EVERY m-PERMUTABLE VARIETY SATISFIES THE CONGRUENCE IDENTITY $\alpha\beta_h = \alpha\gamma_h$

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Abstract. It is known that m-permutable varieties satisfy non-trivial lattice identities; however, the identities discovered so far are rather artificial and seem to have little intrinsic interest.

We show here that every m-permutable variety satisfies the well known and well studied lattice identity $\alpha\beta_h = \alpha\gamma_h$. By the way, in Section 2, we get a new condition equivalent to m-permutability (Proposition 2.4).

1. Introduction

It has been proved about ten years ago that every m-permutable variety $\mathcal{V}$ satisfies a non-trivial lattice identity. Partial results were known before: A. Day and J. B. Nation (see [J, Lemma 3.10]) showed that if some algebra $\mathbf{A}$ is $2m$-permutable, and has a semilattice operation then $\operatorname{Con}\mathbf{A}$ satisfies the identity $\alpha(\beta+\gamma) \leq \alpha\beta_{2m} + \alpha\gamma_{2m}$. As usual, β_n and γ_n are defined as follows:

$$\beta_0 = \gamma_0 = 0$$

$$\beta_{n+1} = \beta + \alpha\gamma_n \quad \gamma_{n+1} = \gamma + \alpha\beta_n$$

G. Czédli [C] weakened to meet semidistributivity the assumption of the existence of a semilattice operation: he showed that an m-permutable variety $\mathcal{V}$ is congruence meet semidistributive if and only if, for some n, $\mathcal{V}$ satisfies the congruence identity $\alpha(\beta+\gamma) = \beta_n$. He also proved the dual result.

D. Hobby and R. McKenzie [HMK, Theorem 9.19] showed that for every locally finite m-permutable variety $\mathcal{V}$ there is a non-trivial lattice identity satisfied in $\mathcal{V}$. Finally, the assumption that $\mathcal{V}$ is locally finite has been removed in [L1]; moreover, the identity found in [L1] depends

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only on m and does not depend on $\mathcal{V}$. More identities have been found in [L2]. However, the identities obtained in [L1, L2] are ad hoc and rather weak. For the most part, such identities simply state that a certain small interval in the congruence lattice is modular; they say almost nothing about the global shape of the congruence lattice.

In the present paper we show that every m-permutable variety satisfies a lattice identity similar to the identities found by A. Day, J. B. Nation and G. Czédli and mentioned at the beginning. It is not the case that all m-permutable varieties satisfy, say, $\alpha(\beta+\gamma)=\beta_n$, since there are non semidistributive m-permutable varieties. However, we show here that every m-permutable variety satisfies the related identity $\alpha\gamma_h=\alpha\beta_h$, for an appropriate h depending only on m (Theorem 3.5). Notice that the three identities $\alpha\gamma_h\leq\beta_h$, $\alpha\gamma_h=\alpha\beta_h$ and $\beta_h=\beta_{h+1}$ are lattice theoretically equivalent.

The terms β_n and γ_n are well known and have been frequently used in lattice theory and universal algebra. B. Jónsson and I. Rival [JR] proved that a variety of lattices is (both meet and join) semidistributive if and only if for some n it satisfies $\alpha(\beta+\gamma)=\alpha\beta_n=\alpha\gamma_n$, as well as the dual identity. The term β_n played a fundamental role in D. Hobby and R. McKenzie's deep analysis of finite algebras and locally finite varieties (see [HMK, Chapter 9]). For example, they proved that a locally finite variety of algebras is congruence meet semidistributive if and only if it satisfies $\alpha(\beta\circ\gamma)\subseteq\beta_n$ for some n.

