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SOME COMPACTNESS PROPERTIES RELATED TO PSEUDOCOMPACTNESS AND ULTRAFILTER CONVERGENCE

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Abstract. We discuss some notions of compactness and convergence relative to a specified family $\mathcal{F}$ of subsets of some topological space X. The two most interesting particular cases of our construction appear to be the following ones.

(1) The case in which $\mathcal{F}$ is the family of all singletons of X, in which case we get back the more usual notions.

(2) The case in which $\mathcal{F}$ is the family of all nonempty open subsets of X, in which case we get notions related to pseudocompactness.

A large part of the results in this note are known in particular case (1); the results are, in general, new in case (2). As an example, we characterize those spaces which are D-pseudocompact, for some ultrafilter D uniform over λ.

1. Introduction

In this note we study various compactness and convergence properties relative to a family $\mathcal{F}$ of subsets of some topological space. In particular, we relativize to $\mathcal{F}$ the notions of D-compactness, CAP_λ,

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and $[\mu, \lambda]$-compactness. The two particular cases which motivate our treatment are when $\mathcal{F}$ is either (1) the family of all singletons of X, or (2) the family of all nonempty open sets of X. As far as case (2) is concerned, we can equivalently consider nonempty elements of some base, and we can also equivalently consider those sets which are the closure of some nonempty open set.

Our results concern the mutual relationship among the above compactness properties, and their behavior with respect to products. Some results which are known in particular case (1) are generalized to the case of an arbitrary family $\mathcal{F}$. Apparently, a few results are new even in particular case (1).

Already in particular case (2), our results appear to be new. For example, we get characterizations of those spaces which are D-pseudocompact, for some ultrafilter D uniform over λ (Corollary 5.5).

Similarly, we get equivalent conditions for the weaker local form asserting that, for every λ-indexed family of nonempty sets of X, there exists some uniform ultrafilter D over λ such that the family has some D-limit point in X (Theorem 4.4). In the particular case $\lambda = \omega$, we get nothing but more conditions equivalent to pseudocompactness (for Tychonoff spaces).

At first reading, the reader might consider only the above particular cases (1) and (2), and look at this note as a generalization to pseudocompactness-like notions of results already known about ultrafilter convergence and complete accumulation points. Of course, it might be the case that our definitions and results can be applied to other situations, apart from the two mentioned particular ones; however, we have not worked details yet.

No separation axiom is assumed, unless explicitly mentioned.

1.1. **Some history and our main aim.** The notion of (pointwise) ultrafilter convergence has proven particularly useful in topology, especially in connection with the study of compactness properties and existence of complete accumulation points, not excluding many other kinds of applications. In particular, ultrafilter convergence is an essential tool in studying compactness properties of products. In a sense made precise in [Li1], the use of ultrafilters is unavoidable in this situation.

Ginsburg and Sack's 1975 paper [GiSa] is a pioneering work in applications of pointwise ultrafilter convergence. In addition, [GiSa] introduces a fundamental new tool, the idea of considering ultrafilter limits of subsets (rather than points) of a topological space. In particular, taking into consideration ultrafilter limits of nonempty open sets provides deep applications to pseudocompactness, as well as the possibility of introducing further pseudocompactness-like notions. Some analogies, as well as some differences between the two cases were already discussed in [GiSa]. Subsequently, [Ga1] analyzed in more details some analogies.

Ginsburg and Sack's work concentrated on ultrafilters uniform over ω. Generalizations and improvements for ultrafilters over larger cardinals appeared, for example, in [Sa] in the case of pointwise D-convergence, and in [Ga2] in the case of D-pseudocompactness.

A new wave of results, partially inspired by seemingly unrelated problems in Mathematical Logic, arose when Caicedo [Ca1, Ca2], using ultrafilters, proved some two-cardinals transfer results for compactness of products. For example, among many other things, Caicedo proved that if all powers of some topological space X are $[\lambda^+, \lambda^+]$-compact, then all powers of X are $[\lambda, \lambda]$-compact. Subsequently, further results along this line appeared in [Li1, Li2, Li3].

The aim of this note is twofold. First, we provide analogues, for pseudocompactness-like notions, of results previously proved only for pointwise convergence; in particular, we provide versions of many results appeared in [Ca1, Ca2, Li1, Li2].

Our second aim is to insert the two above-mentioned kinds of results into a more general framework. Apart from the advantage of a unified treatment of both cases, we hope that this abstract approach will contribute to put in a clearer light the methods and notions used in the more familiar case of pointwise convergence. Moreover, as we mentioned, [GiSa] noticed certain analogies between the two cases, but noticed also that there are asymmetries. In our opinion, our treatment provides a very neat explanation for such asymmetries. See the discussion below in subsection 1.2 relative to Section 5.

Finally, let us mention that, for particular case (1), a large part of the results presented here is well known; however, even in this particular and well studied case, we provide a couple of results which might be new: see, e. g., Propositions 3.3 and 3.5, and Remark 5.4.

1.2. **Synopsis.** In detail, the paper is divided as follows.

In Section 2 we introduce the notion of D-compactness relative to some family $\mathcal{F}$ of subsets of some topological space X. This provides a common generalization both of pointwise D-compactness, and of D-pseudocompactness as introduced by [GiSa, Ga2]. Some trivial facts hold about this notion: for example, we can equivalently consider the family of all the closures of elements of $\mathcal{F}$.

In Section 3 we discuss the notion of a complete accumulation point relative to $\mathcal{F}$. In fact, two version are presented: the first one, starred, dealing with *sequences* of subsets, and the second one, unstarred, dealing with *sets* of subsets. That is, in the starred case repetitions are allowed, while they are not allowed in the unstarred case. The difference between the two cases is essential only when dealing with singular cardinals (Proposition 3.3). In the classical case when $\mathcal{F}$ is the set of all singletons, the unstarred notion is most used in the literature; however, we show that the exact connection between the notion of a D-limit point and the existence of a complete accumulation point holds only for the starred variant (Proposition 4.1).

In Section 4 we introduce a generalization of $[\mu, \lambda]$-compactness which also depends on $\mathcal{F}$, and in Theorem 4.4 we prove the equivalence among many of the $\mathcal{F}$-dependent notions we have defined before.

Section 5 discusses the behavior of the above notions in connection with (Tychonoff) products. Actually, for sake of simplicity only, we mostly deal with powers. Since, in our notions, a topological space X comes equipped with a family $\mathcal{F}$ of subsets attached to it, we have to specify which family should be attached to the power X^δ. In order to get significant results, the right choice is to attach to X^δ the family $\mathcal{F}^\delta$ consisting of all products of δ members of $\mathcal{F}$ (some variations are possible). In the case when $\mathcal{F}$ is the family of all the singletons of X, then $\mathcal{F}^\delta$ turns out to be the family of all singletons of X^δ again, thus we get back the classical results about ultrafilter convergence in products. On the other hand, when $\mathcal{F}$ is the family of all nonempty subsets of X, then $\mathcal{F}^\delta$, in general, contains certain sets which are not open in X^δ; in fact, $\mathcal{F}^\delta$ is a base for the box topology on X^δ, a topology generally strictly finer than the nowadays standardly used Tychonoff topology.

The above fact explains the reason why, in the case of products, there is not a total symmetry between results on compactness and results about pseudocompactness. For example, as already noticed in [GiSa], it is true that all powers of some topological space X are countably compact if and only if X is D-compact, for some ultrafilter D uniform over ω. On the other hand, [GiSa] constructed a topological space X all whose powers are pseudocompact, but for which there exists no ultrafilter D uniform over ω such that X is D-pseudocompact. Our framework not only explains the reason for this asymmetry, but can be used in order to provide a characterization of D-pseudocompact spaces, a characterization parallel to that of D-compact spaces. Indeed, we do find versions for D-pseudocompactness of the classical results about D-convergence (Corollary 5.5). Though statements become a little more involved, we believe that these results have some intrinsic interest.

In Section 6 we show that cardinal transfer results for decomposable ultrafilters deeply affect compactness properties relative to these cardinals. More exactly, if λ and μ are cardinals such that every uniform ultrafilter over λ is μ-decomposable, then every topological space X which is $\mathcal{F}$-D-compact, for some ultrafilter D uniform over λ, is also $\mathcal{F}$-D'-compact, for some ultrafilter D' uniform over μ. Of course, this result applies also to all the equivalent notions discussed in the preceding sections. Since there are highly nontrivial set theoretical results on transfer of ultrafilter decomposability, our theorems provide deep unexpected applications of Set Theory to compactness properties of products. The results in Section 6 generalize some results appeared in [Li1].

