Ordinal Compactness

Paolo Lipparini

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We extend to *ordinal* numbers the standard compactness notion defined in terms of *cardinalities* of open covers.

Background: a topological space is *compact* if every open cover has a finite subcover. Various weakenings have been considered:

• Lindelöf: any open cover has a countable subcover.

• Countable compactness: any countable open cover has a finite subcover.

The most general form is:

• A topological space is $[\mu, \lambda]$-compact if and only if every open cover by at most λ sets has a subcover by $< \mu$ sets. (here, $\mu \leq \lambda$ are cardinals)

By the way, the classical definitions of both weakly compact and strongly compact cardinals had been given in terms of $[\lambda, \lambda]$-compactness.

Throughout, let $\beta \leq \alpha$ be infinite ordinals.

Definition 1 *We say that a topological space is* $[\beta, \alpha]$-compact *if:*

For every sequence $(O_\delta)_{\delta \in \alpha}$ *of open sets such that* $\bigcup_{\delta \in \alpha} O_\delta = X$*, there is* $H \subseteq \alpha$ *with order type* $< \beta$ *and such that* $\bigcup_{\delta \in H} O_\delta = X$*.*

For short: every α-indexed open cover has a subcover indexed by a set of order type $< \beta$. (the order of the initial cover should be respected)

When α and β are cardinals, we get back the classical notion.

Example (κ an infinite regular cardinal.)

κ with the order topology is not $[\kappa+n,\kappa+n]$-compact, for every $n<\omega$. ("$+$" always denotes ordinal sum)

Indeed, consider the following cover $\mathcal{C}=(O_\delta)_{\delta\in\kappa+n}$ defined by:

$O_\delta=(n-1,n+\delta)$, if $\delta<\kappa$

$O_\delta=\{m\}$, if $\delta=\kappa+m$, $m<n$

Of course, $\mathcal{C}$ has cardinality κ, hence every subcover has cardinality κ. However, if we want the original order to be respected, we should have the $\{m\}$'s at the top, hence any subcover is necessarily indexed by $\kappa+n$.

Ordinal compactness "differentiates" spaces which can be hardly distinguished by means of cardinal compactness.

Let $\lambda > \kappa$ be infinite regular cardinals.

• κ with the discrete topology is $[\kappa^+, \lambda]$-compact, and not $[\alpha, \lambda]$-compact, for $\alpha < \kappa^+$.

• (for κ uncountable) κ with the order topology is $[\kappa + \omega, \lambda]$-compact, and not $[\alpha, \lambda]$-compact, for $\alpha < \kappa + \omega$.

• Consider κ with the topology whose open sets are $[0, \gamma)$ $(0 < \gamma \leq \kappa)$. This space is $[\kappa + 1, \lambda]$-compact, and not $[\alpha, \lambda]$-compact, for $\alpha \leq \kappa$.

• The disjoint union of two copies of the above space is $[\kappa + \kappa + 1, \lambda]$-compact, and not $[\alpha, \lambda]$-compact, for $\alpha \leq \kappa + \kappa$.

For cardinals, the only nontrivial relationships between $[\mu, \lambda]$-compactness and $[\mu', \lambda']$-compactness are the following:

• $[\mu, \lambda]$-compactness is equivalent to $[\kappa, \kappa]$-compactness, for every κ with $\mu \leq \kappa \leq \lambda$

• $[\mathrm{cf}\lambda, \mathrm{cf}\lambda]$-compactness implies $[\lambda, \lambda]$-compactness.

On the other hand, there are many more nontrivial implications for ordinal compactness. Some simple examples:

• $[\beta, \alpha]$-compactness implies $[\beta, \alpha+1]$-compactness.

• $[\beta + \alpha, \beta + \alpha]$-compactness implies $[\beta + \alpha + \alpha, \beta + \alpha + \alpha]$-compactness.

• $[\alpha, \alpha]$-compactness implies both $[\beta+\alpha, \beta+\alpha]$-compactness and $[\beta \cdot \alpha, \beta \cdot \alpha]$-compactness.

For T_1 topological spaces, ordinal compactness is generally "invariant" through intervals of countable lengths.

Theorem 2 *Suppose that X is T_1, and β is an ordinal of cofinality ω. Then the following are equivalent.*

1. *X is $[\beta, \beta]$-compact.*

2. *X is $[\beta+\alpha, \beta+\alpha]$-compact, for some countable α.*

3. *X is $[\beta+\alpha, \beta+\alpha]$-compact, for every countable α.*

4. *X is $[\beta, \beta+\alpha]$-compact, for every countable α.*

On spaces of small cardinality, ordinal compactness becomes almost trivial.

Corollary 3 *If $|X| = \omega$, then the following are equivalent.*

1. *X is $[\omega \cdot \omega, \omega \cdot \omega]$-compact.*

2. *X is $[\alpha, \alpha]$-compact, for some ordinal α with $|\alpha| = \omega$.*

3. *X is $[\omega \cdot \omega, \alpha]$-compact, for every $\alpha \geq \omega \cdot \omega$.*

A similar result holds for κ regular, with technical exceptions.

Some problems.

- Behavior with respect to products (for cardinal compactness, there are highly nontrivial results).

- Is there a more refined theory for spaces satisfying higher separation axioms (e. g. normal spaces)?

- Is $[\alpha, \beta]$-compact nontrivial for generalized logics (Abstract Model Theory)?