FROM CONGRUENCE IDENTITIES TO TOLERANCE IDENTITIES

PAOLO LIPPARINI

Dedicated to the 75th Birthday of Béla Csákány

Abstract. We show, under a weak assumption on the term p, that a variety of general algebras satisfies the congruence identity $p(\alpha_1, \dots, \alpha_n) \subseteq q(\alpha_1, \dots, \alpha_n)$ if and only if it satisfies the tolerance identity $p(\Theta_1, \dots, \Theta_n) \subseteq q(\Theta_1, \dots, \Theta_n)$, provided we restrict ourselves to tolerances representable as $R \circ R^-$. Varieties in which every tolerance is representable include all congruence permutable varieties and all varieties of lattices.

For arbitrary tolerances, the congruence identity $p(\alpha_1, \dots, \alpha_n) \subseteq q(\alpha_1, \dots, \alpha_n)$ is equivalent to the identity $p(\Theta_1 \circ \Theta_1, \dots, \Theta_n \circ \Theta_n) \subseteq q(\Theta_1 \circ \Theta_1, \dots, \Theta_n \circ \Theta_n)$. See Theorems 3.1, 3.2 and 3.3.

Our arguments essentially deal with labeled graphs, rather than terms; we try to clarify the role of graphs in the study of Mal'cev conditions (see especially Proposition 7.6 and Theorem 7.7).

1. Introduction

Tolerance identities play an increasingly important role in Universal Algebra. One of the first applications of tolerance identities appears in H. P. Gumm's important monograph [G]. He discovered the Shifting Principle replacing a congruence by a tolerance in the Shifting Lemma, thus getting a great deal of consequences which otherwise could not be obtained; in particular, he simplified and extended commutator theory for congruence modular varieties.

More recently, a stronger tolerance identity, called TIP, has proven particularly useful in the study of congruence modular varieties, and has been applied in order to prove deep new theorems with relatively simple methods: see [CH1, CH2, CHL, CHR]. For example, TIP has been used to show that every congruence identity implying congruence modularity is equivalent to a Mal'cev condition. Moreover, TIP has been used to provide a

2000 *Mathematics Subject Classification.* Primary 08A30, 08B05.

Key words and phrases. Congruence, tolerance identity; Mal'cev condition, labeled graph, regular term, representable tolerance.

The author has received support from MPI and GNSAGA. We acknowledge useful correspondence with G. Czédli and K. Kearnes. In particular, we express our gratitude to G. Czédli for many suggestions which helped understanding varieties with representable tolerances, as well as improving the exposition.

simple proof of a result by R. Freese and B. Jónsson [FJ] asserting that every congruence modular variety is congruence arguesian. In fact, the proof using TIP provides the new result in which the conclusion is strengthened to higher arguesian in the sense of Haiman [H2]. Variations on TIP have been introduced in [L2, L5].

[KK1] deals with another interesting tolerance identity, the Triangular principle, which is connected with join semidistributivity. See [KK1] for the history of the Triangular principle, and for further references. By the way, more conditions equivalent to the Triangular principle are given in the preliminary notes [L3] (cf. [L3, Part II, Theorem 3]).

Tolerances play a prominent role in [KK2], which extends a large part of D. Hobby and R. McKenzie's classification [HMK] of locally finite varieties to arbitrary varieties. Many results from [KK2] are new even in the particular case of locally finite varieties, and provide novel interesting insights. As the title says, the main emphasis in [KK2] is about congruence lattices, however, most theorems and proofs involve tolerances. In particular, [KK2] studies many different kinds of centralizer relations both for congruences and for tolerances. It frequently turns out that, for some given centralizer notion, the omission of abelian tolerances is equivalent to the omission of abelian congruences.

Our results parallel the above results from [KK2] in a different context. In the present work we shall describe a result of a completely general nature which connects congruence identities and tolerance identities. Our main argument Theorem 3.1 deals with terms p and q for the language $\{\circ, \cap\}$ (for short, $\{\circ, \cap\}$-terms), where variables are usually understood to be either congruences α, β, ... or tolerances Θ, Ψ, ... We interpret $\circ$ and $\cap$ as (relational) composition and intersection, the latter usually denoted by juxtaposition. We are able to show that, in many cases, if a variety $\mathcal{V}$ of algebras satisfies a congruence identity $p(\overline{\alpha}) \subseteq q(\overline{\alpha})$, then $\mathcal{V}$ satisfies the tolerance identity $p(\overline{\Theta}) \subseteq q(\overline{\Theta})$. Here, as usual, we say that a variety $\mathcal{V}$ satisfies a congruence identity $p(\overline{\alpha}) \subseteq q(\overline{\alpha})$ if and only if $p(\overline{\alpha}) \subseteq q(\overline{\alpha})$ holds for every algebra $\mathbf{A} \in \mathcal{V}$ and n-tuple of congruences $\overline{\alpha}$ on $\mathbf{A}$. The meaning of "*$\mathcal{V}$ satisfies a tolerance identity*" is similar.

In order to avoid possible misunderstandings, notice that, in the case when only $\circ$ and $\cap$ are allowed in p and q, then $p \subseteq q$ is not able to express *lattice* identities. Of course, $\alpha + \beta$, the join in the lattice of *congruences*, equals $\bigcup_n \underbrace{\alpha \circ \beta \circ \alpha \dots}_{n \text{ factors}}$, so that we can approximate $\alpha + \beta$ by a sequence of terms $p_n(\alpha, \beta) = \alpha \circ \beta \circ \alpha \dots$ (n factors). However, when dealing with *tolerances*, already $p_3(\Theta, \Psi) = \Theta \circ \Psi \circ \Theta$ is generally larger than the join of Θ and Ψ in the lattice of *tolerances*. In other words, we can transfer *identities in $\circ$ and $\cap$* from congruences to tolerances, but such a transfer is not generally possible for *lattice identities*. However, the above remarks suggest that there is a version of our results in which p and q are allowed to contain the symbol $+$. In such a version, $\Theta + \Psi$ has to be interpreted as

$\bigcup_n \Theta \circ \Psi \circ \Theta \dots$, rather than the join of Θ and Ψ in the lattice of tolerances (see Theorems 3.2 and 3.3).

Our methods rely deeply on the theory of (strong) Mal'cev conditions associated with the congruence identity $p(\overline{\alpha}) \subseteq q(\overline{\alpha})$. Though, in principle, the paper is self contained, the reader with a previous knowledge of Mal'cev conditions will probably feel much more comfortable. The general connection between congruence identities and Mal'cev conditions has been developed by R. Wille [Wi] and A. Pixley [P2]. They extended to an abstract setting the particular cases discovered before by Mal'cev itself [M], A. Pixley [P1], B. Jónsson [J1], A. Day [D] and others (see [J2, Section 2] and [CHL] for further references). The reader not familiar with Mal'cev conditions is advised to consider such particular cases first (as found, e.g., in [J2]), since the general theory is better understood by examples. Good introductions to the Wille and Pixley algorithm can be found, e.g., in [HC], or [FMK, Chapter XIII]. Alternative ways to Wille and Pixley's theorem are given in [J2, Ta].

The approach to Mal'cev conditions used in the present paper is by means of labeled graphs associated with terms. This approach is due to G. Czédli, who first employed it in the particular case of congruence join semidistributivity [Cz1]; the general method is then described in [Cz2, Cz3]. Formally, [Cz3] deals only with ternary terms, but the definitions on p. 104-105 are fully general and can be applied to terms of arbitrary arity. The method is also described in [CD], which in fact obtains results more general than the original Wille Pixley theorem. Notice that the graph-theoretical approach is much more than an expository aid. Our core argument essentially deals with labeled graphs, rather than with terms (see Theorem 7.7). As another example, the evaluation of the k_i's in the comment after Proposition 7.6 can be conceivably understood only by means of graphs.