In [L3] we showed that every congruence variety satisfying $\alpha\gamma_h=\alpha\beta_h$ satisfies more identities, which do not follow from it lattice-theoretically. This appears to be the first example of a non trivial congruence implication involving identities weaker than modularity, and, together with the results presented here, confirms the importance of the identity $\alpha\gamma_h=\alpha\beta_h$. Furthermore, we have results suggesting that varieties satisfying $\alpha\gamma_h=\alpha\beta_h$ satisfy many of the good properties of m-permutable varieties (see also Problem 3.6). Notice that $\alpha\gamma_2=\alpha\beta_2$ is an identity equivalent to modularity, thus $\alpha\gamma_h=\alpha\beta_h$ can be seen as a generalization of modularity.

Let us mention that the dual identity $\alpha\gamma^h=\alpha\beta^h$, too, has proven particularly important. K. Kearnes [K1] showed that a locally finite variety $\mathcal{V}$ satisfies some non trivial lattice identity if and only if there is k such that $\mathcal{V}$ satisfies $\alpha\gamma^k=\alpha\beta^k$. Thus, locally finite m-permutable varieties satisfy also $\alpha\gamma^k=\alpha\beta^k$, for some k, by the results proved here, or simply by [HMK, Theorem 9.19]. The k given by the proof seems to depend on $\mathcal{V}$, not just on m. The result we prove is stronger, since there are varieties which for some k satisfy $\alpha\gamma^k=\alpha\beta^k$, but for no n satisfy $\alpha\gamma_n=\alpha\beta_n$ (see [K1, p. 385], [L3, p. 606]). Let us also recall that, in the meantime, many results proved under the assumption of local finiteness have been proved without such an assumption.

Our proof of Theorem 3.5 splits into two neatly separated parts. In the first step, in Section 2, we get an identity similar to $\alpha\beta_h = \alpha\gamma_h$, except that the lattice operation $+$ is replaced by $\circ_3$ (see below for definitions). Theorem 3.5 is then obtained by a quite straightforward application of the commutator theory developed in [L1, L4]. Notice that Section 2 is commutator-free. At the end of each Section some problems are stated.

Here are the notations we use. α, β denote *congruences* on some *algebra* $\mathbf{A}$. Join and meet in the lattice $\operatorname{Con}\mathbf{A}$ of all congruences of $\mathbf{A}$ are denoted, respectively, by $+$ and juxtaposition. We use juxtaposition also to denote intersection.

Relational product is denoted by $\circ$, and $\alpha \circ_n \beta$ is a shorthand for $\alpha \circ \beta \circ \alpha \circ \beta \circ \dots$, with $n+1$ occurrences of $\circ$. Two congruences α, β are said to *m-permute* if and only if $\alpha \circ_m \beta = \beta \circ_m \alpha$ (thus, in particular, $\alpha + \beta = \alpha \circ_m \beta$). An algebra $\mathbf{A}$ is *m-permutable* if and only if every pair of congruences in $\mathbf{A}$ m-permute. A variety $\mathcal{V}$ is *m-permutable* if and only if every algebra in $\mathcal{V}$ is m-permutable. 2-permutability is simply called *permutability.*

2. A nice property of m-permutable varieties

In this section we shall prove that every m-permutable variety satisfies the identity introduced in the following definition.

Definition 2.1. If $\alpha, \beta, \gamma, \delta$ are congruences on some algebra, and m is a natural number, we shall denote by (X_m) the following identity.

$$\alpha(\beta \circ \alpha(\gamma \circ \alpha(\beta \circ \dots \alpha(\gamma^\bullet \circ \alpha(\beta^\bullet \circ \alpha\delta \circ \beta^\bullet) \circ \gamma^\bullet) \dots \circ \beta) \circ \gamma) \circ \beta) =$$
$$\alpha(\gamma \circ \alpha(\beta \circ \alpha(\gamma \circ \dots \alpha(\beta^\bullet \circ \alpha(\gamma^\bullet \circ \alpha\delta \circ \gamma^\bullet) \circ \beta^\bullet) \dots \circ \gamma) \circ \beta) \circ \gamma)$$

with exactly m open brackets (and exactly m closed brackets) on each side, where $\beta^\bullet = \beta$, $\gamma^\bullet = \gamma$ if m is odd, and $\beta^\bullet = \gamma$, $\gamma^\bullet = \beta$ if m is even.