Finally, in Section 7 we discuss still another generalization of $[\lambda, \mu]$-compactness. Again, there are relationships with the other compactness properties introduced before, as well as with further variations on pseudocompactness. The notions introduced in Section 7 are probably worth of further study.

2. D-compactness relative to some family $\mathcal{F}$

Suppose that D is an ultrafilter over some set Z, and X is a topological space.

A family $(x_z)_{z \in Z}$ of (not necessarily distinct) elements of X is said to D-*converge* to some point $x \in X$ if and only if $\{z \in Z \mid x_z \in U\} \in D$, for every neighborhood U of x in X.

The space X is said to be D-compact if and only if every family $(x_z)_{z \in Z}$ of elements of X converges to some point of X.

If $(Y_z)_{z \in Z}$ is a family of (not necessarily distinct) subsets of X, then x is called a D-*limit point* of $(Y_z)_{z \in Z}$ if and only if $\{z \in Z \mid Y_z \cap U \neq \emptyset\} \in D$, for every neighborhood U of x in X.

Since $Y_z \cap U \neq \emptyset$ if and only if $\overline{Y}_z \cap U \neq \emptyset$, we have that x is a D-limit point of $(Y_z)_{z \in Z}$ if and only if x is a D-limit point of $(\overline{Y}_z)_{z \in Z}$.

The space X is said to be D-*pseudocompact* if and only if every family $(O_z)_{z \in Z}$ of nonempty open subsets of X has some D-limit point in X. The above notion is due to [GiSa, Definition 4.1] for non-principal ultrafilters over ω, and appears in [Ga2] for uniform ultrafilters over arbitrary cardinals.

The above notions can be simultaneously generalized as follows.

Definition 2.1. Suppose that D is an ultrafilter over some set Z, X is a topological space, and $\mathcal{F}$ is a specified family of subsets of X.

We say that the space X is $\mathcal{F}$-D-*compact* if and only if every family $(F_z)_{z \in Z}$ of members of $\mathcal{F}$ has some D-limit point in X.

Thus, we get the notion of D-compactness in the particular case when $\mathcal{F}$ is the family of all singletons of X; and we get the notion of D-pseudocompactness in the particular case when $\mathcal{F}$ is the family of all nonempty open subsets of X.

If $\mathcal{G}$ is another family of subsets of X, let us write $\mathcal{F} \rhd \mathcal{G}$ to mean that, for every $F \in \mathcal{F}$, there is $G \in \mathcal{G}$ such that $F \supseteq G$.

With this notation, it is trivial to show that if $\mathcal{F} \rhd \mathcal{G}$ and X is $\mathcal{G}$-D-compact, then X is $\mathcal{F}$-D-compact.

If $\mathcal{F}$ is a family of subsets of X, let $\overline{\mathcal{F}} = \{\overline{F} \mid F \in \mathcal{F}\}$ be the set of all closures of elements of $\mathcal{F}$. With this notation, it is trivial to show that X is $\mathcal{F}$-D-compact if and only if X is $\overline{\mathcal{F}}$-D-compact.

The most interesting cases in Definition 2.1 appear to be the two mentioned ones, that is, when either $\mathcal{F}$ is the set of all singletons of X, or $\mathcal{F}$ is the set of all nonempty open subsets of X.

In the particular case when $\mathcal{F}$ is the set of all singletons, most of the results we prove here are essentially known, except for the technical difference that we deal with sequences, rather than subsets The difference is substantial only when dealing with singular cardinals. See Remark 3.2 and Proposition 3.3.

In the case when $\mathcal{F}$ is the set of all nonempty open subsets of X, most of our results appear to be new.

Remark 2.2. Notice that if X is a topological space, $\mathcal{F}$ is the set of all nonempty open subsets of X, and $\mathcal{B}$ is a base (consisting of nonempty sets) for the topology on X, then both $\mathcal{F} \rhd \mathcal{B}$ and $\mathcal{B} \rhd \mathcal{F}$.

Hence, $\mathcal{F}$-D-compactness is the same as $\mathcal{B}$-D-compactness. A similar remark applies to all compactness properties we shall introduce later (except for those introduced in Section 7).

3. Complete accumulation points relative to $\mathcal{F}$

We are now going to generalize the notion of an accumulation point.

Definition 3.1. If λ is an infinite cardinal, and $(Y_\alpha)_{\alpha \in \lambda}$ is a sequence of subsets of some topological space X, we say that $x \in X$ is a λ-*complete accumulation point* of $(Y_\alpha)_{\alpha \in \lambda}$ if and only if $|\{\alpha \in \lambda \mid Y_\alpha \cap U \neq \emptyset\}| = \lambda$, for every neighborhood U of x in X.

In case $\lambda = \omega$, we get the usual notion of a *cluster point.*

Notice that x is a λ-complete accumulation point of $(Y_\alpha)_{\alpha \in \lambda}$ if and only if x is a λ-complete accumulation point of $(\overline{Y}_\alpha)_{\alpha \in \lambda}$.

If $\mathcal{F}$ is a family of subsets of X, we say that X satisfies $\mathcal{F}$-CAP^*_λ if and only if every sequence $(F_\alpha)_{\alpha \in \lambda}$ of members of $\mathcal{F}$ has a λ-complete accumulation point.

Notice that if X is a Tychonoff space, and $\mathcal{F}$ is the family of all nonempty open sets of X, then a result by Glicksberg [Gl], when reformulated in the present terminology, asserts that $\mathcal{F}$-CAP^*_ω is equivalent to pseudocompactness. See also, e. g., [GiSa, Section 4], [Ga2, St].

If $\mathcal{F} \rhd \mathcal{G}$ and X satisfies $\mathcal{G}$-CAP^*_λ, then X satisfies $\mathcal{F}$-CAP^*_λ.

Moreover, X satisfies $\mathcal{F}$-CAP^*_λ if and only if it satisfies $\overline{\mathcal{F}}$-CAP^*_λ.

Remark 3.2. In the case when each Y_α is a singleton in Definition 3.1, and all such singletons are distinct, we get back the usual notion of a complete accumulation point.

A point $x \in X$ is said to be a *complete accumulation point* of some infinite subset $Y \subseteq X$ if and only if $|Y \cap U| = |Y|$, for every neighborhood U of x in X.

A topological space X satisfies CAP_λ if and only if every subset $Y \subseteq X$ with $|Y| = \lambda$ has a complete accumulation point.

In the case when λ is a singular cardinal, there is some difference between the classic notion of a complete accumulation point and the notion of a λ-complete accumulation point, as introduced in Definition 3.1. This happens because, for our purposes, it is more convenient to deal with sequences, rather than subsets, that is, we allow repetitions. This is the reason for the * in $\mathcal{F}$-CAP^*_λ in Definition 3.1.

As pointed in [Li3, Part VI, Proposition 1], if $\mathcal{F}$ is the family of all singletons, then, for λ regular, $\mathcal{F}$-CAP^*_λ is equivalent to CAP_λ, and, for λ singular, $\mathcal{F}$-CAP^*_λ is equivalent to the conjunction of CAP_λ and $\mathrm{CAP}_{\mathrm{cf}\,\lambda}$.

In fact, a more general result holds for families of nonempty sets. In order to clarify the situation let us introduce the following unstarred variant of $\mathcal{F}$-CAP^*_λ. If $\mathcal{F}$ is a family of subsets of X, we say that X satisfies $\mathcal{F}$-CAP_λ if and only if every family $(F_\alpha)_{\alpha\in\lambda}$ of *distinct* members of $\mathcal{F}$ has a λ-complete accumulation point.

Then we have:

Proposition 3.3. *Suppose that X is a topological space, and $\mathcal{F}$ is a family of nonempty subsets of X.*

(a) If λ is a regular cardinal, then X satisfies $\mathcal{F}$-CAP^*_λ if and only if X satisfies $\mathcal{F}$-CAP_λ.

(b) If λ is a singular cardinal, then X satisfies $\mathcal{F}$-CAP^*_λ if and only if X satisfies both $\mathcal{F}$-CAP_λ and $\mathcal{F}$-$\mathrm{CAP}_{\mathrm{cf}\,\lambda}$.

Proof. It is obvious that $\mathcal{F}$-CAP^*_λ implies $\mathcal{F}$-CAP_λ, for every cardinal λ.