The paper is divided as follows. In Section 2 we recall the definition of the labeled graph associated with a term and we define the notions of a regular term and of a representable and a weakly representable tolerance. In Section 3 we state our main results. Theorem 3.1 deals with $\{\circ, \cap\}$ terms, while in Theorem 3.2 terms are allowed to contain $+$. Theorem 3.3, whose proof relies deeply on [CHL], shows that the assumption that tolerances are representable is not needed in the case of congruence modular varieties, when q is a $\{\cap, +\}$-term. In Section 3 we also present an example which illustrates our two main modifications of the Wille and Pixley argument. Section 4 contains complete proofs of Theorems 3.1 and 3.2. In Section 5 we give immediate applications which deal with congruence identities only. In Section 6 we study the central notions of representable and weakly representable tolerances. We show that the variety of sets, all varieties of lattices and all congruence permutable varieties have the property that every tolerance is weakly representable. On the other side, we construct many examples of algebras in which there exists a non-representable tolerance, among them a set, a semilattice and an algebras with a majority operation. Finally, we show that the property that all tolerances in a variety are representable is not equivalent to a Mal'cev condition. Section 7 contains additional remarks, and some generalizations. We show that the assumption that p is a regular term is necessary in Theorem 3.1; however, we give a version in which this assumption is not needed (Proposition 7.6). Finally, we show that our arguments actually deal with graphs rather than terms; this setting appears to be more general (Theorem 7.7).

2. Preliminaries

In order to state our results we need some definitions. If p is a term for the language $\{\circ, \cap\}$, let us define inductively as follows the sets $\mathbf{L}_p$ resp. $\mathbf{R}_p$ of *variables on the left* resp. *right side of* p.

(i) If $p = X_i$ is a variable then $\mathbf{L}_p = \{X_i\}$ and $\mathbf{R}_p = \{X_i\}$.

(ii) If $p = q \circ r$ then $\mathbf{L}_p = \mathbf{L}_q$ and $\mathbf{R}_p = \mathbf{R}_r$.

(iii) If $p = q \cap r$ then $\mathbf{L}_p = \mathbf{L}_q \cup \mathbf{L}_r$ and $\mathbf{R}_p = \mathbf{R}_q \cup \mathbf{R}_r$.

We will be mainly concerned with regular terms. We define the class of *regular* terms to be the smallest class of terms which

(a) contains all variables;

(b) contains $p = q \circ r$ whenever q and r are regular terms and $\mathbf{R}_q \cap \mathbf{L}_r = \emptyset$;

(c) contains $p = q \cap r$ whenever q and r are regular terms, $\mathbf{L}_q \cap \mathbf{L}_r = \emptyset$ and $\mathbf{R}_q \cap \mathbf{R}_r = \emptyset$.

Thus, for example, $X \cap (Y \circ Z)$ and $X \cap (Y \circ (Z \cap (X \circ Z \circ Y)))$ are regular, while $X \circ X$ and $(X \circ Y) \cap (Z \circ Y)$ are not regular. Notice that most terms which have found applications in universal algebra are in fact regular terms. The above definitions, as well as some further remarks, are better understood by means of the notion of the labeled graph associated with a term. See e.g. [Cz1, Cz2, Cz3, Cz4, Cz5, CD, H1] for further information and references on graphs associated with terms.

If p is a term for the language $\{\circ, \cap\}$, the *labeled graph* $\mathbf{G}_p$ *associated with* p has a left distinguished vertex, a right distinguished vertex and each of its edges has a label. $\mathbf{G}_p$ is defined inductively as follows.

(i) If $p = X_i$ is a variable, then $\mathbf{G}_p$ consists of two vertices (which are the distinguished ones) connected by one edge labeled by X_i.

(ii) If $p = q \circ r$ then $\mathbf{G}_p$ is obtained by putting a copy of $\mathbf{G}_q$ on the left, a copy of $\mathbf{G}_r$ on the right, and by attaching the two graphs by joining together the right distinguished vertex of $\mathbf{G}_q$ with the left distinguished vertex of $\mathbf{G}_r$ (the above vertices join into one). The left distinguished vertex of $\mathbf{G}_p$ is the left distinguished vertex of $\mathbf{G}_q$, and the right distinguished vertex of $\mathbf{G}_p$ is the right distinguished vertex of $\mathbf{G}_r$.

(iii) If $p = q \cap r$ then $\mathbf{G}_p$ is obtained by putting a copy of $\mathbf{G}_q$ above a copy of $\mathbf{G}_r$ and by attaching the two graphs by joining together the right distinguished vertices, as well as by joining together the left distinguished vertices. The new vertices obtained by such unions are the distinguished vertices of $\mathbf{G}_p$.

The reader is advised to draw pictures or look, for example, at [Cz2, p. 47, Figure 1] or [Cz3, p. 105]. Graph theoretically, a term p is regular if and only if in the graph $\mathbf{G}_p$ no vertex has two distinct adjacent edges labeled by the same name. In the present context the importance of the notion of the labeled graph associated with a term arises from the following observation, coming from [Cz3, Claim 1] (see also [CD, Proposition 3.1]). Here we deal with a simpler version, due to the fact that $+$ is not allowed in the terms p and q. However, our version deals with symmetric relations, rather than congruences. For sake of notational simplicity, if R is a binary relation, we shall sometimes write $a\,R\,b$ in place of $(a,b) \in R$.

Proposition 2.1. *Suppose that $R_1, \dots, R_n$ are symmetric relations on some set A and $p(X_1, \dots, X_n)$ is a term for the language $\{\circ, \cap\}$. Let $\mathbf{G}_p$ be the labeled graph associated with p. Let V denote the set of vertices of $\mathbf{G}_p$ and let v_ℓ and v_r be its distinguished left and right vertices.*

If $a, b \in A$, then $(a,b) \in p(R_1, \dots, R_n)$ if and only if there exists some function $c : V \to A$ sending $v \in V$ to $c_v \in A$ such that (i) $a = c_{v_\ell}$ and $c_{v_r} = b$ and (ii) whenever two vertices $v, w \in V$ are connected by an edge labeled by X_i, then $c_v\, R_i\, c_w$.

Following [Cz3, p. 106], we say that a and b can be *connected by the graph* $\mathbf{G}_p$, if the situation in the statement of Proposition 2.1 occurs.

We now introduce a key definition for our results.

Definition 2.2. A tolerance Θ of some algebra $\mathbf{A}$ is *representable* if and only if there exists a compatible and reflexive relation R on $\mathbf{A}$ such that $\Theta = R \circ R^-$ (here, R^- denotes the converse of R).

A tolerance Θ of some algebra $\mathbf{A}$ is *weakly representable* if and only if there exists a set K (possibly infinite) and there are compatible and reflexive relations R_k ($k \in K$) on $\mathbf{A}$ such that $\Theta = \bigcap_{k \in K} (R_k \circ R_k^-)$.

Notice that if R is a compatible and reflexive relation, then $R \circ R^-$ is always a tolerance. Thus a tolerance is weakly representable if and only if it is the intersection of some family of representable tolerances.

3. Statement of the main theorems and an illustrative example

Theorem 3.1. *Suppose that $\mathcal{V}$ is a variety and that p and q are terms of the same arity for the language $\{\circ, \cap\}$. If p is regular, then the following are equivalent.*

(i) $\mathcal{V}$ satisfies the congruence identity $p(\alpha_1, \dots, \alpha_n) \subseteq q(\alpha_1, \dots, \alpha_n)$.

(ii) $\mathcal{V}$ satisfies the strong Mal'cev condition $M(p \subseteq q)$ (see Definition 4.1 below).

(iii) The tolerance identity $p(\Theta_1, \dots, \Theta_n) \subseteq q(\Theta_1, \dots, \Theta_n)$ holds for every algebra $\mathbf{A}$ in $\mathcal{V}$ and for all representable tolerances $\Theta_1, \dots, \Theta_n$ of $\mathbf{A}$.

