COMPACTNESS OF POWERS OF ω

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Abstract. We characterize exactly the compactness properties of the product of κ copies of the space ω with the discrete topology. The characterization involves uniform ultrafilters, the existence of certain nonstandard elements, and infinitary languages. We also have results involving products of possibly uncountable regular cardinals.

Mycielski [My], extending previous results by Ehrenfeucht, Erdös, Hajnal, Łoś and Stone, showed that ω^κ is not (finally) κ-compact, for every infinite cardinal κ strictly less than the first weakly inaccessible cardinal. Here ω denotes a countable topological space with the discrete topology; products (and powers) are endowed with the Tychonoff topology, and a topological space is said to be *finally κ-compact* if any open cover has a subcover of cardinality strictly less than κ.

On the other direction, Mrówka [Mr1, Mr2] showed that if $\mathcal{L}_{\omega_1,\omega}$ is (κ,κ)-compact, then ω^κ is indeed finally κ-compact (in particular, this holds if κ is weakly compact). As usual, $\mathcal{L}_{\lambda,\mu}$ is the *infinitary language* which allows conjunctions and disjunctions of $< \lambda$ formulas, and universal or existential quantification over $< \mu$ variables; (κ,κ)-*compactness* means that any κ-satisfiable set of $|\kappa|$-many sentences is satisfiable.

To the best of our knowledge, the gap between Mycielski's and Mrówka's results has never been exactly filled. It follows from [Mr2, Theorem 1] and Čudnovskiĭ [Ču, Theorem 2] that $\mathcal{L}_{\kappa,\omega}$ is (κ,κ)-compact if and only if every product of $|\kappa|$-many discrete spaces, each of cardinality $< \kappa$, is finally κ-compact (the proofs build also on work by Hanf, Keisler, Monk, Scott, Tarski, Ulam and others; earlier versions and variants were known under inaccessibility conditions). No matter how satisfying the above result is, it adds nothing about powers of ω, since it deals with possibly uncountable factors.

In this note we show that Mrówka gives the exact estimation, namely, that ω^κ is finally κ-compact if and only if $\mathcal{L}_{\omega_1,\omega}$ is (κ,κ)-compact. More

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generally, we find necessary and sufficient conditions for ω^κ being finally λ-compact, or, even, just being $[\lambda, \lambda]$-compact. Our methods involve intermediate steps of independent interest, dealing with uniform ultrafilters and extensions of models by means of "λ-nonstandard" elements. The equivalences we find in such intermediate steps hold for arbitrary regular cardinals, not only for ω; in particular, compactness properties of products of regular cardinals (with the order topology) are characterized.

Throughout, λ, μ, κ and ν are infinite cardinals, X is a topological space, and D is an ultrafilter. Cardinals are also considered as topological spaces endowed with the order topology.

The space X is $[\mu, \lambda]$-*compact* if every open cover of X by at most λ sets has a subcover by less than μ sets. It is easy to show that final κ-compactness is equivalent to $[\nu, \nu]$-compactness, for every $\nu \geqslant \kappa$, or, more generally, that $[\mu, \lambda]$-compactness is equivalent to $[\nu, \nu]$-compactness, for every ν such that $\mu \leqslant \nu \leqslant \lambda$. If λ is regular, a space X is $[\lambda, \lambda]$-compact if and only if every subset of X of cardinality λ has a complete accumulation point. If D is an ultrafilter over some set I, a sequence $(x_i)_{i \in I}$ of elements of X is said to *D-converge* to $x \in X$ if $\{i \in I \mid x_i \in U\} \in D$, for every open neighborhood U of x. If $f : I \to J$ is a function, $f(D)$ is the ultrafilter over J defined by $Y \in f(D)$ if and only if $f^{-1}(Y) \in D$.

Definition 1. We shall denote by $\lambda \not\Rightarrow (\mu_\gamma)_{\gamma \in \kappa}$ the following statement.

(*) For every sequence of functions $(f_\gamma)_{\gamma \in \kappa}$, such that $f_\gamma : \lambda \to \mu_\gamma$ for $\gamma \in \kappa$, there is some uniform ultrafilter D over λ such that, for no $\gamma \in \kappa$, $f_\gamma(D)$ is uniform over μ_γ.

We shall write $\lambda \overset{\kappa}{\not\Rightarrow} \mu$ when all the μ_γ's in (*) are equal to μ.

The negation of $\lambda \overset{\kappa}{\not\Rightarrow} \mu$ is denoted by $\lambda \overset{\kappa}{\Rightarrow} \mu$.

