{sup}_{i \in Y} g_{\alpha(i)}(j) < \lambda_j \). Hence, if \( i \in Y \), then \( f(\hat{j}, i) \leq g_{\alpha(i)}(\hat{j}) \leq \beta < \lambda_j \). Thus, \( \{0, \beta\} < \lambda_j \), but \( \{i \in I | f(\hat{j}, i) \in [0, \beta]\} \subseteq Y \in F \), and this contradicts the assumption that \( f(\hat{j}, -) \) is a \( \lambda_j \) decomposition for \( F \). \( \Box \)
Proposition 3 has not the most general form: we have results dealing with the cofinality $\mu$ of reduced products $\prod E \lambda_j$, where $E$ a filter on $J$. We shall not need this more general version here.

Recall from [She] that if $a$ is a set of regular cardinals, then $pcf a$ is the set of regular cardinals which can be obtained as $\prod E a$, for some ultrafilter $E$ on $a$.

**Corollary 4**  If $a$ is a set of infinite regular cardinals, $|a|^+ < \min a$, and the filter $F$ is $\lambda$-decomposable for all $\lambda \in a$, then $F$ is $\mu'$-decomposable for some $\mu'$ with $\sup a \leq \mu' \leq \max pcf a$.

**Proof** By [She, II, Lemma 3.1], if $|a|^+ < \min a$ then $\max pcf a = \prod a \lambda$, thus the conclusion is immediate from Proposition 3. □

Recall that an ultrafilter $D$ is $(\mu, \lambda)$-regular if and only if there is a family of $\lambda$ members of $D$ such that the intersection of any $\mu$ members of the family is empty. We list below the properties of decomposability and regularity we shall need. Much more is known: see [DD; F], [W, p. 427-431] for recent results. See [L2; L4] for more references.

**Properties 5**  
(a) Every $\lambda$-decomposable ultrafilter is $cf \lambda$-decomposable.  
(b) Every $cf \lambda$-decomposable ultrafilter is $(\lambda, \lambda)$-regular.  
(c) If $\mu' \geq \mu$ and $\lambda' \leq \lambda$ then every $(\mu, \lambda)$-regular ultrafilter is $(\mu', \lambda')$-regular.  
(d) [CC, Theorem 1] [KP, Theorem 2.1] If $\lambda$ is singular, $D$ is a $\lambda^+$-decomposable ultrafilter, and $D$ is not $cf \lambda$-decomposable then $D$ is $(\lambda', \lambda^+)$-regular for some $\lambda' < \lambda$.  
(e) [K, Corollary 2.4] If $\lambda$ is singular then every $\lambda^+$-decomposable ultrafilter is either $cf \lambda$-decomposable or $(\lambda', \lambda)$-regular for some $\lambda' < \lambda$.  
(f) [L1, Corollary 1.4] If $\lambda$ is singular then every $(\lambda, \lambda)$-regular ultrafilter is either $cf \lambda$-decomposable or $(\lambda', \lambda)$-regular for some $\lambda' < \lambda$.  
(g) If $\lambda$ is regular then an ultrafilter is $\lambda$-decomposable if and only if it is $(\lambda, \lambda)$-regular.

**Theorem 6**  Suppose that $\lambda$ is a singular cardinal, $F$ is a filter, and either  
(a) there is $\lambda' < \lambda$ such that $F$ is $\kappa^+$-decomposable for all regular cardinals $\kappa$ with $\lambda' < \kappa < \lambda$, or  
(b) $cf \lambda > \omega$ and $S = \{ \kappa < \lambda | F \text{ is } \kappa^+\text{-decomposable} \}$ is stationary in $\lambda$.  
Then $F$ is either $\lambda$-decomposable, or $\lambda^+$-decomposable.  
If $F = D$ is an ultrafilter, then $D$ is $(\lambda, \lambda)$-regular. Moreover, $D$ is either  
(i) $\lambda$-decomposable, or  
(ii) $(\lambda', \lambda^+)$-regular for some $\lambda' < \lambda$, or  
(iii) $cf \lambda$-decomposable and $(\lambda, \lambda^+)$-regular.

**Proof**  If $cf \lambda = \nu > \omega$ then by [She, II, Claim 2.1] there is a sequence $(\lambda_\alpha)_{\alpha \in \nu}$ closed and unbounded in $\lambda$ and such that, letting $a = \{ \lambda_\alpha \mid \alpha \in \nu \}$, we have $\lambda^+ = \max pcf a$. If $cf \lambda = \omega$ then we have $\lambda^+ = \max pcf a$ for some $a$ of order type $\omega$ unbounded in $\lambda$ as a consequence of [She, II, Theorem 1.5] (since $a$ has order type $\omega$, any ultrafilter over $a$ is either principal, or extends the dual of the ideal of bounded subsets of $a$).
Letting $b = a \cap [\lambda', \lambda)$ in case (a), and $b = a \cap \{\kappa | \kappa \in S\}$ in case (b), we still have $\text{max pcf} b = \lambda^+$, because $b$ is unbounded in $\lambda$, hence $\text{max pcf} b \geq \lambda^+$, and because $\text{max pcf} b \leq \text{max pcf} a = \lambda^+$, since $b \subseteq a$.

Assume, without loss of generality, that $\lambda' > (\text{cf} \lambda)^+$ in (a), and that $\inf S > (\text{cf} \lambda)^+$ in (b). Since $|b| \leq |a| = \text{cf} \lambda$, then $|b|^+ < \min b$, hence Corollary 4 with $b$ in place of $a$ implies that $F$ is either $\lambda$-decomposable, or $\lambda^+$-decomposable.