(iii)′ The tolerance identity $p(\Theta_1, \dots, \Theta_n) \subseteq q(\Theta_1, \dots, \Theta_n)$ holds for every algebra $\mathbf{A}$ in $\mathcal{V}$ and for all weakly representable tolerances $\Theta_1, \dots, \Theta_n$ of $\mathbf{A}$.

(iv) $\mathcal{V}$ *satisfies the tolerance identity* $p(\Theta_1 \circ \Theta_1, \dots, \Theta_n \circ \Theta_n) \subseteq q(\Theta_1 \circ \Theta_1, \dots, \Theta_n \circ \Theta_n)$.

The word *identity* in the statement of Theorem 3.1 is justified since the inclusion $p \subseteq q$ is equivalent to the identity $p = pq$ (recall that juxtaposition denotes intersection).

In the following theorem we consider terms which can contain the operation $+$. Here $+$ is always interpreted to be the operation on reflexive and admissible relations on some algebra defined by: $R + S$ is the smallest transitive relation containing both R and S. Thus $R + S = \bigcup_m R \circ_m S$, where $R \circ_m S$ denotes $\underbrace{R \circ S \circ R \dots}_{m \text{ factors}}$. Notice that if α and β are congruences, then $\alpha + \beta$ is the join of α and β in the lattice of congruences, while if Θ and Φ are tolerances, then $\Theta + \Phi$ is much larger than the join of Θ and Φ in the lattice of tolerances. In fact $\Theta + \Phi$ turns out to be the smallest congruence containing both Θ and Φ.

If p is a $\{\circ, \cap, +\}$-term and n is an integer, let us denote by p_n the $\{\circ, \cap\}$-term obtained from p by substituting every occurrence of $+$ with $\circ_n$ (see [CHL] for more details).

Theorem 3.2. *Suppose that* p *and* q *are* $\{\circ, \cap, +\}$*-terms. If either* p_3 *or* p_4 *is regular, then Conditions (i), (iii), (iii)′ and (iv) in Theorem 3.1 are equivalent (provided* $+$ *is interpreted in the above sense).*

In the particular case when p is a $\{\circ, \cap\}$-term then p_h coincides with p, and we obtain back Theorem 3.1. When p and q are $\circ$-free then we are dealing with a lattice identity (in the congruence lattice); notice that the majority of important lattice identities belong to the scope of Theorem 3.2. However recall that, as we mentioned, Theorems 3.1 and 3.2 *do not* deal with lattice identities in *tolerance* lattices.

In congruence modular varieties there is a version of Theorems 3.1 and 3.2 in which the assumption of representability of tolerances is not necessary. In what follows * denotes transitive closure.

Theorem 3.3. *Suppose that* $\mathcal{V}$ *is a congruence modular variety,* p *is a* $\{\circ, \cap, +\}$*-term and* q *is a* $\{\cap, +\}$*-term. Then the following are equivalent.*

(i) $\mathcal{V}$ *satisfies the congruence identity* $p(\alpha_1, \dots, \alpha_n) \subseteq q(\alpha_1, \dots, \alpha_n)$.

(ii) $\mathcal{V}$ *satisfies the congruence identity* $p_2(\alpha_1, \dots, \alpha_n) \subseteq q(\alpha_1, \dots, \alpha_n)$.

(iii) $\mathcal{V}$ *satisfies the tolerance identity* $p(\Theta_1, \dots, \Theta_n) \subseteq (q(\Theta_1, \dots, \Theta_n))^*$.

(iv) $\mathcal{V}$ *satisfies the tolerance identity* $p_2(\Theta_1, \dots, \Theta_n) \subseteq (q(\Theta_1, \dots, \Theta_n))^*$.

Proof. (i) $\Leftrightarrow$ (ii) is proved in [CHL, Theorem 3].

(iii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (ii) are trivial, since $q(\alpha_1, \dots, \alpha_n)$ is a congruence, hence $(q(\alpha_1, \dots, \alpha_n))^* = q(\alpha_1, \dots, \alpha_n)$.

In order to prove (i) $\Rightarrow$ (iii) recall that congruence modular varieties satisfy the following tolerance identity TIP $(\Theta \cap \Phi)^* = \Theta^* \cap \Phi^*$. Using TIP it is easy to see by induction that $(q(\Theta_1, \dots, \Theta_n))^* = q(\Theta_1^*, \dots, \Theta_n^*)$ (cf., for example, the proof of [CHL, Lemma 1]). Since $\Theta_1^*, \dots, \Theta_n^*$ are congruences, we have, by (i) and the above remark, $p(\Theta_1, \dots, \Theta_n) \subseteq p(\Theta_1^*, \dots, \Theta_n^*) \subseteq q(\Theta_1^*, \dots, \Theta_n^*) = q(\Theta_1, \dots, \Theta_n)^*$.

The proof of (ii) $\Rightarrow$ (iv) is identical to (i) $\Rightarrow$ (iii). $\square$

The main result of [CH1] asserts that 3.3 (i) $\Leftrightarrow$ (iii) holds for two particular identities. Notice however that [CH1] cannot be obtained as a corollary of Theorem 3.3, since the proof of 3.3 relies on [CHL], which in turn relies on the methods discovered in [CH1]. Notice also that the proof of 3.3 does not rely on other results proved here.

We now sketch the new main ideas in the proof of Theorem 3.1. The equivalence (i) $\Leftrightarrow$ (ii) in Theorem 3.1 is a classical result by Wille and Pixley [Wi, P2] (and does not need the assumption that p is regular). Our proof of 3.1 (ii) $\Rightarrow$ (iii) is modeled after the original Wille and Pixley proof of 3.1 (ii) $\Rightarrow$ (i) with two new ideas added. Suppose that (a, b) belongs to $p(\Theta_1, \dots, \Theta_n)$. We have to show that (a, b) belongs to $q(\Theta_1, \dots, \Theta_n)$. By Proposition 2.1, that (a, b) belongs to $p(\Theta_1, \dots, \Theta_n)$ is witnessed by elements $c_v \in A$, where v varies among the vertices of the graph $\mathbf{G}_p$ associated with p. Let us enumerate the c_v's as $c_1, \dots, c_m$. The strong Mal'cev condition $M(p \subseteq q)$ provides certain terms t_w ($w \in W$), which satisfy certain identities. Notice that we use p and q to denote terms whose variables range over congruences or tolerances, while the t_w's are terms of $\mathcal{V}$, and variables of t_w range over elements belonging to some algebra in $\mathcal{V}$. The Wille Pixley proof goes on by using the terms t_w and the identities they satisfy in order to show that the elements $t_w(c_1, \dots, c_m)$ ($w \in W$) witness that (a, b) belongs to $q(\alpha_1, \dots, \alpha_n)$. This is accomplished as follows. Suppose, for sake of simplicity, to have the following identity

$$(^*) \qquad t_w(x, y, z, z, u, u, v) = t_{w'}(x, y, z, z, u, u, v),$$

and suppose that $c_3 \,\alpha\, c_4$ and $c_5 \,\alpha\, c_6$. Then

$$t_w(c_1, c_2, c_3, c_4, c_5, c_6, c_7) \,\alpha\, t_w(c_1, c_2, c_3, c_3, c_5, c_5, c_7) = \\ t_{w'}(c_1, c_2, c_3, c_3, c_5, c_5, c_7) \,\alpha\, t_{w'}(c_1, c_2, c_3, c_4, c_5, c_6, c_7),$$

thus

$$t_w(c_1, c_2, c_3, c_4, c_5, c_6, c_7) \,\alpha\, t_{w'}(c_1, c_2, c_3, c_4, c_5, c_6, c_7).$$

In this way, from each identity of a form similar to (*) (intended to be satisfied in $\mathcal{V}$), one gets a relation of the form $t_w(c_1, \dots, c_m) \,\alpha_i\, t_{w'}(c_1, \dots, c_m)$ which holds in our fixed algebra $\mathbf{A}$. Putting together the above relations, one gets that (a, b) belongs to $q(\alpha_1, \dots, \alpha_n)$, using again Proposition 2.1 (applied to the graph $\mathbf{G}_q$).