Suppose that λ is regular, that $\mathcal{F}$-CAP_λ holds, and that $(F_\alpha)_{\alpha\in\lambda}$ is a sequence of elements of $\mathcal{F}$. If some subsequence consists of λ-many distinct elements, then, by $\mathcal{F}$-CAP_λ, this subsequence has some λ-complete accumulation point which necessarily is also a λ-complete accumulation point for $(F_\alpha)_{\alpha\in\lambda}$. Otherwise, since λ is regular, there exists some $F \in \mathcal{F}$ which appears λ-many times in $(F_\alpha)_{\alpha\in\lambda}$. Since, by assumption, F is nonempty, just take some

$x \in F$ to get a λ-complete accumulation point for $(F_\alpha)_{\alpha\in\lambda}$. Thus we have proved that $\mathcal{F}$-CAP_λ implies $\mathcal{F}$-CAP^*_λ, for λ regular.

Now suppose that λ is singular and that both $\mathcal{F}$-CAP_λ and $\mathcal{F}$-$\mathrm{CAP}_{\mathrm{cf}\,\lambda}$ hold. We are going to show that $\mathcal{F}$-CAP^*_λ holds. Let $(F_\alpha)_{\alpha\in\lambda}$ be a sequence of elements of $\mathcal{F}$. There are three cases. (i) There exists some $F \in \mathcal{F}$ which appears λ-many times in $(F_\alpha)_{\alpha\in\lambda}$. In this case, as above, it is enough to choose some element from F. (ii) Some subsequence of $(F_\alpha)_{\alpha\in\lambda}$ consists of λ-many distinct elements. Then, as above, apply $\mathcal{F}$-CAP_λ to this subsequence. (iii) Otherwise, $(F_\alpha)_{\alpha\in\lambda}$ consists of $< \lambda$ different elements, each one appearing $< \lambda$ times. Moreover, if $(\lambda_\beta)_{\beta\in\mathrm{cf}\,\lambda}$ is a sequence of cardinals $< \lambda$ whose supremum is λ, then, for every $\beta \in \mathrm{cf}\,\lambda$, there is $F_\beta \in \mathcal{F}$ appearing at least λ_β-many times. Since, for each β, F_β appears $< \lambda$ times, we can choose $cf\lambda$-many distinct F_β's as above. Applying $\mathcal{F}$-$\mathrm{CAP}_{\mathrm{cf}\,\lambda}$ to those F_β's, we get a λ-complete accumulation point for $(F_\alpha)_{\alpha\in\lambda}$.

It remains to show that $\mathcal{F}$-CAP^*_λ implies $\mathcal{F}$-$\mathrm{CAP}_{\mathrm{cf}\,\lambda}$. Let $(\lambda_\beta)_{\beta\in\mathrm{cf}\,\lambda}$ be a sequence of cardinals $< \lambda$ whose supremum is λ. If $(F_\beta)_{\beta\in\mathrm{cf}\,\lambda}$ is a sequence of distinct members of $\mathcal{F}$, let $(G_\alpha)_{\alpha\in\lambda}$ be a sequence defined in such a way that, for every $\beta \in \mathrm{cf}\,\lambda$, $G_\alpha = F_\beta$ for exactly λ_β-many α's. By $\mathcal{F}$-CAP^*_λ, $(G_\alpha)_{\alpha\in\lambda}$ has a λ-complete accumulation point x. It is immediate to show that x is also a $\mathrm{cf}\,\lambda$-complete accumulation point for $(F_\beta)_{\beta\in\mathrm{cf}\,\lambda}$. □

If D is an ultrafilter, Y is a D-compact Hausdorff space, and $X \subseteq Y$, then there is the smallest D-compact subspace Z of Y containing X. This is because the intersection of any family of D-compact subspaces of Y is still D-compact, since, in a Hausdorff space, the D-limit of a sequence is unique (if it exists). Such a Z can be also constructed by an iteration procedure in $|I|^+$ stages, if D is over I. This is similar to, e. g., [GiSa, Theorem 2.12], or [Ga2].

If X is a Tychonoff space, and $Y = \beta(X)$ is the Stone-Čech compactification of X, the smallest D-compact subspace of $\beta(X)$ containing X is called the *D-compactification* of X, and is denoted by $\beta_D(X)$. See, e. g., [Ga1, p. 14], [Ga2], or [GiSa] for further references and alternative definitions of the D-compactification (sometimes also called D-compact reflection).

Example 3.4. (a) If λ is singular, then $\mathrm{cf}\,\lambda$, endowed with either the order topology or the discrete topology, fails to satisfy $\mathrm{CAP}_{\mathrm{cf}\,\lambda}$, but trivially satisfies CAP_λ.

(b) Suppose that λ is singular, and X is any Tychonoff space. If D is an ultrafilter uniform over $\mathrm{cf}\,\lambda$, then the D-compactification $\beta_D(X)$ of X satisfies $\mathrm{CAP}_{\mathrm{cf}\,\lambda}$, by Theorem 4.4 (d) $\Rightarrow$ (c) and Proposition 3.3 (a).

(c) If X is λ with the discrete topology, then X does not satisfy CAP_λ. By (b) above, if D is an ultrafilter uniform over $\mathrm{cf}\,\lambda$, then the D-compactification $\beta_D(X)$ of X satisfies $\mathrm{CAP}_{\mathrm{cf}\,\lambda}$. However, $\beta_D(X)$ does not satisfy CAP_λ. Thus, we have a space satisfying $\mathrm{CAP}_{\mathrm{cf}\,\lambda}$, but not satisfying CAP_λ.

(d) In order to get an example as (c) above, it is not sufficient to take any space X which does not satisfy CAP_λ. Indeed, if X is λ with the order topology, then $\beta_D(X)$ does satisfy CAP_λ, if D is an ultrafilter uniform over $\mathrm{cf}\,\lambda$.

The next proposition shows that, for λ a singular cardinal, $\mathrm{CAP}_{\mathrm{cf}\,\lambda}$ implies $\mathcal{F}$-CAP^*_λ, provided that $\mathcal{F}$-CAP_μ holds for a set of cardinals unbounded in λ.

Proposition 3.5. *Suppose that X is a topological space, $\mathcal{F}$ is a family of nonempty subsets of X, λ is a singular cardinal, and $(\lambda_\beta)_{\beta\in\mathrm{cf}\,\lambda}$ is a sequence of cardinals $<\lambda$ such that* $\sup_{\beta\in\mathrm{cf}\,\lambda}\lambda_\beta=\lambda$.

If X satisfies $\mathrm{CAP}_{\mathrm{cf}\,\lambda}$, *and* $\mathcal{F}$-$\mathrm{CAP}_{\lambda_\beta}$, *for every $\beta\in\mathrm{cf}\,\lambda$, then X satisfies* $\mathcal{F}$-CAP^*_λ.

In particular, if X satisfies $\mathrm{CAP}_{\mathrm{cf}\,\lambda}$, *and* $\mathrm{CAP}_{\lambda_\beta}$, *for every $\beta\in\mathrm{cf}\,\lambda$, then X satisfies* CAP^*_λ.

Proof. We first prove that X satisfies $\mathcal{F}$-CAP_λ. The proof takes some ideas from [Sa, proof of the proposition on p. 94]. So, let $(F_\alpha)_{\alpha\in\lambda}$ be a sequence of distinct elements of $\mathcal{F}$. For every $\beta\in\mathrm{cf}\,\lambda$, by $\mathcal{F}$-$\mathrm{CAP}_{\lambda_\beta}$, we get some element x_β which is a λ_β-complete accumulation point for $(F_\alpha)_{\alpha\in\lambda_\beta}$. By $\mathrm{CAP}^*_{\mathrm{cf}\,\lambda}$ (which follows from $\mathrm{CAP}_{\mathrm{cf}\,\lambda}$, by Proposition 3.3(a)), the sequence $(x_\beta)_{\beta\in\mathrm{cf}\,\lambda}$ has some $\mathrm{cf}\,\lambda$-complete accumulation point x. It is now easy to see that x is a λ-complete accumulation point for $(F_\alpha)_{\alpha\in\lambda}$.

Since the members of $\mathcal{F}$ are nonempty, $\mathrm{CAP}_{\mathrm{cf}\,\lambda}$ implies $\mathcal{F}$-$\mathrm{CAP}_{\mathrm{cf}\,\lambda}$, hence $\mathcal{F}$-CAP^*_λ follows from $\mathcal{F}$-CAP_λ, by Proposition 3.3(b).