The last statements follow from Properties 5(a)-(e). □

**Corollary 7** If $\lambda$ is a singular cardinal and the ultrafilter $D$ is not $\text{cf} \lambda$-decomposable, then the following conditions are equivalent:

(a) There is $\lambda' < \lambda$ such that $D$ is $\kappa$-decomposable for all regular cardinals $\kappa$ with $\lambda' < \kappa < \lambda$.

(a') (Only in case $\text{cf} \lambda > \omega$) $\{\kappa < \lambda | F^+ \text{ is } \kappa^+\text{-decomposable}\}$ is stationary in $\lambda$.

(b) $D$ is $\lambda^+$-decomposable.

(c) There is $\lambda' < \lambda$ such that $D$ is $(\lambda', \lambda^+)$-regular.

(d) $D$ is $(\lambda, \lambda)$-regular.

(e) There is $\lambda' < \lambda$ such that $D$ is $(\lambda', \lambda)$-regular.

(f) There is $\lambda' < \lambda$ such that $D$ is $(\lambda'', \lambda''')$-regular for every $\lambda''$ with $\lambda' < \lambda'' < \lambda$.

**Proof** (a) $\Rightarrow$ (b) and (a') $\Rightarrow$ (b) are immediate from Theorem 6 and Property 5(a). In case $\text{cf} \lambda > \omega$, (a) $\Rightarrow$ (a') is trivial.

(b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (f) $\Rightarrow$ (a) are given, respectively, by Properties 5(d)(c)(f)(c)(g). □

**Proof of Corollary 2** Immediate from Corollary 7(d)$\Rightarrow$(b) and Properties 5(b)-(d). □

A topological space is $[\mu, \lambda]$-compact if and only if every open cover by $\lambda$ many sets has a subcover by $< \mu$ many sets. A family $\mathcal{F}$ of topological spaces is productively $[\mu, \lambda]$-compact if and only if every (Tychonoff) product of members of $\mathcal{F}$ (allowing repetitions) is $[\mu, \lambda]$-compact.

**Corollary 8** If $\lambda$ is a singular cardinal, then a family of topological spaces is productively $[\lambda, \lambda]$-compact if and only if it is either productively $[\text{cf} \lambda, \text{cf} \lambda]$-compact or productively $[\lambda^+, \lambda^+]$-compact.

**Proof** [C, Theorem 1.7] proved that, for every infinite cardinals $\mu$ and $\lambda$, a family $\mathcal{F}$ of topological spaces is productively $[\mu, \lambda]$-compact if and only if there exists a ($\mu, \lambda$)-regular ultrafilter $D$ such that every member of $\mathcal{F}$ is $D$-compact (see [C] for the definition and references). The corollary is then immediate from Corollary 2, using Property 5(g). □

Henceforth, by a **logic**, we mean a **regular logic** in the sense of [E]. Typical examples of regular logics are infinitary logics, or extensions of first-order logic obtained by adding new quantifiers; e. g., cardinality quantifiers asserting “there are at least $\omega_\alpha x$’s such that . . . ”.

A logic $L$ is $[\lambda, \mu]$-compact if and only if for every pair of sets $\Gamma$ and $\Sigma$ of sentences of $L$, if $|\Sigma| \leq \lambda$ and if $\Gamma \cup \Sigma'$ has a model for every $\Sigma' \subseteq \Sigma$ with
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\(|\Sigma| < \mu\), then \(\Gamma \cup \Sigma\) has a model (see [C; M] for some history and further comments).

**Corollary 9** If \(\lambda\) is a singular cardinal, then a logic is \([\lambda, \lambda]\)-compact if and only if it is either \([\cf \lambda, \cf \lambda]\)-compact or \([\lambda^+, \lambda^+]\)-compact.

**Proof** J. Makowski and S. Shelah defined what it means for an ultrafilter to be related to a logic, and showed that a logic \(\mathcal{L}\) is \([\lambda, \mu]\)-compact if and only if there exists some \((\mu, \lambda)\)-regular ultrafilter related to \(\mathcal{L}\) (see [M, Theorem I.4.4]; notice that the order of the parameters is reversed in the definition of \((\lambda, \mu)\)-regularity as given by [M]). The corollary is then immediate from Corollary 2 and Property 5(g).

**Theorem 10** Suppose that \((\lambda_i)_{i \in I}\) and \((\mu_j)_{j \in J}\) are sets of infinite cardinals. Then the following are equivalent:

(i) For every \(i \in I\) there is a \((\lambda_i, \lambda_i)\)-regular ultrafilter which for no \(j \in J\) is \((\mu_j, \mu_j)\)-regular.

(ii) There is a logic which is \([\lambda_i, \lambda_i]\)-compact for every \(i \in I\), and which for no \(j \in J\) is \([\mu_j, \mu_j]\)-compact.

(iii) For every \(i \in I\) there is a \([\lambda_i, \lambda_i]\)-compact logic which for no \(j \in J\) is \([\mu_j, \mu_j]\)-compact.

The logics in (ii) and (iii) can be chosen to be generated by at most \(2 \cdot |J|\) cardinality quantifiers (at most \(|J|\) cardinality quantifiers if all \(\mu_j\)'s are regular).

**Proof** In the case when all the \(\mu_j\)'s are regular, the theorem is proved in [L1, Theorem 4.1]. The general case follows from the above particular case, by applying Corollaries 2 and 9.

**References**


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