The following observation by Saks [Sa, Fact (i) on pp. 80–81] (building also on ideas of Bernstein and Ginsburg) will play a fundamental role in the present note. We shall assume that λ is regular, so that we do not need the assumption that sequences are faithfully indexed. See Caicedo [Ca, Section 3] for a variation for the case when λ is singular.

Proposition 2. [Sa] *If λ is regular, then X is $[\lambda, \lambda]$-compact if and only if, for every sequence $(x_\alpha)_{\alpha \in \lambda}$ of elements of X, there is an ultrafilter D uniform over λ such that $(x_\alpha)_{\alpha \in \lambda}$ D-converges to some $x \in X$.*

Theorem 3. *If λ and $(\mu_\gamma)_{\gamma \in \kappa}$ are regular cardinals, then $\prod_{\gamma \in \kappa} \mu_\gamma$ is $[\lambda, \lambda]$-compact if and only if $\lambda \not\Rightarrow (\mu_\gamma)_{\gamma \in \kappa}$.*

Proof. Let $X = \prod_{\gamma \in \kappa} \mu_\gamma$, and, for $\gamma \in \kappa$, let $\pi_\gamma : X \to \mu_\gamma$ be the natural projection. A sequence of functions as in the first line of (*) can be naturally identified with a sequence $(x_\alpha)_{\alpha \in \lambda}$ of elements of X, by posing $\pi_\gamma(x_\alpha) = f_\gamma(\alpha)$. By Proposition 2, X is $[\lambda, \lambda]$-compact if and only if, for every sequence $(x_\alpha)_{\alpha \in \lambda}$ of elements of X, there is an ultrafilter D uniform over λ such that $(x_\alpha)_{\alpha \in \lambda}$ D-converges in X. As well known, this happens if and only if, for each $\gamma \in \kappa$, $(\pi_\gamma(x_\alpha))_{\alpha \in \lambda}$ D-converges in μ_γ, and this happens if and only if, for each $\gamma \in \kappa$, there is $\delta_\gamma \in \mu_\gamma$ such that $\{\alpha \in \lambda \mid \pi_\gamma(x_\alpha) < \delta_\gamma\} \in D$. Under the mentioned identification, and since every μ_γ is regular, this means exactly that each $f_\gamma(D)$ fails to be uniform over μ_γ. $\square$

We now consider models of the form $\mathfrak{A} = \langle \lambda, <, \alpha, \dots \rangle_{\alpha \in \lambda}$ (here, by abuse of notation, we do not distinguish between a symbol and its interpretation). If $\mathfrak{B} \equiv \mathfrak{A}$ (that is, $\mathfrak{B}$ is *elementarily equivalent* to $\mathfrak{A}$), we say that $b \in B$ is λ-*nonstandard* if $\alpha < b$ holds in $\mathfrak{B}$, for every $\alpha \in \lambda$. Similarly, for $\mu < \lambda$, we say that $c \in B$ is μ-*nonstandard* if $c < \mu$ and $\beta < c$ hold in $\mathfrak{B}$, for every $\beta \in \mu$. Of course, in the case $\lambda = \omega$, we get the usual notion of a nonstandard element. The importance of λ-nonstandard elements in Model Theory has been stressed by C. C. Chang and H. J. Keisler; see [Ch, pp. 115–118]. (About the terminology: a μ-nonstandard element c in the above sense is said to *realize* μ in [Ch], and to *bound* μ in [Li1].)

Theorem 4. *If $\mu \leqslant \lambda$ are regular cardinals and $\kappa \geqslant \lambda$, then $\lambda \overset{\kappa}{\not\Rightarrow} \mu$ if and only if, for every expansion $\mathfrak{A}$ of $\langle \lambda, <, \alpha \rangle_{\alpha \in \lambda}$ with at most κ new symbols (equivalently, symbols and sorts), there is $\mathfrak{B} \equiv \mathfrak{A}$ such that $\mathfrak{B}$ has a λ-nonstandard element but no μ-nonstandard element.*