The last statement follows by taking $\mathcal{F}$ to be the family of all singletons of X. $\square$

The last statement in Proposition 3.5 has appeared in [Li3, Part VI, p. 2].

4. Relationship among compactness properties

In the next proposition we deal with the fundamental relationship, for a given sequence, between the existence of a λ-complete accumulation point and the existence of a D-limit point, for D uniform over λ. Then in Theorem 4.4 we shall present more equivalent formulations referring to various compactness properties.

Proposition 4.1. *Suppose that λ is an infinite cardinal, and $(Y_\alpha)_{\alpha\in\lambda}$ is a sequence of subsets of some topological space X.*

Then $x \in X$ is a λ-complete accumulation point of $(Y_\alpha)_{\alpha\in\lambda}$ if and only if there exists an ultrafilter D uniform over λ such that x is a D-limit point of $(Y_\alpha)_{\alpha\in\lambda}$.

In particular, $(Y_\alpha)_{\alpha\in\lambda}$ has a λ-complete accumulation point if and only if $(Y_\alpha)_{\alpha\in\lambda}$ has a D-limit point, for some ultrafilter D uniform over λ.

Proof. If $x \in X$ is a λ-complete accumulation point of $(Y_\alpha)_{\alpha\in\lambda}$, then the family $\mathcal{H}$ consisting of the sets $\{\alpha \in \lambda \mid Y_\alpha \cap U \neq \emptyset\}$ (U a neighborhood of x) and $\lambda \setminus Z$ ($|Z| < \lambda$) has the finite intersection property, indeed, the intersection of any finite set of members of $\mathcal{H}$ has cardinality λ. Hence $\mathcal{H}$ can be extended to some ultrafilter D, which is necessarily uniform over λ. It is trivial to see that, for such a D, x is a D-limit point of $(Y_\alpha)_{\alpha\in\lambda}$.

The converse is trivial, since the ultrafilter D is assumed to be uniform over λ. □

The particular case of Proposition 4.1 in which all Y_α's are distinct one-element sets is well-known. See [Sa, pp. 80–81].

Definition 4.2. If X is a topological space, and $\mathcal{F}$ is a family of subsets of X, we say that X is $\mathcal{F}$-$[\mu, \lambda]$-*compact* if and only if the following holds.

For every family $(C_\alpha)_{\alpha\in\lambda}$ of closed sets of X, if, for every $Z \subseteq \lambda$ with $|Z| < \mu$, there exists $F \in \mathcal{F}$ such that $\bigcap_{\alpha\in Z} C_\alpha \supseteq F$, then $\bigcap_{\alpha\in\lambda} C_\alpha \neq \emptyset$.

Of course, in the particular case when $\mathcal{F}$ is the set of all the singletons, $\mathcal{F}$-$[\mu, \lambda]$-compactness is the usual notion of $[\mu, \lambda]$-compactness.

Remark 4.3. Trivially, if $\mathcal{F} \rhd \mathcal{G}$, and X is $\mathcal{G}$-$[\mu, \lambda]$-compact, then X is $\mathcal{F}$-$[\mu, \lambda]$-compact.

Recall that if $\mathcal{F}$ is a family of subsets of X, we have defined $\overline{\mathcal{F}} = \{\overline{F} \mid F \in \mathcal{F}\}$. It is trivial to observe that X is $\mathcal{F}$-$[\mu, \lambda]$-compact if and only if X is $\overline{\mathcal{F}}$-$[\mu, \lambda]$-compact.

Theorem 4.4. *Suppose that X is a topological space, $\mathcal{F}$ is a family of subsets of X, and λ is a regular cardinal. Then the following conditions are equivalent.*

(a) X is $\mathcal{F}$-$[\lambda, \lambda]$-compact.

(b) Suppose that $(C_\alpha)_{\alpha \in \lambda}$ is a family of closed sets of X such that $C_\alpha \supseteq C_\beta$, whenever $\alpha \leq \beta < \lambda$. If, for every $\alpha \in \lambda$, there exists $F \in \mathcal{F}$ such that $C_\alpha \supseteq F$, then $\bigcap_{\alpha \in \lambda} C_\alpha \neq \emptyset$.

(b_1) Suppose that $(C_\alpha)_{\alpha \in \lambda}$ is a family of closed sets of X such that $C_\alpha \supseteq C_\beta$, whenever $\alpha \leq \beta < \lambda$. Suppose further that, for every $\alpha \in \lambda$, C_α is the closure of the union of some set of members of $\mathcal{F}$. If, for every $\alpha \in \lambda$, there exists $F \in \mathcal{F}$ such that $C_\alpha \supseteq F$, then $\bigcap_{\alpha \in \lambda} C_\alpha \neq \emptyset$.

(b_2) Suppose that $(C_\alpha)_{\alpha \in \lambda}$ is a family of closed sets of X such that $C_\alpha \supseteq C_\beta$, whenever $\alpha \leq \beta < \lambda$. Suppose further that, for every $\alpha \in \lambda$, C_α is the closure of the union of some set of $\leq \lambda$ members of $\mathcal{F}$. If, for every $\alpha \in \lambda$, there exists $F \in \mathcal{F}$ such that $C_\alpha \supseteq F$, then $\bigcap_{\alpha \in \lambda} C_\alpha \neq \emptyset$.

(c) Every sequence $(F_\alpha)_{\alpha \in \lambda}$ of elements of $\mathcal{F}$ has a λ-complete accumulation point (that is, X satisfies $\mathcal{F}$-CAP^*_λ).

(d) For every sequence $(F_\alpha)_{\alpha \in \lambda}$ of elements of $\mathcal{F}$, there exists some ultrafilter D uniform over λ such that $(F_\alpha)_{\alpha \in \lambda}$ has a D-limit point.

(e) For every λ-indexed open cover $(O_\alpha)_{\alpha \in \lambda}$ of X, there exists $Z \subseteq \lambda$, with $|Z| < \lambda$, such that, for every $F \in \mathcal{F}$, $F \cap \bigcup_{\alpha \in Z} O_\alpha \neq \emptyset$.

(f) For every λ-indexed open cover $(O_\alpha)_{\alpha \in \lambda}$ of X, such that $O_\alpha \subseteq O_\beta$ whenever $\alpha \leq \beta < \lambda$, there exists $\alpha \in \lambda$ such that O_α intersects each $F \in \mathcal{F}$.

In each of the above conditions we can equivalently replace $\mathcal{F}$ by $\overline{\mathcal{F}}$.

If $\mathcal{F} \rhd \mathcal{G}$ and $\mathcal{G} \rhd \mathcal{F}$, then in each of the above conditions we can equivalently replace $\mathcal{F}$ by $\mathcal{G}$.

Proof. (a) $\Rightarrow$ (b) is obvious, since λ is regular.

Conversely, suppose that (b) holds, and that $(C_\alpha)_{\alpha\in\lambda}$ are closed sets of X such that, for every $Z \subseteq \lambda$ with $|Z| < \mu$, there exists $F \in \mathcal{F}$ such that $\bigcap_{\alpha\in Z} C_\alpha \supseteq F$.

For $\alpha \in \lambda$, define $D_\alpha = \bigcap_{\beta<\alpha} C_\beta$. The D_α's are closed sets of X, and satisfy the assumption in (b), hence $\bigcap_{\alpha\in\lambda} D_\alpha \neq \emptyset$. But $\bigcap_{\alpha\in\lambda} C_\alpha = \bigcap_{\alpha\in\lambda} D_\alpha \neq \emptyset$, thus (a) is proved.

(b) $\Rightarrow$ (b_1) $\Rightarrow$ (b_2) are trivial.

(b_2) $\Rightarrow$ (c) Suppose that (b_2) holds, and that $(F_\alpha)_{\alpha\in\lambda}$ are elements of $\mathcal{F}$. For $\alpha \in \lambda$, let C_α be the closure of $\bigcup_{\beta>\alpha} F_\beta$. The C_α's satisfy the assumptions in (b_2), hence $\bigcap_{\alpha\in\lambda} C_\alpha \neq \emptyset$. Let $x \in \bigcap_{\alpha\in\lambda} C_\alpha$. We want to show that x is a λ-complete accumulation point for $(F_\alpha)_{\alpha\in\lambda}$. Indeed, suppose by contradiction that $|\{\alpha \in \lambda \mid F_\alpha \cap U \neq \emptyset\}| < \lambda$, for some neighborhood U of x in X. If $\beta = \sup\{\alpha \in \lambda \mid F_\alpha \cap U \neq \emptyset\}$, then $\beta < \lambda$, since λ is regular, and we are taking the supremum of a set of cardinality $< \lambda$. Thus, $F_\alpha \cap U = \emptyset$, for every $\alpha > \beta$, hence $U \cap \bigcup_{\alpha>\beta} F_\alpha = \emptyset$, and $x \notin C_\beta$, a contradiction.