If in the above argument α is replaced by a tolerance Θ, the argument breaks, since we only get

$$t_w(c_1, c_2, c_3, c_4, c_5, c_6, c_7) \,\Theta \circ \Theta\, t_{w'}(c_1, c_2, c_3, c_4, c_5, c_6, c_7).$$

Here is our key modification of the argument, which uses the assumption that Θ is representable. Suppose, as above, that the identity (*) holds in

$\mathcal{V}$, and suppose that $c_3 \,\Theta\, c_4$ and $c_5 \,\Theta\, c_6$, for some representable tolerance $\Theta = R \circ R^-$. Thus there are elements b' and b'' such that $c_3 \, R \, b' \, R^- \, c_4$ and $c_5 \, R \, b'' \, R^- \, c_6$. Then, since R is reflexive and compatible,

$$t_w(c_1, c_2, c_3, c_4, c_5, c_6, c_7) \, R \, t_w(c_1, c_2, b', b', b'', b'', c_7) = \\ t_{w'}(c_1, c_2, b', b', b'', b'', c_7) \, R^- \, t_{w'}(c_1, c_2, c_3, c_4, c_5, c_6, c_7).$$

That is,

$$t_w(c_1, c_2, c_3, c_4, c_5, c_6, c_7) \, R \circ R^- \, t_{w'}(c_1, c_2, c_3, c_4, c_5, c_6, c_7),$$

hence

$$t_w(c_1, c_2, c_3, c_4, c_5, c_6, c_7) \,\Theta\, t_{w'}(c_1, c_2, c_3, c_4, c_5, c_6, c_7),$$

since $\Theta = R \circ R^-$, by assumption.

The reason why we can perform the above procedure is that the variable z appears only two times on each side in the identity $t_w(x, y, z, z, u, u, v) = t_{w'}(x, y, z, z, u, u, v)$, and similarly the variable u appears only two times on each side, while the other variables appear just one time. Were we dealing with identities of the form $t_w(x, z, z, z) = t_{w'}(x, z, z, z)$, we could not have performed the above trick. Here is where the assumption that p is regular comes into play: such an assumption implies that each variable appears at most twice on each side of the identities given by the Mal'cev condition $M(p \subseteq q)$, so that we can actually proceed as above.

4. Proofs of Theorems 3.1 and 3.2

In the present section we develop the above arguments in a more detailed way.

Definition 4.1. Suppose that p and q are terms in n variables $\alpha_1, \dots, \alpha_n$ for the language $\{\circ, \cap\}$. Let us consider the labeled graphs $\mathbf{G}_p$ and $\mathbf{G}_q$ associated with p and q. Let V resp. W denote the set of vertices of $\mathbf{G}_p$ resp. $\mathbf{G}_q$. Let v_ℓ and w_ℓ denote the distinguished left vertices of $\mathbf{G}_p$ and $\mathbf{G}_q$, and let v_r and w_r denote the distinguished right vertices.

For each i with $1 \leq i \leq n$, let $\sim_i$ be the least equivalence relation on V such that $v \sim_i v'$ whenever v and v' are vertices of V which are connected by some edge labeled by α_i. For each i with $1 \leq i \leq n$, fix π_i to be any function from V to an arbitrary set of variables with the property that $\ker \pi_i = \sim_i$.

The *strong Mal'cev condition* $M(p \subseteq q)$ associated with the inclusion $p \subseteq q$ involves operations t_w ($w \in W$) depending on $|V|$ variables (in fact, we shall identify the variables of t_w with the vertices of $\mathbf{G}_p$). Given a fixed arbitrary enumeration $v_1, \dots, v_m$ of V, the identities of $M(p \subseteq q)$ are the following:

$$(\ell) \qquad v_\ell = t_{w_\ell}(v_1, v_2, \dots, v_m)$$

$$(\mathrm{r}) \qquad t_{w_r}(v_1, v_2, \dots, v_m) = v_r$$

plus all the identities:

$(\mathrm{m}_{w,w',i})$ $\quad t_w(\pi_i(v_1), \pi_i(v_2), \dots, \pi_i(v_m)) = t_{w'}(\pi_i(v_1), \pi_i(v_2), \dots, \pi_i(v_m)),$

whenever w and w' are vertices of $\mathbf{G}_q$ connected by an edge labeled α_i.

Having defined $M(p \subseteq q)$, we proceed to give the proof of Theorem 3.1.

Proof of Theorem 3.1. As we mentioned, (i) $\Rightarrow$ (ii) is due to Wille and Pixley.

We now prove (ii) $\Rightarrow$ (iii). We suppose that $\mathcal{V}$ has terms satisfying all the identities given by $M(p \subseteq q)$ and that, in some algebra $\mathbf{A} \in \mathcal{V}$, (a, b) belongs to $p(\Theta_1, \dots, \Theta_n)$. We want to show that (a, b) belongs to $q(\Theta_1, \dots, \Theta_n)$. That (a, b) belongs to $p(\Theta_1, \dots, \Theta_n)$ is witnessed by elements c_v $(v \in V)$ satisfying Proposition 2.1. Let us write c_j in place of c_{v_j}. We shall show that the elements $t_w(c_1, \dots, c_m)$ $(w \in W)$ witness that (a, b) belongs to $q(\Theta_1, \dots, \Theta_n)$. For this it is enough to show that the function d which assigns $w \in W$ to $d_w = t_w(c_1, \dots, c_m) \in A$ satisfies the conditions in Proposition 2.1 applied to the graph $\mathbf{G}_q$, with the labels α_i substituted for Θ_i.

The conditions $a = d_{w_\ell}$ and $d_{w_r} = b$ follow immediately from the identities (ℓ) and (r), since $a = c_{v_\ell}$ and $b = c_{v_r}$ by Proposition 2.1. It is thus enough to show that if the vertices $w, w' \in W$ of $\mathbf{G}_q$ are connected by an edge labeled by α_i, then $d_w \,\Theta_i\, d_{w'}$. If we had to show only (ii) $\Rightarrow$ (i) (that is, in the case all $\Theta_i = \alpha_i$ are congruences) then this would follow easily from the identities $(\mathrm{m}_{w,w',i})$, as in the original Wille Pixley proof. Since we have to show (iii), we have to use the additional arguments we have indicated in the preceding section.