If (a_0, b_0) belongs to the left-hand side of (X_m) then $a_0 \alpha b_0$, and there are elements a_1, b_1 such that $a_0 \beta a_1$, $b_1 \beta b_0$ and $(a_1, b_1) \in \alpha(\gamma \circ \alpha(\beta \circ \dots \alpha(\gamma^\bullet \circ \alpha(\beta^\bullet \circ \alpha\delta \circ \beta^\bullet) \circ \gamma^\bullet) \dots \circ \beta) \circ \gamma)$, with $m-1$ open brackets. Repeating this argument m times, we get that (a_0, b_0) belongs to the left-hand side of (X_m) if and only if there are further elements $a_1, a_2, \dots, a_m$ and $b_1, b_2, \dots, b_m$ such that

$$
\begin{array}{lll}
 & a_i \alpha b_i, & \text{for } i = 0, \dots, m, \\
 & a_m \delta b_m, & \\
a_i \beta a_{i+1}, & b_i \beta b_{i+1}, & \text{for } i \text{ even, } 0 \leq i \leq m-1, \\
a_i \gamma a_{i+1}, & b_i \gamma b_{i+1}, & \text{for } i \text{ odd, } 0 \leq i \leq m-1.
\end{array}
$$

The conditions asserting that (a_0, b_0) belongs to the right-hand side of (X_m) are similar, with β and γ interchanged.

The situation is better represented by a diagram:

$$
\begin{array}{ccc}
\begin{array}{ccc}
a_0 & \overset{\alpha}{\text{———}} & b_0 \\
\beta \big| & & \big| \beta \\
a_1 & \overset{\alpha}{\text{———}} & b_1 \\
\gamma \big| & & \big| \gamma \\
a_2 & \overset{\alpha}{\text{———}} & b_2 \\
\beta \big| & & \big| \beta \\
 & \dots & \\
\gamma^\bullet \big| & & \big| \gamma^\bullet \\
a_{m-1} & \overset{\alpha}{\text{———}} & b_{m-1} \\
\beta^\bullet \big| & & \big| \beta^\bullet \\
a_m & \overset{\alpha\delta}{\text{———}} & b_m
\end{array}
& \Leftrightarrow &
\begin{array}{ccc}
a_0 & \overset{\alpha}{\text{———}} & b_0 \\
\gamma \big| & & \big| \gamma \\
c_1 & \overset{\alpha}{\text{———}} & d_1 \\
\beta \big| & & \big| \beta \\
c_2 & \overset{\alpha}{\text{———}} & d_2 \\
\gamma \big| & & \big| \gamma \\
 & \dots & \\
\beta^\bullet \big| & & \big| \beta^\bullet \\
c_{m-1} & \overset{\alpha}{\text{———}} & d_{m-1} \\
\gamma^\bullet \big| & & \big| \gamma^\bullet \\
c_m & \overset{\alpha\delta}{\text{———}} & d_m
\end{array}
\end{array}
$$

where, as above, $\beta^\bullet = \beta$, $\gamma^\bullet = \gamma$ if m is odd, and $\beta^\bullet = \gamma$, $\gamma^\bullet = \beta$ if m is even. (X_m) asserts that the pair (a_0, b_0) can be extended to a sequence (a_i, b_i) $(0 \leq i \leq m)$ satisfying the conditions represented in the left-hand side of the above diagram if and only if (a_0, b_0) can be extended to a sequence as in the right-hand side.

We say that the algebra $\mathbf{A}$ satisfies (X_m) if and only if (X_m) holds for every congruences $\alpha, \beta, \gamma, \delta$ of $\mathbf{A}$, and we say that a variety $\mathcal{V}$ satisfies (X_m) if and only if every algebra in $\mathcal{V}$ satisfies (X_m). Notice that (X_m) is not a *lattice* identity, due to the occurrence of composition in it.

Theorem 2.2. *If every subalgebra of* $\mathbf{A}^2$ *is* m*-permutable then* $\mathbf{A}$ *satisfies* (X_m).