(c) $\Rightarrow$ (b) Suppose that (c) holds, and that $(C_\alpha)_{\alpha\in\lambda}$ satisfies the premise of (b). For each $\alpha \in \lambda$, choose $F_\alpha \in \mathcal{F}$ with $F_\alpha \subseteq C_\alpha$. By (c), $(F_\alpha)_{\alpha\in\lambda}$ has a λ-complete accumulation point x. Hence, for every neighborhood U of x, there are arbitrarily large $\alpha < \lambda$ such that U intersects F_α, so there are arbitrarily large $\alpha < \lambda$ such that U intersects C_α, hence U intersects every C_α, since the C_α's form a decreasing sequence. In conclusion, for every $\alpha \in \lambda$, every neighborhood of x intersects C_α, that is, $x \in C_\alpha$, since C_α is closed.

(c) $\Leftrightarrow$ (d) is immediate from Proposition 4.1.

(e) and (f) are obtained from (a) and (b), respectively, by taking complements.

It follows from preceding remarks that we get equivalent conditions when we replace $\mathcal{F}$ by $\overline{\mathcal{F}}$, or by $\mathcal{G}$, if $\mathcal{F} \rhd \mathcal{G}$ and $\mathcal{G} \rhd \mathcal{F}$. $\square$

In the particular case when $\mathcal{F}$ is the set of all singletons, the equivalence of the conditions in Theorem 4.4 (except perhaps for conditions (b_1) (b_2)) is well-known and, for the most part, dates back already to Alexandroff and Urysohn's classical survey [AlUr]. See, e.g., [Va1, Va2] for further comments and references.

Remark 4.5. In the particular case when $\lambda = \omega$, X is Tychonoff and $\mathcal{F}$ is the family of all nonempty sets of X, in Theorem 4.4 we get conditions equivalent to pseudocompactness, since, as we mentioned, a result by Glicksberg implies that, for Tychonoff spaces, $\mathcal{F}$-CAP$^*_\omega$ is equivalent to pseudocompactness. Some of these equivalences are known: for example, Condition (e) becomes Condition (C_5) in [St].

Corollary 4.6. *Suppose that X is a topological space, $\mathcal{F}$ is a family of subsets of X, and λ is a regular cardinal. If X is $\mathcal{F}$-D-compact, for some ultrafilter D uniform over λ, then all the conditions in Theorem 4.4 hold.*

Proof. If X is $\mathcal{F}$-D-compact, for some ultrafilter D uniform over λ, then Condition 4.4 (d) holds, hence all the other equivalent conditions hold. □

5. Behavior with respect to products

We now discuss the behavior of $\mathcal{F}$-D-compactness with respect to products.

Proposition 5.1. *Suppose that $(X_i)_{i\in I}$ is a family of topological spaces, and let $X = \prod_{i\in I} X_i$, with the Tychonoff topology. Let D be an ultrafilter over λ.*

(a) Suppose that, for each $i \in I$, $(Y_{i,\alpha})_{\alpha\in\lambda}$ is a sequence of subsets of X_i. Then some point $x = (x_i)_{i\in I}$ is a D-limit point of $(\prod_{i\in I} Y_{i,\alpha})_{\alpha\in\lambda}$ in X if and only if, for each $i \in I$, x_i is a D-limit point of $(Y_{i,\alpha})_{\alpha\in\lambda}$ in X_i.

In particular, $(\prod_{i\in I} Y_{i,\alpha})_{\alpha\in\lambda}$ has a D-limit point in X if and only if, for each $i \in I$, $(Y_{i,\alpha})_{\alpha\in\lambda}$ has a D-limit point in X_i.

(b) Suppose that, for each $i \in I$, $\mathcal{F}_i$ is a family of subsets of X_i, and let $\mathcal{F}$ be either

- the family of all subsets of X of the form $\prod_{i\in I} F_i$, where each F_i belongs to $\mathcal{F}_i$, or

- for some fixed cardinal $\nu > 1$, the family of all subsets of X of the form $\prod_{i\in I} F_i$, where, for some $I' \subseteq I$ with $|I'| < \nu$, F_i belongs to $\mathcal{F}_i$, for $i \in I'$, and $F_i = X_i$, for $i \in I \setminus I'$.

Then X is $\mathcal{F}$-D-compact if and only if X_i is $\mathcal{F}_i$-D-compact, for every $i \in I$.

Theorem 5.2. *Suppose that X is a topological space, and that $\mathcal{F}$ is a family of subsets of X. For every cardinal δ, let X^δ be the δ^{th} power of X, endowed with the Tychonoff topology, and let $\mathcal{F}^\delta$ be the* *family of all products of* δ *members of* $\mathcal{F}$. *Then, for every cardinal* λ, *the following are equivalent.*

(1) *There exists some ultrafilter* D *uniform over* λ *such that* X *is* $\mathcal{F}$-D-*compact.*
(2) *There exists some ultrafilter* D *uniform over* λ *such that, for every cardinal* δ, *the space* X^δ *is* $\mathcal{F}^\delta$-D-*compact.*
(3) X^δ *satisfies* $\mathcal{F}^\delta$-CAP^*_λ, *for every cardinal* δ *(if* λ *is regular, then all the equivalent conditions in Theorem 4.4 hold, for* X^δ *and* $\mathcal{F}^\delta$*).*
(4) X^δ *satisfies* $\mathcal{F}^\delta$-CAP^*_λ, *for* $\delta = \min\{2^{2^\lambda}, |\mathcal{F}|^\lambda\}$ *(if* λ *is regular, then all the equivalent conditions in Theorem 4.4 hold, for* X^δ *and* $\mathcal{F}^\delta$*).*

Proof. (1) $\Rightarrow$ (2) follows from Proposition 5.1(b).

(2) $\Rightarrow$ (3) follows from Proposition 4.1.

(3) $\Rightarrow$ (4) is trivial.

(4) $\Rightarrow$ (1) We first consider the case $\delta = |\mathcal{F}|^\lambda$. Thus, there are δ-many λ-indexed sequences of elements of $\mathcal{F}$. Let us enumerate them as $(F_{\beta,\alpha})_{\alpha\in\lambda}$, β varying in δ.

In X^δ, consider the sequence $(\prod_{\beta\in\delta} F_{\beta,\alpha})_{\alpha\in\lambda}$ of elements of $\mathcal{F}^\delta$. By (4), the above sequence has a λ-complete accumulation point and, by Proposition 4.1, there exists some ultrafilter D uniform over λ such that $(\prod_{\beta\in\delta} F_{\beta,\alpha})_{\alpha\in\lambda}$ has a D-limit point x in X^δ. Say, $x = (x_\beta)_{\beta\in\delta}$. By Proposition 5.1(a), for every $\beta \in \delta$, x_β is a D-limit point of $(F_{\beta,\alpha})_{\alpha\in\lambda}$ in X.

Since every λ-indexed sequence of elements of $\mathcal{F}$ has the form $(F_{\beta,\alpha})_{\alpha\in\lambda}$, for some $\beta \in \delta$, we have that every λ-indexed sequence of elements of $\mathcal{F}$ has some D-limit point in X, that is, X is $\mathcal{F}$-D-compact.

Now we consider the case $\delta = 2^{2^\lambda}$. We shall prove that if $\delta = 2^{2^\lambda}$ and (1) fails, then (4) fails. If (1) fails, then, for every ultrafilter D uniform over λ, there is a sequence $(F_\alpha)_{\alpha\in\lambda}$ of elements in $\mathcal{F}$ which has no D-limit point. Since there are δ-many ultrafilters over λ, we can enumerate the above sequences as $(F_{\beta,\alpha})_{\alpha\in\lambda}$, β varying in δ.

