Ordinal Compactness

Paolo Lipparini

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We extend to ordinal numbers the standard compactness notion defined in terms of cardinalities of open covers.

Background: a topological space is compact if every open cover has a finite subcover. Various weakenings have been considered:

- Lindelöf: any open cover has a countable subcover.
- Countable compactness: any countable open cover has a finite subcover.

The most general form is:

- A topological space is \([\mu, \lambda]\)-compact if and only if every open cover by at most \(\lambda\) sets has a subcover by \(<\mu\) sets. (here, \(\mu \leq \lambda\) are cardinals)

By the way, the classical definitions of both weakly compact and strongly compact cardinals had been given in terms of \([\lambda, \lambda]\)-compactness.
Throughout, let $\beta \leq \alpha$ be infinite ordinals.

**Definition 1** We say that a topological space is $[\beta, \alpha]$-compact if:

For every sequence $(O_\delta)_{\delta \in \alpha}$ of open sets such that $\bigcup_{\delta \in \alpha} O_\delta = X$, there is $H \subseteq \alpha$ with order type $< \beta$ and such that $\bigcup_{\delta \in H} O_\delta = X$.

For short: every $\alpha$-indexed open cover has a subcover indexed by a set of order type $< \beta$. (the order of the initial cover should be respected)

When $\alpha$ and $\beta$ are cardinals, we get back the classical notion.
Example ($\kappa$ an infinite regular cardinal.)

$\kappa$ with the order topology is not $[\kappa + n, \kappa + n]$-compact, for every $n < \omega$. ("+" always denotes ordinal sum)

Indeed, consider the following cover $\mathcal{C} = (O_\delta)_{\delta \in \kappa + n}$ defined by:

$O_\delta = (n - 1, n + \delta)$, if $\delta < \kappa$

$O_\delta = \{m\}$, if $\delta = \kappa + m$, $m < n$

Of course, $\mathcal{C}$ has cardinality $\kappa$, hence every subcover has cardinality $\kappa$. However, if we want the original order to be respected, we should have the $\{m\}$'s at the top, hence any subcover is necessarily indexed by $\kappa + n$. 
Ordinal compactness “differentiates” spaces which can be hardly distinguished by means of cardinal compactness.

Let $\lambda > \kappa$ be infinite regular cardinals.

- $\kappa$ with the discrete topology is $[\kappa^+, \lambda]$-compact, and not $[\alpha, \lambda]$-compact, for $\alpha < \kappa^+$.

- (for $\kappa$ uncountable) $\kappa$ with the order topology is $[\kappa + \omega, \lambda]$-compact, and not $[\alpha, \lambda]$-compact, for $\alpha < \kappa + \omega$.

- Consider $\kappa$ with the topology whose open sets are $[0, \gamma)$ ($0 < \gamma \leq \kappa$). This space is $[\kappa + 1, \lambda]$-compact, and not $[\alpha, \lambda]$-compact, for $\alpha \leq \kappa$.

- The disjoint union of two copies of the above space is $[\kappa + \kappa + 1, \lambda]$-compact, and not $[\alpha, \lambda]$-compact, for $\alpha \leq \kappa + \kappa$. 
For cardinals, the only nontrivial relationships between $[\mu, \lambda]$-compactness and $[\mu', \lambda']$-compactness are the following:

- $[\mu, \lambda]$-compactness is equivalent to $[\kappa, \kappa]$-compactness, for every $\kappa$ with $\mu \leq \kappa \leq \lambda$

- $[\text{cf}\lambda, \text{cf}\lambda]$-compactness implies $[\lambda, \lambda]$-compactness.

On the other hand, there are many more nontrivial implications for ordinal compactness. Some simple examples:

- $[\beta, \alpha]$-compactness implies $[\beta, \alpha+1]$-compactness.

- $[\beta + \alpha, \beta + \alpha]$-compactness implies $[\beta + \alpha + \alpha, \beta + \alpha + \alpha]$-compactness.

- $[\alpha, \alpha]$-compactness implies both $[\beta + \alpha, \beta + \alpha]$-compactness and $[\beta \cdot \alpha, \beta \cdot \alpha]$-compactness.
For $T_1$ topological spaces, ordinal compactness is generally “invariant” through intervals of countable lengths.

**Theorem 2** Suppose that $X$ is $T_1$, and $\beta$ is an ordinal of cofinality $\omega$. Then the following are equivalent.

1. $X$ is $[\beta, \beta]$-compact.

2. $X$ is $[\beta+\alpha, \beta+\alpha]$-compact, for some countable $\alpha$.

3. $X$ is $[\beta+\alpha, \beta+\alpha]$-compact, for every countable $\alpha$.

4. $X$ is $[\beta, \beta+\alpha]$-compact, for every countable $\alpha$. 
On spaces of small cardinality, ordinal compactness becomes almost trivial.

**Corollary 3** If \(|X| = \omega\), then the following are equivalent.

1. \(X\) is \([\omega \cdot \omega, \omega \cdot \omega]\)-compact.

2. \(X\) is \([\alpha, \alpha]\)-compact, for some ordinal \(\alpha\) with \(|\alpha| = \omega\).

3. \(X\) is \([\omega \cdot \omega, \alpha]\)-compact, for every \(\alpha \geq \omega \cdot \omega\).

A similar result holds for \(\kappa\) regular, with technical exceptions.
Some problems.

- Behavior with respect to products (for cardinal compactness, there are highly nontrivial results).

- Is there a more refined theory for spaces satisfying higher separation axioms (e.g. normal spaces)?

- Is $[\alpha, \beta]$-compact nontrivial for generalized logics (Abstract Model Theory)?