Proof. Suppose $\lambda \overset{\kappa}{\not\Rightarrow} \mu$ and let $\mathfrak{A}$ be an expansion of $\langle \lambda, <, \alpha \rangle_{\alpha \in \lambda}$ with at most κ new symbols and sorts. Without loss of generality, we can assume that $\mathfrak{A}$ has Skolem functions, since this adds at most $\kappa \geqslant \lambda$ new symbols. Enumerate as $(f_\gamma)_{\gamma \in \kappa}$ all the functions from λ to μ which are definable in $\mathfrak{A}$ (repeat occurrences, if necessary), and let D be the ultrafilter given by $\lambda \overset{\kappa}{\not\Rightarrow} \mu$. Let $\mathfrak{C}$ be the ultrapower $\prod_D \mathfrak{A}$. Since D is uniform over λ, $b = [Id]_D$, the D-class of the identity on λ, is a λ-nonstandard element in $\mathfrak{C}$. Let $\mathfrak{B}$ be the Skolem hull of $\{b\}$ in $\mathfrak{C}$; thus $\mathfrak{B} \equiv \mathfrak{C} \equiv A$, and b is a λ-nonstandard element of $\mathfrak{B}$. Had $\mathfrak{B}$ a μ-nonstandard element c, there would be $\gamma \in \kappa$ such that $c = f_\gamma(b)$, by the definition of $\mathfrak{B}$. Thus $c = f_\gamma([Id]_D) = [f_\gamma]_D$, but this would imply that $f_\gamma(D)$ is uniform over μ (since μ is regular), contradicting the choice of D.

For the converse, suppose that $(f_\gamma)_{\gamma\in\kappa}$ is a sequence of functions from λ to μ. Let $\mathfrak{A}$ be the expansion of $\langle \lambda, <, \alpha\rangle_{\alpha\in\lambda}$ obtained by adding the f_γ's as unary functions. By assumption, there is $\mathfrak{B} \equiv \mathfrak{A}$ with a λ-nonstandard element b but without μ-nonstandard elements. For every formula $\varphi(y)$ in the similarity type of $\mathfrak{A}$ and with exactly one free variable y, let $Z_\varphi = \{\alpha \in \lambda \mid \varphi(\alpha) \text{ holds in } \mathfrak{A}\}$. Put $E = \{Z_\varphi \mid \varphi \text{ is as above, and } \varphi(b) \text{ holds in } \mathfrak{B}\}$. E has trivially the finite intersection property, thus it can be extended to some ultrafilter D over λ. Since λ is regular and, for every $\alpha \in \lambda$, $(\alpha, \lambda) \in E \subseteq D$, we get that D is uniform. Let $\gamma \in \kappa$. Since $\mathfrak{B}$ has no μ-nonstandard element, there is $\beta < \mu$ such that $f_\gamma(b) < \beta$ holds in $\mathfrak{B}$. Letting $\varphi(y)$ be $f_\gamma(y) < \beta$, we get that $Z_\varphi = \{\alpha \in \lambda \mid f_\gamma(\alpha) < \beta\} \in E \subseteq D$, proving that $f_\gamma(D)$ is not uniform over μ. $\square$

If Σ and Γ are sets of sentences of $\mathcal{L}_{\omega_1,\omega}$, we say that Γ is λ-*satisfiable relative to* Σ if $\Sigma \cup \Gamma'$ is satisfiable, for every $\Gamma' \subseteq \Gamma$ of cardinality $< \lambda$. We say that $\mathcal{L}_{\omega_1,\omega}$ is κ-(λ,λ)-*compact* if $\Sigma \cup \Gamma$ is satisfiable, whenever $|\Sigma| \leqslant \kappa$, $|\Gamma| \leqslant \lambda$, and Γ is λ-satisfiable relative to Σ. The above notion has been introduced in [Li1] for arbitrary logics, extending notions by Chang, Keisler, Makowsky, Shelah and Tarski and others. Clearly, if $\kappa \leqslant \lambda$, then κ-(λ,λ)-compactness reduces to (λ,λ)-compactness.

Theorem 5. *If $\kappa \geqslant \lambda$ and λ is regular, the following conditions are equivalent.*

(1) ω^κ is $[\lambda,\lambda]$-compact.
(2) The language $\mathcal{L}_{\omega_1,\omega}$ is κ-(λ,λ)-compact.
(3) $\lambda \overset{\kappa}{\not\Rightarrow} \omega$.

In particular, if λ is regular, then ω^λ is finally λ-compact if and only if $\mathcal{L}_{\omega_1,\omega}$ is (λ,λ)-compact.