Now the sequence $(\prod_{\beta\in\delta} F_{\beta,\alpha})_{\alpha\in\lambda}$ in X^δ has no λ-complete accumulation point in X^δ since, otherwise, by Proposition 4.1, for some ultrafilter D uniform over λ, it would have some D-limit point in

X^δ. However, this contradicts Proposition 5.1(a) since, by assumption, there is a β such that $(F_{\beta,\alpha})_{\alpha\in\lambda}$ has no D-limit point. $\square$

Remark 5.3. Suppose that $\mathcal{F}$ in Theorem 5.2 is the family of all nonempty open subsets of X. Then in (3) and (4) we cannot replace $\mathcal{F}^\delta$ by the family $\mathcal{G}^\delta$ of all nonempty open subsets of X^δ. Indeed, if X is a Tychonoff space, and we take $\lambda = \omega$, then $\mathcal{G}^\delta$-CAP$^*_\omega$ for X^δ is equivalent to the pseudocompactness of X^δ. However, [GiSa, Example 4.4] constructed a Tychonoff space all whose powers are pseudocompact, but which for no uniform ultrafilter D over ω is D-pseudocompact. Thus, (3) $\Rightarrow$ (1) becomes false, in general, if we choose $\mathcal{G}^\delta$ instead of $\mathcal{F}^\delta$.

Remark 5.4. In the particular case when $\lambda = \omega$ and $\mathcal{F}$ is the set of all singletons of X, the equivalence of (1), (3) and (4) in Theorem 5.2 is due to Ginsburg and Saks [GiSa, Theorem 2.6], here in equivalent form via Theorem 4.4. See also [ScSt, Theorem 5.6] for a related result.

More generally, when $\mathcal{F}$ is the set of all singletons of X, the equivalence of (1) and (3) in Theorem 5.2 is due to [Sa, Theorem 6.2]. See also [Ga1, Corollary 2.15], [Ca1] and [Ca2, Theorem 3.4].

Let us mention the special case of Theorem 5.2 dealing with D-pseudocompactness.

Corollary 5.5. *Let X be a topological space, and λ be an infinite cardinal. For every cardinal δ, let $\mathcal{F}^\delta$ be either the family of all members of X^δ which are the products of δ nonempty open sets of X, or the family of the nonempty open sets of X^δ in the box topology. (Thus, the former family is a base for the topology given by the latter family) Then the following are equivalent.*

1. *There exists some ultrafilter D uniform over λ such that X is D-pseudocompact.*
2. *There exists some ultrafilter D uniform over λ such that, for every cardinal δ, every λ-indexed sequence of members of $\mathcal{F}^\delta$ has some D-limit point in X^δ (X^δ is endowed with the Tychonoff topology).*
3. *For every cardinal δ, in X^δ (endowed with the Tychonoff topology), every λ-indexed sequence of members of $\mathcal{F}^\delta$ has a λ-complete accumulation point.*

(4) *Let* $\delta = \min\{2^{2^\lambda}, \kappa^\lambda\}$, *where* κ *is the weight of* X. *In* X^δ *(endowed with the Tychonoff topology), every* λ*-indexed sequence of members of* $\mathcal{F}^\delta$ *has a* λ*-complete accumulation point.*

(5) *(provided* λ *is regular) For every cardinal* δ, X^δ *(endowed with the Tychonoff topology) is* $\mathcal{F}^\delta$*-*$[\lambda, \lambda]$*-compact.*

(6) *(provided* λ *is regular) Suppose that* δ *is a cardinal,* $(C_\alpha)_{\alpha\in\lambda}$ *is a family of closed sets of* X^δ *(endowed with the Tychonoff topology) and* $C_\alpha \supseteq C_\beta$, *whenever* $\alpha \le \beta < \lambda$. *If, for every* $\alpha \in \lambda$, *there exists* $F \in \mathcal{F}^\delta$ *such that* $C_\alpha \supseteq F$, *then* $\bigcap_{\alpha\in\lambda} C_\alpha \neq \emptyset$.

Proof. In order to prove the equivalence of conditions (1)-(3), just take $\mathcal{F}$ in Theorem 5.2 to be the family of all nonempty sets of X, to get the result when $\mathcal{F}^\delta$ is the family of all members of X^δ which are the products of nonempty open sets of X.

In order to get the right bound in Condition (4), recall that if $\mathcal{B}$ is a base (consisting of nonempty sets) of X, then, by Remark 2.2, $\mathcal{F} \rhd \mathcal{B}$ and $\mathcal{B} \rhd \mathcal{F}$. Notice also that $\mathcal{F}^\delta \rhd \mathcal{B}^\delta$ and $\mathcal{B}^\delta \rhd \mathcal{F}^\delta$ as well. Thus, we can apply Theorem 5.2 with $\mathcal{B}$ in place of $\mathcal{F}$, getting the right bound in which $|\mathcal{B}| = \kappa$ is the weight of X.

If $\mathcal{F}'^\delta$ is the family of the open sets of X^δ in the box topology, then, by Remark 2.2, trivially both $\mathcal{F}'^\delta \rhd \mathcal{F}^\delta$ and $\mathcal{F}'^\delta \rhd \mathcal{F}^\delta$, thus the corollary holds for $\mathcal{F}'^\delta$, too.

If λ is regular, then Conditions (5) and (6) are equivalent to (3), by Theorem 4.4. $\square$

When λ is regular, we can use Theorem 4.4 in order to get still more conditions equivalent to (3) and (4) above.

6. Two cardinals transfer results

We are now going to show that there are very non trivial cardinal transfer properties for the conditions dealt with in Theorem 5.2.

Let D be an ultrafilter over λ, and let $f : \lambda \to \mu$. The ultrafilter $f(D)$ over μ is defined by $Y \in f(D)$ if and only if $f^{-1}(Y) \in D$.

Fact 6.1. *Suppose that* X *is a topological space,* $\mathcal{F}$ *is a family of subsets of* X, D *is an ultrafilter over* λ, *and* $f : \lambda \to \mu$.

If X *is* $\mathcal{F}$*-*D*-compact, then* X *is* $\mathcal{F}$*-*$f(D)$*-compact,* If D is an ultrafilter over some set Z, and μ is a cardinal, D is said to be μ-decomposable if and only if there exists a function $f : Z \to \mu$ such that $f(D)$ is uniform over μ.

The next corollary implies that if every ultrafilter uniform over λ is μ-decomposable and the conditions in Theorem 5.2 hold for the cardinal λ, then they hold for the cardinal μ, too.

Corollary 6.2. *Suppose that λ is an infinite cardinal, and K is a set of infinite cardinals, and suppose that every uniform ultrafilter over λ is μ-decomposable, for some $\mu \in K$.*

If X is a topological space, $\mathcal{F}$ is a family of subsets of X and one (and hence all) of the conditions in Theorem 5.2 hold for λ, then there is $\mu \in K$ such that the conditions in Theorem 5.2 hold when λ is everywhere replaced by μ.

The same applies with respect to Corollary 5.5.

Proof. Suppose that the conditions in Theorem 5.2 hold for λ. By Condition 5.2 (1), there exists some ultrafilter D uniform over λ such that X is $\mathcal{F}$-D-compact. By assumption, there exist $\mu \in K$ and $f : \lambda \to \mu$ such that $D' = f(D)$ is uniform over μ. By Fact 6.1, X is $\mathcal{F}$-D'-compact, hence Condition 5.2 (1) holds for the cardinal μ. $\square$

There are many results asserting that, for some cardinal λ and some set K, the assumption in Corollary 6.2 holds. In order to state some of these results in a more concise way, let us denote by $\lambda \stackrel{\infty}{\Rightarrow} K$, for K a set of infinite cardinals, the statement that the assumption in Corollary 6.2 holds. That is, $\lambda \stackrel{\infty}{\Rightarrow} K$ means that every uniform ultrafilter over λ is μ-decomposable, for some $\mu \in K$. In the case when $K = \{\mu\}$, we simply write $\lambda \stackrel{\infty}{\Rightarrow} \mu$ in place of $\lambda \stackrel{\infty}{\Rightarrow} K$. The reason for the superscript ∞ is only to keep the notation consistent with the notation used in former papers (e. g. [Li3]). Notice that many conditions equivalent to $\lambda \stackrel{\infty}{\Rightarrow} K$ can be obtained from [Li3, Part VI, Theorems 8 and 10], by letting $\kappa = 2^\lambda$ there (equivalently, letting κ be arbitrarily large) there.

The following are trivial facts about the relation $\lambda \stackrel{\infty}{\Rightarrow} K$. If $\lambda \in K$, then $\lambda \stackrel{\infty}{\Rightarrow} K$ holds. In particular, $\lambda \stackrel{\infty}{\Rightarrow} \lambda$ holds. If $\lambda \stackrel{\infty}{\Rightarrow} K$ holds, and $K' \supseteq K$, then $\lambda \stackrel{\infty}{\Rightarrow} K'$ holds, too.