First observe that if p is a regular term then, for every $1 \le i \le n$, all equivalence classes of $\sim_i$ have cardinality ≤ 2. Thus, for every i and $v_j, v_h \in V$, it happens that $\pi_i(v_j) = \pi_i(v_h)$ if and only if either $v_j = v_h$ or v_j and v_h are connected by an edge labeled by α_i. For every i, define $\phi_i : \{c_1, \dots, c_m\} \to A$ as follows. If $\{c_j\}$ is a $\sim_i$-equivalence class, let $\phi_i(c_j) = c_j$. If $\{c_j, c_h\}$ is a $\sim_i$-equivalence class, then $c_j \,\Theta_i\, c_h$. Since, by assumption, Θ_i is representable as $\Theta_i = R_i \circ R_i^-$, there is some $b_{ijh} \in A$ such that $c_j \, R_i \, b_{ijh} \, R_i^- \, c_h$, hence $c_h \, R_i \, b_{ijh}$. In this case, define $\phi_i(c_j) = \phi_i(c_h) = b_{ijh}$. Thus if $w, w' \in W$ are connected by an edge labeled by α_i, then, by the definition of ϕ_i, and since R_i is compatible and reflexive

$$t_w(c_1, \dots, c_m) \, R_i \, t_w(\phi_i(c_1), \dots, \phi_i(c_m)).$$

Moreover,

$$t_w(\phi_i(c_1), \dots, \phi_i(c_m)) = t_{w'}(\phi_i(c_1), \dots, \phi_i(c_m)),$$

by identity $(\mathrm{m}_{w,w',i})$ above, and since if $\pi_i(v_j) = \pi_i(v_h)$ then $v_j \sim_i v_h$, which implies $\phi_i(c_j) = \phi_i(c_h)$. Again by the definition of ϕ_i

$$t_{w'}(\phi_i(c_1), \dots, \phi_i(c_m)) \, R_i^- \, t_{w'}(c_1, \dots, c_m).$$

Putting the last three identities together, we get

$$d_w = t_w(c_1, \dots, c_m) \, R_i \circ R_i^- \, t_{w'}(c_1, \dots, c_m) = d_{w'},$$

that is, our desired relation $d_w \,\Theta_i\, d_{w'}$.

The proof of (ii) $\Rightarrow$ (iii)$'$ is entirely similar to (ii) $\Rightarrow$ (iii). If $c_j \sim_i c_h$ and $\Theta = \bigcap_{k \in K_i} (R_k \circ R_k^-)$, let us define as above $\phi_{ik}(c_j) = \phi_{ik}(c_h) = b_{ijhk}$ for some b_{ijhk} such that $c_j \, R_k \, b_{ijhk} \, R_k^- \, c_h$. Then, for every pair (w, w'), apply identity $(\mathrm{m}_{w,w',i})$ $|K_i|$-many times.

The implication (iii)$'$ $\Rightarrow$ (iii) is trivial.

We now prove (iii) $\Rightarrow$ (iv). If Θ is a tolerance, then $\Theta \circ \Theta$ is a tolerance, too; moreover, $\Theta \circ \Theta$ is representable (take $R = \Theta$, and observe that $R = \Theta = \Theta^- = R^-$). Thus (iv) is obtained by applying (iii) to the tolerances $\Theta_1 \circ \Theta_1, \dots, \Theta_n \circ \Theta_n$ in place of the tolerances $\Theta_1, \dots, \Theta_n$.

The implication (iv) $\Rightarrow$ (i) is trivial, since every congruence α is also a tolerance, and $\alpha = \alpha \circ \alpha$.

The proof of Theorem 3.1 is thus complete. $\square$

Proof of Theorem 3.2. It is easy to check that if p_4 is regular then all p_h's with h even are regular and that if p_3 is regular then all p_h's with h odd are regular. The general theory of Mal'cev conditions shows that if $\mathcal{V}$ satisfies the congruence identity $p(\overline{\alpha}) \subseteq q(\overline{\alpha})$, then for every $h \geq 2$ there exists $k \geq 2$ such that $\mathcal{V}$ satisfies the $\{\circ, \cap\}$-congruence identity $p_h(\overline{\alpha}) \subseteq q_k(\overline{\alpha})$ (see [CHL] for references). For all odd resp. even integers $h > 2$, Theorem 3.1 (i) $\Rightarrow$ (iii) implies that the tolerance identity $p_h(\overline{\Theta}) \subseteq q_k(\overline{\Theta})$ holds in $\mathcal{V}$ for all representable tolerances. Since, for every h, $p_h \subseteq p_{h+1}$, we have that for every $h \geq 2$ there exists $k' \geq 2$ such that $\mathcal{V}$ satisfies $p_h(\overline{\Theta}) \subseteq q_{k'}(\overline{\Theta})$ for representable tolerances. Because of the interpretation we have chosen for $+$, this easily implies that $\mathcal{V}$ satisfies $p(\overline{\Theta}) \subseteq q(\overline{\Theta})$ for representable tolerances. Thus we have proved (i) $\Rightarrow$ (iii).

The implication (i) $\Rightarrow$ (iii)$'$ is similar, and relies on the corresponding implication in Theorem 3.1. All other implications are trivial, and are obtained as the corresponding ones in Theorem 3.1. $\square$

Notice that if $p = \gamma(\alpha + \beta) + \alpha\delta$, then p_2 is regular, but for every $i > 2$ p_i is not regular. However, in the above situation, we can equivalently consider the term $p' = \gamma(\beta + \alpha) + \alpha\delta$. In this case, Theorem 3.2 can be applied, since p'_3 is regular.

5. Applications.

Though dealing with tolerances, Theorem 3.1 has an immediate application to congruence identities. Recall that $R \circ_m S = R \circ S \circ R \circ \dots$ with $m-1$ occurrences of $\circ$.

Corollary 5.1. *Suppose that $\mathcal{V}$ is a variety and that p and q are terms for the language $\{\circ, \cap\}$. If p is regular then the following are equivalent.*

(i) $\mathcal{V}$ satisfies the congruence identity $p(\alpha_1, \dots, \alpha_n) \subseteq q(\alpha_1, \dots, \alpha_n)$.

(ii) $\mathcal{V}$ satisfies the congruence identity $p(\beta_1 \circ \gamma_1 \circ \beta_1, \dots, \beta_n \circ \gamma_n \circ \beta_n) \subseteq q(\beta_1 \circ \gamma_1 \circ \beta_1, \dots, \beta_n \circ \gamma_n \circ \beta_n)$.

(iii) For every (equivalently, some) odd $m \geq 1$, $\mathcal{V}$ satisfies the congruence identity $p(\beta_1 \circ_m \gamma_1, \dots, \beta_n \circ_m \gamma_n) \subseteq q(\beta_1 \circ_m \gamma_1, \dots, \beta_n \circ_m \gamma_n)$.

Proof. If m is odd and β_i and γ_i are congruences then $\Theta_i = \beta_i \circ_m \gamma_i$ is a representable tolerance, since $\Theta_i = R_i \circ R_i^-$, for $R = \beta_i \circ_h \gamma_i$, with $h = \frac{m+1}{2}$. Thus, by Theorem 3.1 (i) $\Rightarrow$ (iii), (i) implies that (iii) holds for every odd m.

On the other side, if (iii) holds for some odd $m \geq 1$, we get (i) by applying (iii) in the particular case $\beta_i = \alpha_i$, $\gamma_i = 0$.

(ii) is just a particular case of (iii). □

Corollary 5.1 may be seen as a generalization of A. Day's characterization of congruence modular varieties.

Corollary 5.2. [D] *A variety $\mathcal{V}$ is congruence modular if and only if there exists some integer k such that $\mathcal{V}$ satisfies the congruence identity $\alpha(\beta \circ \alpha\gamma \circ \beta) \subseteq \alpha\beta \circ_k \alpha\gamma$.*

Proof. In order to prove the nontrivial inclusion, suppose that $\mathcal{V}$ satisfies $\alpha(\beta \circ \alpha\gamma \circ \beta) \subseteq \alpha\beta \circ_k \alpha\gamma$. By Corollary 5.1 (i) $\Rightarrow$ (iii), for every odd $m \geq 1$, $\mathcal{V}$ satisfies the congruence identity $\alpha((\beta' \circ_m \beta'') \circ \alpha\gamma \circ (\beta' \circ_m \beta'')) \subseteq \alpha(\beta' \circ_m \beta'') \circ_k \alpha\gamma$. By taking $\beta' = \beta$ and $\beta'' = \alpha\gamma$, we have $\alpha(\beta \circ_{2m+1} \alpha\gamma) = \alpha((\beta \circ_m \alpha\gamma) \circ \alpha\gamma \circ (\beta \circ_m \alpha\gamma)) \subseteq \alpha(\beta \circ_m \alpha\gamma) \circ_k \alpha\gamma$, for odd $m \geq 1$. It is now easy to show by induction on m that $\alpha(\beta \circ_m \alpha\gamma) \subseteq \alpha\beta + \alpha\gamma$, for every m. Hence $\alpha(\beta + \alpha\gamma) \leq \alpha\beta + \alpha\gamma$. □