Proof. The equivalence of (1) and (3) is the particular case of Theorem 3 when all μ_γ's equal ω. In view of Theorem 4, it is enough to prove that (2) is equivalent to the necessary and sufficient condition given there for $\lambda \overset{\kappa}{\not\Rightarrow} \omega$. This is Theorem 3.12 in [Li1] and, anyway, it is a standard argument. We sketch a proof for the non trivial direction. So, suppose that the condition in Theorem 4 holds. For models without ω-nonstandard elements, a formula of $\mathcal{L}_{\omega_1,\omega}$ of the form $\bigwedge_{n\in\omega} \varphi_n(\bar{x})$ is equivalent to $\forall y < \omega R(y,\bar{x})$, for a newly introduced relation R such that $R(n,\bar{x}) \Leftrightarrow \varphi_n(\bar{x})$, for every $n \in \omega$. Thus, working within such models, and appropriately extending the vocabulary, we may assume that Σ and Γ are sets of first order sentences. If $|\Sigma| \leqslant \kappa$, and $\Gamma = \{\gamma_\alpha \mid \alpha \in \lambda\}$ is λ-satisfiable relative to Σ, construct a model

$\mathfrak{A}$ which contains $\langle \lambda, <, \alpha \rangle_{\alpha \in \lambda}$, and with a relation S such that, for every $\beta < \lambda$, $\{z \in A \mid S(\beta, z)\}$ models $\Sigma \cup \{\gamma_\alpha \mid \alpha < \beta\}$. This is possible, since Γ is λ-satisfiable relative to Σ. If $\mathfrak{B} \equiv \mathfrak{A}$ is given by $\lambda \overset{\kappa}{\not\Rightarrow} \omega$, and $b \in B$ is λ-nonstandard, then $\{z \in B \mid S(b, z)\}$ models $\Sigma \cup \Gamma$.

The last statement follows from the trivial fact that ω^λ is finally λ^+-compact, since it has a base of cardinality λ; hence ω^λ is finally λ-compact if and only if it is $[\lambda, \lambda]$-compact. □

The assumption that λ is regular in Theorem 5 is only for simplicity: we can devise a modified principle, call it $(\lambda, \lambda) \overset{\kappa}{\not\Rightarrow} \omega$, which involves (λ, λ)-regular ultrafilters [Li2], and functions $f_\gamma : [\lambda]^{<\lambda} \to \omega$. All the arguments carry over to get a result corresponding to Theorem 5. In particular, the equivalence of (1) and (2) holds with no assumption on λ. To keep this note within the limits of a reasonable length, we shall present details elsewhere.

A remark is in order here, about the principle $\lambda \overset{\kappa}{\not\Rightarrow} \mu$. Since there are μ^λ functions from λ to μ, we get that if $\kappa, \kappa' \geqslant \mu^\lambda$, then $\lambda \overset{\kappa}{\not\Rightarrow} \mu$ is equivalent to $\lambda \overset{\kappa'}{\not\Rightarrow} \mu$, and it is also equivalent to the statement "there is some ultrafilter D uniform over λ such that for no function $f : \lambda \to \mu$, $f(D)$ is uniform over μ". This property has been widely studied by set theorists, generally under the terminology "D over λ is μ-indecomposable". In this sense, the particular case $\mu = \omega$ considered in Theorem 5 incorporates some simple results involving measurable and related cardinals. For example, if λ is regular, all powers of ω are $[\lambda, \lambda]$-compact if and only if ω^{2^λ} is $[\lambda, \lambda]$-compact, if and only if λ carries some ω_1-complete uniform ultrafilter. In particular, we get a classical result by Łoś [Ło], asserting that ω^{2^λ} is not finally λ-compact, provided that λ is regular and there is no measurable cardinal $\leqslant \lambda$. Moreover, we get that, for λ regular, all powers of ω are finally λ-compact if and only if every $\lambda' \geqslant \lambda$ carries an ω_1-complete uniform ultrafilter (in particular, this holds if λ is strongly compact).

Many results about μ-indecomposable ultrafilters over λ generalize to properties of $\lambda \overset{\kappa}{\not\Rightarrow} \mu$, for $\kappa < \mu^\lambda$, usually with more involved proofs. We initiated this project in [Li1, Li2]. Applications to powers of ω are presented in the next two corollaries. Notice that in [Li1] the definition of $\lambda \overset{\kappa}{\not\Rightarrow} \omega$ is given directly by means of the condition in Theorem 4. The two definitions do not necessarily coincide for $\kappa < \lambda$; however, here $\kappa \geqslant \lambda$ is always assumed.