In the next Theorem we reformulate, according to the present terminology, some of the results on decomposability of ultrafilters collected in [Li4]. In order to state the theorem, we need to introduce some notational conventions. By λ^{+n} we denote the n^{th} successor of λ, that is, $\lambda^{+n} = \lambda^{\underbrace{+\cdots+}_{n \text{ times}}}$. By $\beth_n(\lambda)$ we denote the n^{th} iteration of the power set of λ; that is, $\beth_0(\lambda) = \lambda$, and $\beth_{n+1}(\lambda) = 2^{\beth_n(\lambda)}$. As usual, $[\mu, \lambda]$ denotes the interval $\{\nu \mid \mu \leq \nu \leq \lambda\}$.

Theorem 6.3. *The following hold.*

(1) *If λ is a regular cardinal, then $\lambda^+ \stackrel{\infty}{\Rightarrow} \lambda$.*
(2) *More generally, if λ is a regular cardinal, then $\lambda^{+n} \stackrel{\infty}{\Rightarrow} \lambda$.*
(3) *If λ is a singular cardinal, then $\lambda \stackrel{\infty}{\Rightarrow} \operatorname{cf} \lambda$.*
(4) *If λ is a singular cardinal, then $\lambda^+ \stackrel{\infty}{\Rightarrow} \{\operatorname{cf} \lambda\} \cup K$, for every set K of regular cardinals $< \lambda$ such that K is cofinal in λ.*
(5) *$\nu^{\kappa^{+n}} \stackrel{\infty}{\Rightarrow} [\kappa, \nu^\kappa]$.*
(6) *If $m \geq 1$, then $\beth_m(\kappa^{+n}) \stackrel{\infty}{\Rightarrow} [\kappa, 2^\kappa]$.*
(7) *If κ is a strong limit cardinal, then $\beth_m(\kappa^{+n}) \stackrel{\infty}{\Rightarrow} \{\operatorname{cf} \kappa\} \cup [\kappa', \kappa)$, for every $\kappa' < \kappa$.*
(8) *If λ is smaller than the first measurable cardinal (or no measurable cardinal exists), then $\lambda \stackrel{\infty}{\Rightarrow} \omega$.*
(9) *More generally, for every infinite cardinal λ, we have that $\lambda \stackrel{\infty}{\Rightarrow} \{\omega\} \cup M$, where M is the set of all measurable cardinals $\leq \lambda$.*
(10) *If there is no inner model with a measurable cardinal, and $\lambda \geq \mu$ are infinite cardinals, then $\lambda \stackrel{\infty}{\Rightarrow} \mu$.*

In particular, Corollary 6.2 applies in each of the above cases.

Remark 6.4. Notice that, by [Li4, Properties 1.1(iii),(x)], and arguing as in [Li4, Consequence 1.2], the relation $\lambda \stackrel{\infty}{\Rightarrow} \mu$ is equivalent to "every λ-decomposable ultrafilter is μ-decomposable".

Similarly, $\lambda \stackrel{\infty}{\Rightarrow} K$ is equivalent to "every λ-decomposable ultrafilter is μ-decomposable, for some $\mu \in K$".

Proof of Theorem 6.3. (1)-(4) and (8)-(9) are immediate from classical results about ultrafilters; see, e. g., the comments after Problem 6.8 in [Li4].

(5)-(7) follow from [Li4, Theorem 4.3 and Property 1.1(vii)].

(10) is immediate from [Do, Theorem 4.5], by using [Li4, Properties 1.1 and Remark 1.5(b)]. □

By Remark 6.4, we get the following transitivity properties of the relation $\lambda \overset{\infty}{\Rightarrow} K$.

Proposition 6.5. *The following hold.*

(1) *If* $\lambda \overset{\infty}{\Rightarrow} \mu$ *and* $\mu \overset{\infty}{\Rightarrow} K$, *then* $\lambda \overset{\infty}{\Rightarrow} K$.
(2) *More generally, suppose that* $\lambda \overset{\infty}{\Rightarrow} K$ *and, for every* $\mu \in K$, *it happens that* $\mu \overset{\infty}{\Rightarrow} H_\mu$, *for some set* H_μ *depending on* μ. *Then* $\lambda \overset{\infty}{\Rightarrow} \bigcup_{\mu \in K} H_\mu$.
(3) *Suppose that* $\lambda \overset{\infty}{\Rightarrow} K$, $\mu \in K$, *and* $\mu \overset{\infty}{\Rightarrow} K'$, *for some set* $K' \subseteq K$ *such that* $\mu \notin K'$. *Then* $\lambda \overset{\infty}{\Rightarrow} K \setminus \{\mu\}$.
(4) *More generally, suppose that* $\lambda \overset{\infty}{\Rightarrow} K$, $H \subseteq K$ *and, for every* $\mu \in H$, *it happens that* $\mu \overset{\infty}{\Rightarrow} K \setminus H$. *Then* $\lambda \overset{\infty}{\Rightarrow} K \setminus H$.

Proof. (1) and (2) follow from Remark 6.4.

(4) is immediate from (2), by taking $H_\mu = K \setminus H$, if $\mu \in H$, and taking $H_\mu = \{\mu\}$, if $\mu \in K \setminus H$, since, trivially $\mu \overset{\infty}{\Rightarrow} \mu$.

(3) is a particular case of (4), since $K' \subseteq K \setminus \{\mu\}$. □

Corollary 6.6. *Suppose that* $\kappa < \nu$ *are infinite cardinals, and that either* $K = [\kappa, \nu]$, *or* $K = [\kappa, \nu)$.

(a) If $\lambda \overset{\infty}{\Rightarrow} K$, *then* $\lambda \overset{\infty}{\Rightarrow} S$, *where* S *is the set containing* κ, *containing all limit cardinals of* K, *and containing all cardinals of* K *which are successors of singular cardinals.*

(b) More generally, if $\lambda \overset{\infty}{\Rightarrow} K$, *then* $\lambda \overset{\infty}{\Rightarrow} L$, *where* L *is the set of all* $\mu \in K$ *such that either*

(1) $\mu = \kappa$, *or*
(2) μ *is singular and* $\mathrm{cf}\, \mu < \kappa$, *or*
(3) $\mu = \varepsilon^+$, *for some singular* ε *such that* $\mathrm{cf}\, \varepsilon < \kappa$, *or*
(4) μ *is weakly inaccessible.*

In particular, the above statements can be used to refine Theorem 6.3(5)-(6).

Proof. Clearly, (a) follows from (b). In order to prove (b), let $H = K \setminus L$, thus $L = K \setminus H$.

By Proposition 6.5(4), it is enough to show that if $\mu \in H$, then $\mu \overset{\infty}{\Rightarrow} L$.

This is trivial if $H = \emptyset$. Otherwise, suppose by contradiction that there is some $\mu \in H$ such that $\mu \overset{\infty}{\Rrightarrow} L$ fails. Let μ_0 be the least such μ.

We now show that there is some $\mu' < \mu_0$ such that $\mu' \geq \kappa$ and $\mu_0 \overset{\infty}{\Rrightarrow} \mu'$. This follows from Theorem 6.3(1), if μ_0 is the successor of some regular cardinal, since $\mu_0 > \kappa \notin H$, by Clause (1). The existence of μ' follows from Theorem 6.3(4), if $\mu_0 = \varepsilon^+$ with ε singular such that $\operatorname{cf} \varepsilon \geq \kappa$. Finally, the existence of μ' follows from Theorem 6.3(3), if μ_0 is singular and $\operatorname{cf} \mu_0 \geq \kappa$. By Clauses (2)-(4), no other possibility can occur for μ_0, since $\mu_0 \in H$, that is, $\mu_0 \notin L$.

Since $\kappa \leq \mu' < \mu_0$, then $\mu' \overset{\infty}{\Rrightarrow} L$. This is trivial if $\mu' \in L$; and follows from the minimality of μ_0, if $\mu' \notin L$, which means $\mu' \in H = K \setminus L$.

From $\mu_0 \overset{\infty}{\Rrightarrow} \mu'$, and $\mu' \overset{\infty}{\Rrightarrow} L$, we infer $\mu_0 \overset{\infty}{\Rrightarrow} L$, by applying Proposition 6.5(1). We have reached the desired contradiction. $\square$

Some more results about the relation $\lambda \overset{\infty}{\Rrightarrow} K$ follow from results in [Li4]. See [Li5]. See also the comments after [Li4, Problem 6.8], in particular, for some open problems concerning transfer of decomposability for ultrafilters.