6. Representability of tolerances

In this section we study the notion of a (weakly) representable tolerance in more detail. In particular, we shall give examples of representable and not representable tolerances. Notice that examples of representable tolerances abound: every congruence α is trivially representable, since $\alpha = \alpha \circ \alpha$. More generally, if Θ is a tolerance, then the tolerance $\Theta \circ \Theta$ is representable. Since in a variety $\mathcal{V}$ all tolerances are congruences if and only if $\mathcal{V}$ is congruence permutable (Proposition 6.5), our main result Theorem 3.1 can be seen as a generalization to the class of all varieties of some results valid in permutable varieties. However, there are non-congruence permutable varieties in which every tolerance is representable, for example, any variety of lattices (Proposition 6.3). We shall also show that a Mal'cev condition implies that every tolerance is representable if and only if it implies congruence permutability (Corollary 6.6). We first show that all tolerances in algebras without operations are weakly representable.

Proposition 6.1. *If $\mathbf{A}$ is an algebra belonging to the variety of sets (that is, an algebra without operations) then every tolerance of $\mathbf{A}$ is weakly representable.*

Proof. Let $\mathbf{A}$ be an algebra without operations. For every pair of distinct elements $a, b \in A$ let Θ_{ab} be the reflexive and symmetric relation defined by

$(x, y) \in \Theta_{ab}$ if and only if $\{x, y\} \neq \{a, b\}$. Θ_{ab} is representable: define R by $x \, R \, y$ if and only if either $x = y = a$, or $x = y = b$, or $x \notin \{a, b\}$. R is clearly reflexive, and is compatible since $\mathbf{A}$ has no operation. It is easy to see that $\Theta_{ab} = R \circ R^{-}$. If Θ is any tolerance of $\mathbf{A}$ then Θ is weakly representable, since $\Theta = \bigcap_{(a,b) \notin \Theta} \Theta_{ab}$. □

In contrast to Proposition 6.1, in algebras without operations there can be non-representable tolerances. Such tolerances remain non-representable when we add a certain kind of operations. Recall that a *majority operation* is a ternary operation f satisfying $x = f(x, x, y) = f(x, y, x) = f(y, x, x)$.

Proposition 6.2. *(i) In the 5-element algebra without operations there is a non-representable tolerance.*

(ii) There exists a 7-element semilattice with a non-representable tolerance.

(iii) There exists a 7-element algebra with a majority operation and with a non-representable tolerance.

Proof. (i) Let a, b_1, b_2, b_3 and c denote the elements of the 5-element algebra without operations. Let Θ be the smallest reflexive and symmetric relation such that $a \, \Theta \, b_i$ and $b_i \, \Theta \, c$ for $i = 1, 2, 3$. Θ is a tolerance since the algebra has no operation. It is easy to see that Θ is not representable. Indeed, suppose by contradiction that R is reflexive and $\Theta = R \circ R^{-}$. Then $R \subseteq \Theta$ and $R^{-} \subseteq \Theta$, hence either $a \, R \, b_1$ or $b_1 \, R \, a$. Suppose that $a \, R \, b_1$ (the case $b_1 \, R \, a$ is similar). If $c \, R \, b_1$ then $a \, R \circ R^{-} \, c$, that is, $a \, \Theta \, c$. This is false, hence necessarily $b_1 \, R \, c$. Continuing in the same way we obtain $c \, R \, b_2$ and $c \, R \, b_3$. Going further, we get both $b_2 \, R \, a$ and $b_3 \, R \, a$, which imply $b_2 \, R \circ R^{-} \, b_3$, hence $b_2 \, \Theta \, b_3$, contradiction.

(ii) Consider the join semilattice S with 6 minimal elements a, b_1, b_2, b_3, b_4, c and with a largest element 1. Let Θ be the smallest reflexive and symmetric relation such that 1 is Θ-related to all elements of S and such that $a \, \Theta \, b_i$ and $b_i \, \Theta \, c$ for $i = 1, 2, 3, 4$. It is easy to check that Θ is a tolerance. Suppose by contradiction that Θ is representable as $R \circ R^{-}$. If x and y are minimal elements of S and both $x \, R \, 1$ and $y \, R \, 1$, then $x \, R \circ R^{-} \, y$, hence $x \, \Theta \, y$. Thus $|\{x \in S | x \text{ is minimal and } x \, R \, 1\}| \leq 2$, since in S there do not exist 3 pairwise Θ-connected minimal elements. We can now repeat the arguments in (i) restricting ourselves to minimal elements x such that not $x \, R \, 1$.

(iii) Consider the lattice $\langle L, +, \cdot \rangle$ with 6 atoms a, b_1, b_2, b_3, b_4, c and with a largest element 1 and a smallest element 0. If f is the ternary operation defined by $f(x, y, z) = (x + y)(x + z)(y + z)$ then $\langle L \setminus \{0\}, f \rangle$ is an algebra, since $L \setminus \{0\}$ is closed under f. We have that f is a majority operation and the same tolerance as in (ii) is not representable. □

Even if we have showed that a majority term does not necessarily imply representability, we can show that lattices have representable tolerances.

Proposition 6.3. *Suppose that the algebra $\mathbf{A}$ has binary terms $\vee$ and $\wedge$ such that $\vee$ defines a join-semilattice operation. Suppose further that the* *identities $a \wedge (a \vee b) = a$ and $(a \vee b) \wedge b = b$ are satisfied for all elements $a, b \in A$ and that the semilattice order $\leq$ induced by $\vee$ is a compatible relation on $\mathbf{A}$. Then all tolerances of $\mathbf{A}$ are representable.*

In particular, all tolerances in a lattice are representable.

Proof. If Θ is a tolerance of $\mathbf{A}$ let $R = \Theta \cap \leq$. R is compatible since both Θ and $\leq$ are compatible. If $a \,\Theta\, b$ then $a = a \vee a \,\Theta\, a \vee b$ and $a \leq a \vee b$, thus $a \,R\, a \vee b$. Similarly $b \,R\, a \vee b$, that is, $a \vee b \,R^-\, b$, thus $\Theta \subseteq R \circ R^-$.

Conversely, if $(a, b) \in R \circ R^-$, say $a \,R\, c \,R^-\, b$, then $a \leq c$, thus $c = a \vee c$, hence $a = a \wedge (a \vee c) = a \wedge c$. Similarly, $c \wedge b = b$. Hence $a = a \wedge c \,\Theta\, c \wedge b = b$, since both $R \subseteq \Theta$ and $R^- \subseteq \Theta$. Thus $a \,\Theta\, b$. We have proved $R \circ R^- \subseteq \Theta$. $\square$

We now proceed to show that if $\mathbf{A}$ has a tolerance Θ which is not a congruence, then we can add operations to $\mathbf{A}$ in such a way that, in the expanded algebra, Θ remains a tolerance, but Θ is not even weakly representable. As a consequence, for every Mal'cev condition $\mathcal{M}$, $\mathcal{M}$ implies that every tolerance is representable if and only if $\mathcal{M}$ implies congruence permutability (Corollary 6.6).