Corollary 6. *Let κ be given, and suppose that there is some $\lambda \leqslant \kappa$ such that ω^κ is $[\lambda, \lambda]$-compact. If λ is the first such cardinal, then $\mathcal{L}_{\lambda,\omega}$ is κ-(λ, λ)-compact; in particular, λ is weakly inaccessible (actually, very high in the* *weak Mahlo hierarchy). If, in addition, $2^{<\lambda} \leqslant \kappa$, then λ is weakly compact; and if $2^\lambda \leqslant \kappa$, then λ is measurable.*

Proof. From Theorem 5 and Theorem 3.9 in [Li1], applied in the particular case of $\mathcal{L}_{\omega_1,\omega}$. $\square$

As a consequence of Corollary 6, if there is no measurable cardinal and the Generalized Continuum Hypothesis holds, then ω^λ is finally λ-compact if and only if λ is weakly compact; moreover, ω^κ is never $[\lambda, \lambda]$-compact, for $\kappa > \lambda$ (only special consequences of GCH are needed in the above statements: respectively, that every weakly Mahlo cardinal is inaccessible, and that GCH holds at weakly Mahlo cardinals). The assumptions are necessary: as we mentioned, if λ is measurable, then all powers of ω are $[\lambda, \lambda]$-compact. Moreover, if μ is μ^+-compact, then there is an ω_1-complete ultrafilter uniform over μ^+, hence, by a previous remark, all powers of ω are $[\mu^+, \mu^+]$-compact; however, μ^+ is not weakly compact. With less stringent large cardinal assumptions, Boos [Bo], extending results by Kunen, Solovay and others, constructed models in which GCH fails and $\mathcal{L}_{\lambda,\omega}$ (hence also $\mathcal{L}_{\omega_1,\omega}$) are (λ, λ)-compact but λ is not weakly compact, not even inaccessible.

For μ, λ regular cardinals, the principle E^μ_λ asserts that λ has a nonreflecting stationary set consisting of ordinals of cofinality μ. The next corollary applies not only to powers of ω, but also to powers of regular cardinals (always endowed with the order topology).

Corollary 7. *If $\mu < \lambda$ are regular, and E^μ_λ, then μ^λ is not $[\lambda, \lambda]$-compact.*
If $\square_\lambda$, then μ^{λ^+} is not $[\lambda^+, \lambda^+]$-compact, for every regular $\mu \leqslant \lambda$.

Proof. By [Li1, Theorem 4.1], in the present notation, $\lambda \overset{\lambda}{\Rightarrow} \mu$ (this was denoted by $\lambda \Rightarrow \mu$ in [Li1], a notation not consistent with the present one). The first statement is immediate from Theorem 3. The second statement follows from the well known fact that $\square_\lambda$ implies $E^\mu_{\lambda^+}$, for every regular $\mu < \lambda$. (We need not bother with the case $\lambda = \omega$, since $E^\omega_{\omega_1}$ is a theorem in ZFC.) $\square$

Mycielski [My] has also considered the property that ω^κ contains a closed discrete subset of cardinality κ. Clearly, if this is the case, then ω^κ is not κ-finally compact, not even $[\kappa, \kappa]$-compact, and not $[\kappa', \kappa']$-compact, for every $\kappa' \leqslant \kappa$. A variation on the methods of the present note can be used to show that if $\lambda' \leqslant \kappa$, then ω^κ contains a closed discrete subset of cardinality λ' if and only if there is no $\lambda \leqslant \lambda'$ such that $\mathcal{L}_{\omega_1,\omega}$ is κ-(λ, λ)-compact, if and only if (by Corollary 6) there is no $\lambda \leqslant \lambda'$ such that $\mathcal{L}_{\lambda,\omega}$ is κ-(λ, λ)-compact, if and only if (by Theorem 5) for no $\lambda \leqslant \lambda'$ ω^κ is $[\lambda, \lambda]$-compact.

Finally, let us notice that, though we have stated our results in terms of powers of ω, they can be reformulated in a way which involves arbitrary T_1 spaces.

Proposition 8. *For given λ and κ, the following conditions are equivalent.*

(1) ω^κ is not $[\lambda, \lambda]$-compact.

(2) For every product $X = \prod_{i \in I} X_j$ of T_1 topological spaces, if X is $[\lambda, \lambda]$-compact, then $|\{i \in I \mid X_i$ is not countably compact$\}| < \kappa$.

Proof. (2) $\Rightarrow$ (1) is trivial. For the converse, notice that if a T_1 topological spaces is not countably compact, then it contains a countable discrete closed subset, that is, a closed copy of ω. $\square$

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