In the particular case when $\mathcal{F}$ is the set of all singletons, many versions of Corollary 6.2 are known, and are usually stated by means of conditions involving $[\lambda, \lambda]$-compactness (for regular cardinals, the conditions are equivalent by Theorem 4.4). Caicedo [Ca1] and [Ca2, Corollary 1.8(ii)] proved, among other, that every productively $[\lambda^+, \lambda^+]$-compact family of topological spaces is productively $[\lambda, \lambda]$-compact. More generally, among other, we proved in [Li2, Theorem 16] that if a product of topological spaces is $[\lambda^+, \lambda^+]$-compact, then all but at most λ factors are $[\lambda, \lambda]$-compact. Results related to Corollary 6.2 appear in [Ca1, Ca2, Li1] and [Li4, Corollary 4.6]: generally, they deal with (λ, μ)-regularity of ultrafilters, which is a notion tightly connected to decomposability, since, for λ a regular cardinal, an ultrafilter is λ-decomposable if and only if it is (λ, λ)-regular. Stronger related results appear in [Li3], dealing also with equivalent notions from Model Theory and Set Theory: in particular, see [Li3, Part VI, Theorem 8]. Even in the case when $\mathcal{F}$ is the set of all singletons, some consequences of Theorem 6.3 and Corollaries 6.6 and 6.2 appear to be new, particularly, in the case of singular cardinals.

Already the special case $\mu = \omega$ for pseudocompactness of Corollary 6.2 appears to have some interest.

Corollary 6.7. *Suppose that λ is an infinite cardinal, and suppose that every uniform ultrafilter over λ is ω-decomposable (for example, this happens when either* $\operatorname{cf} \lambda = \omega$*, or when λ is less than the first measurable cardinal, or if there exists no inner model with a measurable cardinal).*

Suppose that X is a topological space satisfying one of the conditions in Corollary 5.5. Then X is D-pseudocompact, for some ultrafilter D uniform over ω. In particular, if X is Tychonoff, then X is pseudocompact, and, furthermore, all powers of X are pseudocompact.

Proof. Immediate from Remark 4.5. $\square$

Garcia-Ferreira [Ga2] contains results related to Corollary 6.7. In particular, [Ga2] analyzes the relationship between D-(pseudo)compactness and D'-(pseudo)compactness for various ultrafilters D, D'.

7. $[\mu, \lambda]$-compactness relative to a family $\mathcal{F}$

We can generalize the notion of $[\mu, \lambda]$-compactness in another direction.

Definition 7.1. If X is a topological space, and $\mathcal{G}$ is a family of subsets of X, we say that X is $[\mu, \lambda]$-*compact relative to* $\mathcal{G}$ if and only if the following holds.

For every family $(G_\alpha)_{\alpha \in \lambda}$ of elements of $\mathcal{G}$, if, for every $Z \subseteq \lambda$ with $|Z| < \mu$, $\bigcap_{\alpha \in Z} G_\alpha \neq \emptyset$, then $\bigcap_{\alpha \in \lambda} G_\alpha \neq \emptyset$.

The usual notion of $[\mu, \lambda]$-compactness can be obtained from the above definition in the particular case when $\mathcal{G}$ is the family of all closed sets of X.

If $\mathcal{G}$ is the family of all zero sets of some Tychonoff space X, then X is $[\omega, \lambda]$-compact relative to $\mathcal{G}$ if and only if X is λ-pseudocompact. See, e. g., [Ga2, St] for results about λ-pseudocompactness, equivalent formulations, and further references. Notice that [Ga2] shows that it is possible, under some set-theoretical assumptions, to construct a space which is not ω_1-pseudocompact, but which is D-pseudocompact, for some ultrafilter D uniform over ω_1.

Proposition 7.2. *Suppose that X is a topological space, and $\mathcal{G}$ is a family of subsets of X. Then the following are equivalent.*

(a) X is $[\mu, \lambda]$-compact relative to $\mathcal{G}$.

(b) X is $[\kappa, \kappa]$-compact relative to $\mathcal{G}$, for every κ with $\mu \leq \kappa \leq \lambda$.

Proof. Similar to the proof of the classical result for $[\mu, \lambda]$-compactness, see, e. g., [Li2, Proposition 8]. □

There is some connection between the compactness properties introduced in Definitions 4.2 and 7.1. In order to deal with the relationship between the two properties, it is convenient to introduce a common generalization.

Definition 7.3. If X is a topological space, $\mathcal{F}$ and $\mathcal{G}$ are families of subsets of X, we say that X is $\mathcal{F}$-$[\mu, \lambda]$-*compact relative to* $\mathcal{G}$ if and only if the following holds.

For every family $(G_\alpha)_{\alpha \in \lambda}$ of elements of $\mathcal{G}$, if, for every $Z \subseteq \lambda$ with $|Z| < \mu$, there exists $F \in \mathcal{F}$ such that $\bigcap_{\alpha \in Z} G_\alpha \supseteq F$, then $\bigcap_{\alpha \in \lambda} G_\alpha \neq \emptyset$.

Thus, $\mathcal{F}$-$[\mu, \lambda]$-compactness is $\mathcal{F}$-$[\mu, \lambda]$-compactness relative to $\mathcal{G}$, when $\mathcal{G}$ is the family of all closed subsets of X.

On the other hand, $[\mu, \lambda]$-compactness relative to $\mathcal{G}$ is $\mathcal{F}$-$[\mu, \lambda]$-compactness relative to $\mathcal{G}$, when $\mathcal{F}$ is the set of all singletons of X.

Proposition 7.4. *Suppose that λ and μ are infinite cardinals, and let $\kappa = \sup\{\lambda^{\mu'} \mid \mu' < \mu\}$. Suppose that X is a topological space, and $\mathcal{F}$ is a family of subsets of X. Let $\mathcal{F}^*$ ($\mathcal{F}^*_{\leq \kappa}$, resp.) be the family of all subsets of X which are the closure of the union of some family of ($\leq \kappa$, resp.) sets in $\mathcal{F}$. Then:*

(1) *The following conditions are equivalent.*
   (a) X is $\mathcal{F}$-$[\mu, \lambda]$-compact.
   (b) X is $\mathcal{F}$-$[\mu, \lambda]$-compact relative to $\mathcal{F}^*$.
   (c) X is $\mathcal{F}$-$[\mu, \lambda]$-compact relative to $\mathcal{F}^*_{\leq \kappa}$.

(2) *Suppose in addition that all members of $\mathcal{F}$ are nonempty. If X is $[\mu, \lambda]$-compact relative to $\mathcal{F}^*_{\leq \kappa}$, then X is $\mathcal{F}$-$[\mu, \lambda]$-compact.*

Proof. In (1), the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are trivial.

In order to show that (c) $\Rightarrow$ (a) holds, let $(C_\alpha)_{\alpha \in \lambda}$ be a family of closed sets of X such that, for every $Z \subseteq \lambda$ with $|Z| < \mu$, there exists $F_Z \in \mathcal{F}$ such that $\bigcap_{\alpha \in Z} C_\alpha \supseteq F_Z$.

For $\alpha \in \lambda$, let C'_α be the closure of $\bigcup_{\alpha \in Z} F_Z$. Clearly, for every $\alpha \in \lambda$, we have $C_\alpha \supseteq C'_\alpha$. Since there are κ subsets of λ of cardinality $< \mu$, that is, we can choose Z in κ-many ways, we have that each C'_α is the closure of the union of $\leq \kappa$ elements from $\mathcal{F}$. Thus we can apply (c) in order to get $\bigcap_{\alpha \in \lambda} C_\alpha \supseteq \bigcap_{\alpha \in \lambda} C'_\alpha \neq \emptyset$.

(2) is immediate from (1) (c) $\Rightarrow$ (a), since if $\mathcal{F}$ is a family of *nonempty* subsets of X, then $[\mu, \lambda]$-compactness relative to some family $\mathcal{G}$ implies $\mathcal{F}$-$[\mu, \lambda]$-compactness relative to $\mathcal{G}$. □

Remark 7.5. The value $\kappa = \sup\{\lambda^{\mu'} \mid \mu' < \mu\}$ in Proposition 7.4 can be improved to $\kappa =$ the cofinality of the partial order $S_\mu(\lambda)$ (see [Li4]).

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