Proposition 6.4. *Let $\mathbf{A}$ be any algebra and let Θ be a tolerance of $\mathbf{A}$. Then there is an expansion $\mathbf{A}^+$ of $\mathbf{A}$ by unary operations such that Θ is a tolerance of $\mathbf{A}^+$ and any nontrivial reflexive compatible relation of $\mathbf{A}^+$ contains Θ. If in addition Θ is not a congruence of $\mathbf{A}$ then Θ is not weakly representable in $\mathbf{A}^+$.*

Proof. Let $\mathbf{A}^+$ be obtained from $\mathbf{A}$ by adding, for every $a, b \in A$ such that $a \,\Theta\, b$ and for every function $f : A \to \{a, b\}$, a new unary operation which represents the function. Since we are considering only pairs (a, b) such that $a \,\Theta\, b$, we have that Θ is a tolerance of $\mathbf{A}^+$. If R is a nontrivial reflexive compatible relation of $\mathbf{A}^+$ there exist $c \neq d \in A$ such that $c \,R\, d$. For every $a \,\Theta\, b$ there is a function f such that $f(c) = a$ and $f(d) = b$. Thus $a = f(c) \,R\, f(d) = b$, since R is compatible in $\mathbf{A}^+$. This proves that $R \supseteq \Theta$. Finally, if Θ is not transitive, then $\Theta \subset \Theta \circ \Theta \subseteq R \circ R^-$ yields that Θ is not weakly representable. $\square$

Part (b) in the following Proposition is stated as Theorem 1 in [Ch]. We sketch a proof for the reader's convenience.

Proposition 6.5. *(a) If $\mathbf{A}$ is an algebra and every tolerance of $\mathbf{A}$ is a congruence then all congruences of $\mathbf{A}$ permute.*

(b) A variety $\mathcal{V}$ is congruence permutable if and only if every tolerance of every algebra in $\mathcal{V}$ is a congruence.

Proof. (a) If α and β are congruences of $\mathbf{A}$, let $\overline{\alpha \cup \beta}$ denote the smallest tolerance containing α and β, which is the smallest admissible relation containing $\alpha \cup \beta$. Notice that $\overline{\alpha \cup \beta} \subseteq \beta \circ \alpha$. By assumption, $\overline{\alpha \cup \beta}$ is a congruence. Then $\alpha \circ \beta \subseteq \overline{\alpha \cup \beta} \circ \overline{\alpha \cup \beta} = \overline{\alpha \cup \beta} \subseteq \beta \circ \alpha$.

(b) is immediate from (a) and the well known result that in permutable varieties every reflexive and admissible relation is a congruence [We]. $\square$

Trivially every congruence α is representable, since $\alpha = \alpha \circ \alpha$. By Proposition 6.5(b), congruence permutability, for varieties, implies that every tolerance is representable. The next result shows that if a Mal'cev condition $\mathcal{M}$ implies that every tolerance is representable, then $\mathcal{M}$ implies congruence permutability.

Corollary 6.6. *Let $\mathcal{M}$ be either a Mal'cev condition, or a weak Mal'cev condition, or a strong Mal'cev condition. The following are equivalent.*

(i) $\mathcal{M}$ implies congruence permutability.

(ii) $\mathcal{M}$ implies that every tolerance is representable.

(iii) $\mathcal{M}$ implies that every tolerance is weakly representable.

Proof. (i) $\Rightarrow$ (ii). Suppose that (i) holds. If $\mathcal{V}$ satisfies $\mathcal{M}$ then by Proposition 6.5(b) every tolerance in every algebra in $\mathcal{V}$ is a congruence, hence is representable. Thus (ii) holds.

(ii) $\Rightarrow$ (iii) is trivial.

We shall prove (iii) $\Rightarrow$ (i) by contradiction. Suppose that (i) fails. Then there exists some variety $\mathcal{V}$ which satisfies $\mathcal{M}$ but which is not congruence permutable. By Proposition 6.5(b) there is an algebra $\mathbf{A} \in \mathcal{V}$ with a tolerance Θ which is not a congruence. By Corollary 6.4 $\mathbf{A}$ can be expanded to an algebra $\mathbf{A}^+$ in which Θ is a tolerance which is not weakly representable. By well known properties of Mal'cev conditions, the variety generated by $\mathbf{A}^+$ still satisfies $\mathcal{M}$ and this contradicts (iii). □

Corollary 6.7. *The class of varieties $\mathcal{V}$ such that every tolerance in every algebra in $\mathcal{V}$ is representable (resp. weakly representable) cannot be characterized by a weak Mal'cev condition.*

Proof. If any of those classes could be characterized by some weak Mal'cev condition $\mathcal{M}$ then by Corollary 6.6 $\mathcal{M}$ would imply permutability. This is a contradiction, since Propositions 6.1 and 6.3 provide examples of non-permutable varieties in which every tolerance is (weakly) representable. □

7. Further remarks and generalizations

Remark 7.1. The assumption that p is regular is necessary in Theorem 3.1. Indeed, every algebra in every variety satisfies the congruence identity $\alpha \circ \alpha \subseteq \alpha$, while a tolerance satisfying $\Theta \circ \Theta \subseteq \Theta$ is necessarily a congruence. Notice that there are representable tolerances which are not congruences (e. g., by Proposition 6.3).

Remark 7.2. The proof of Theorem 3.1 gives slightly more. Let us call an edge of $\mathbf{G}_q$ an *outer edge* in case it is adjacent to one of the distinguished vertices v_ℓ and v_r. Let us call an occurrence of α_i or Θ_i in q an *outer occurrence* if it corresponds to an outer edge. In condition 3.1(iii) we do not need the assumption that the tolerance Θ_i is representable, for the outer occurrences of Θ_i in q. More precisely, if $\mathcal{V}$ satisfies the congruence identity $p(\alpha_1, \dots, \alpha_n) \subseteq q(\alpha_1, \dots, \alpha_n)$, then $\mathcal{V}$ satisfies $p(\Theta_1, \dots, \Theta_n) \subseteq q'$, where

q' is obtained from q by substituting every occurrence of α_i for Θ_i in case either Θ_i is representable, or the occurrence of α_i is an outer occurrence. Otherwise, the occurrence of α_i should be substituted for $\Theta_i \circ \Theta_i$.

Indeed, for outer edges, we simply use identities (ℓ) and (r) and we do not need the main trick in the proof of 3.1 (i) $\Rightarrow$ (iii), where representability is used. In more detail, everything goes as in the proof of 3.1 except when we deal with outer edges. If, say, the vertex w is adjacent to the vertex v_r and they are connected by an edge labeled by α_i, let us define $\phi_i : \{c_1, \dots, c_m\} \to \{c_1, \dots, c_m\}$ as follows: if $\{c_j\}$ is a $\sim_i$-equivalence class, let $\phi_i(c_j) = c_j$. If $\{c_j, c_h\}$ is a $\sim_i$-equivalence class, then choose one element c belonging to $\{c_j, c_h\}$, and let let $\phi_i(c_j) = \phi_i(c_h) = c$. The choice of c is arbitrary, except for the case when one of the vertices c_j and c_h is c_{v_r}. In this case we should choose $c = c_{v_r}$. Thus, by identities ($\mathrm{m}_{w,w_r,i}$) and (r), we get

$$t_w(c_1, \dots, c_m) \, \Theta_i \, t_w(\phi_i(c_1), \dots, \phi_i(c_m)) = t_{w_r}(\phi_i(c_1), \dots, \phi_i(c_m)) = \phi_i(c_{v_r}) = c_{v_r}$$

Notice that, in most cases and for appropriate choices of the c's, in the above argument we do not even use the symmetry of Θ, it is enough to deal with a reflexive compatible relation.

The remark is better illustrated by an example. If $\mathcal{V}$ satisfies the congruence identity $\alpha(\beta \circ \gamma) \subseteq \alpha\beta \circ \alpha\gamma \circ \alpha\beta$ then the above remark implies that $\mathcal{V}$ satisfies the tolerance identity $\Gamma(\Phi \circ \Psi) \subseteq \Gamma\Phi \circ (\Gamma \circ \Gamma)(\Psi \circ \Psi) \circ \Gamma\Phi$. If, in addition, say, Γ is representable, then $\Gamma(\Phi \circ \Psi) \subseteq \Gamma\Phi \circ \Gamma(\Psi \circ \Psi) \circ \Gamma\Phi$. Moreover $\mathcal{V}$ satisfies the identity $R(S \circ T) \subseteq RS \circ (R \circ R^-)(T \circ T^-) \circ RS$, where R, S, T are intended to be variables for reflexive and compatible relations. Notice that the proof of [CH1, Theorem 1] shows that $\mathcal{V}$ satisfies also the tolerance identity $\Gamma(\Phi \circ \Psi) \subseteq \Gamma\Phi \circ \Gamma\Psi \circ \Gamma\Psi \circ \Gamma\Phi$.

Remark 7.3. In the particular case of the simpler identity $\alpha(\beta \circ \gamma) \subseteq \alpha\beta \circ \alpha\gamma$ the above remark (and the proof of [CH1, Theorem 1] as well) show that if a variety $\mathcal{V}$ satisfies the identity for congruences then $\mathcal{V}$ satisfies the same identity for tolerances. We expect that the arguments from [CH1], as well as the above remarks, can be extended further, but we have not worked out details.

Theorem 3.1, Proposition 6.3 and Remark 7.3 lead to the following problem.

Problem 7.4. Characterize those identities ε such that, for every variety $\mathcal{V}$, $\mathcal{V}$ satisfies ε for congruences if and only if $\mathcal{V}$ satisfies ε for tolerances.

Remark 7.5. Without any particular change, the classical proof of Wille and Pixley's Theorem 3.1 (i) $\Leftrightarrow$ (ii) can be used to show the following. If p is regular then a variety $\mathcal{V}$ satisfies the congruence identity $p(\alpha_1, \dots, \alpha_n) \subseteq q(\alpha_1, \dots, \alpha_n)$ if and only if all algebras in $\mathcal{V}$ satisfy $p(R_1, \dots, R_n) \subseteq q(R_1 \circ R_1^-, \dots, R_n \circ R_n^-)$, where the R_i's range among reflexive and compatible relations. However, Theorem 3.1 is more general, since Condition 3.1 (iii), applied to the *representable tolerances* $R_i \circ R_i^-$, gives the stronger inclusion $p(R_1 \circ R_1^-, \dots, R_n \circ R_n^-) \subseteq q(R_1 \circ R_1^-, \dots, R_n \circ R_n^-)$.

However, the Wille Pixley argument gives the following (the assumption that p is regular is unnecessary here).

Proposition 7.6. *For every variety $\mathcal{V}$ and for every pair of terms p and q for the language $\{\circ, \cap\}$ the following are equivalent.*

(i) $\mathcal{V}$ satisfies the congruence identity $p(\alpha_1, \dots, \alpha_n) \subseteq q(\alpha_1, \dots, \alpha_n)$.

(ii) There exist integers k_i (depending only on p) such that $\mathcal{V}$ satisfies the tolerance identity $p(\Theta_1, \dots, \Theta_n) \subseteq q(\Theta_1^{k_1}, \dots, \Theta_n^{k_n})$.

In the above proposition Θ^k denotes $\Theta \circ \Theta \circ \dots \circ \Theta$ with k occurrences of Θ. For each i, the integer k_i is equal to $2k'_i$, where k'_i is the smallest integer such that, for every equivalence class X of $\sim_i$ in the graph $\mathbf{G}_p$, there is an element $x \in X$ such that every other element of X can be connected to x by a path of length $\leq k_i$ contained in X.

In particular, Proposition 7.6 implies that every congruence identity is equivalent to some tolerance identity. The main point in Theorem 3.1 is that from a congruence identity we obtain *the very same* tolerance identity.

Theorem 3.1 can be generalized further.

Let $\mathbf{G}$ be a graph with h distinguished vertices, $h > 1$, and with edges labeled by the set of labels $\{X_1, ..., X_n\}$. For reflexive, symmetric and compatible relations $R_1, ..., R_n$ on some algebra $\mathbf{A}$ one can naturally define an h-ary (compatible) relation $\mathbf{G}(R_1, ..., R_n)$ by declaring $a_1, \dots, a_h \in \mathbf{G}(R_1, ..., R_n)$ if and only if a situation analogue to the last sentence in Proposition 2.1 occurs. In particular, for every $\{\circ, \cap\}$-term p, $(a_1, a_2) \in p(R_1, ..., R_n)$ if and only if $(a_1, a_2) \in \mathbf{G}_p(R_1, ..., R_n)$. If $\mathbf{G}'$ is another graph of the same type then it makes sense to say that $\mathbf{G}(R_1, ..., R_n) \subseteq \mathbf{G}'(R_1, ..., R_n)$ for certain symmetric relations $R_1, ..., R_n$ of $\mathbf{A}$ and that $\mathbf{G} \subseteq \mathbf{G}'$ holds for congruences (tolerances) of $\mathbf{A}$. The Mal'cev condition $M(\mathbf{G} \subseteq \mathbf{G}')$ and the equivalence relation $\sim_i$ are defined as in Definition 4.1 (see [CD, L4] for more details). We say that $\mathbf{G}$ is *regular* if and only if for every $1 \leq i \leq n$ all equivalence classes of $\sim_i$ have cardinality ≤ 2. The methods we have used so far imply the following statement.

Theorem 7.7. *Theorem 3.1, Corollary 5.1, Remark 7.2 and Proposition 7.6 remain valid if we replace the terms p and q by the graphs $\mathbf{G}$ and $\mathbf{G}'$.*

Remarks 7.8. (a) We cannot expect to generalize Theorem 3.1 to the effect that from a congruence identity we get the same identity in which congruences are replaced by reflexive admissible relations.

For example, Polin's variety satisfies the congruence identity $\alpha(\beta \circ \gamma) \subseteq \alpha\beta \circ \alpha(\gamma \circ \beta) \circ \alpha\gamma$ ([L1, p. 167]; see also [DF], [J2, p. 383]). On the other hand, Polin's variety does not even satisfy the identity $\alpha(\beta \circ R) \subseteq \alpha\beta \circ \alpha(R \circ \beta) \circ \alpha R$. Indeed, by taking R to be the admissible reflexive relation $\alpha\gamma \circ \beta$, we obtain from the above inclusion: $\alpha(\beta \circ \alpha\gamma \circ \beta) \subseteq \alpha\beta \circ \alpha(\alpha\gamma \circ \beta \circ \beta) \circ \alpha(\alpha\gamma \circ \beta) = \alpha\beta \circ \alpha\gamma \circ \alpha\beta \circ \alpha\gamma \circ \alpha\beta$, which implies congruence modularity. Hence $\alpha(\beta \circ R) \subseteq \alpha\beta \circ \alpha(R \circ \beta) \circ \alpha R$ does not hold in Polin's variety, which is not congruence modular.

See also [J2, Ts] concerning the relationship between congruence identities and identities involving arbitrary admissible reflexive relations.

(b) We know further applications (still unpublished) of the main trick used in the proof of Theorem 3.1 (ii) $\Rightarrow$ (iii).

(c) The assumption of representability in Theorem 3.1(iii) can be somewhat relaxed. It is enough to assume that, for every i and for every pair of vertices $w, w' \in W$ connected by an edge labeled α_i, there exists a relation R_i such that $\Theta_i = R_i \circ R_i^-$ and R_i is compatible in the algebra $\langle A, t_w, t_{w'} \rangle$. Moreover, when dealing with varieties, it is enough that the above weaker property holds in free algebras, and just for tolerances generated by a finite sets of disjoint pairs of variables.

It is likely that the above remarks can be used in order to obtain many instances of identities satisfying the condition in Problem 7.4.

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Dipartimento di Matematica, Viale della Ricerca Scientifica, II Università di Roma (Tor Vergatina), ROME ITALY

URL: `http://www.mat.uniroma2.it/~lipparin`