Proof. Suppose that every subalgebra of $\mathbf{A}^2$ is m-permutable, and $\alpha, \beta, \gamma, \delta \in \operatorname{Con} \mathbf{A}$.

Suppose that $a_0, b_0 \in \mathbf{A}$, and that (a_0, b_0) belongs to the left-hand side of (X_m). It is enough to show that (a_0, b_0) belongs to the right-hand side. The reverse inclusion is obtained by symmetry.

Since (a_0, b_0) belongs to the left-hand side of (X_m) we have elements $a_1, a_2, \dots, a_m$ and $b_1, b_2, \dots, b_m \in \mathbf{A}$ as in the left-hand side of the diagram in Definition 2.1. We want to obtain elements $c_1, c_2, \dots, c_m$ and $d_1, d_2, \dots, d_m$ as in the right-hand side.

Let $\mathbf{B}$ be the congruence α, considered as a subalgebra of $\mathbf{A}^2$, that is, $\mathbf{B} = \{(a, b) | a, b \in A, \text{ and } a \alpha b\}$. Notice that the pairs $(a_0, b_0), (a_1, b_1)$, $\dots$,(a_m, b_m) belong to $\mathbf{B}$. Moreover, working in $\mathbf{B}$, $\big((a_i, b_i), (a_{i+1}, b_{i+1})\big) \in (\beta \times \beta)_{|\mathbf{B}}$ for i even, $0 \leq i < m$, and $\big((a_i, b_i), (a_{i+1}, b_{i+1})\big) \in (\gamma \times \gamma)_{|\mathbf{B}}$ for i odd, $0 \leq i < m$. Thus, $\big((a_0, b_0), (a_m, b_m)\big) \in (\beta \times \beta)_{|\mathbf{B}} \circ_m (\gamma \times \gamma)_{|\mathbf{B}}$. Since, by the assumption, $\mathbf{B}$ is m-permutable, we have $\big((a_0, b_0), (a_m, b_m)\big) \in (\gamma \times \gamma)_{|\mathbf{B}} \circ_m (\beta \times \beta)_{|\mathbf{B}}$. This means that in $\mathbf{B}$ there are pairs $(c_0, d_0) = (a_0, b_0)$, (c_1, d_1), (c_2, d_2), $\dots$, (c_{m-1}, d_{m-1}), $(c_m, d_m) = (a_m, b_m)$, such that $\big((c_i, d_i), (c_{i+1}, d_{i+1})\big) \in (\gamma \times \gamma)_{|\mathbf{B}}$ for i even, and $\big((c_i, d_i), (c_{i+1}, d_{i+1})\big) \in (\beta \times \beta)_{|\mathbf{B}}$ for i odd, $0 \leq i < m$.

Translating the above relations in the algebra $\mathbf{A}$ we get that $c_i \gamma c_{i+1}$ and $d_i \gamma d_{i+1}$ for i even, as well as $c_i \beta c_{i+1}$ and $d_i \beta d_{i+1}$ for i odd. Moreover, $c_i \alpha d_i$ for $0 \leq i \leq m$, by the definition of $\mathbf{B}$, and since $(c_i, d_i) \in \mathbf{B}$. Finally, $c_0 = a_0$, $d_0 = b_0$, and $c_m = a_m \delta b_m = d_m$, thus the elements c_i, d_i satisfy the desired relations. □

The above proof gives slightly more.

Theorem 2.3. *(i) If every congruence of* $\mathbf{A}$*, thought of as a subalgebra of* $\mathbf{A}^2$*, is m-permutable then* $\mathbf{A}$ *satisfies* (X_m)*.*

(ii) If every subalgebra of $\mathbf{A}^2$ *generated by $m+1$ elements is m-permutable then* $\mathbf{A}$ *satisfies* (X_m)*; actually,* $\mathbf{A}$ *satisfies the stronger version of* (X_m) *in which α is only supposed to be a compatible relation on* $\mathbf{A}$*, and δ is any relation on* $\mathbf{A}$*.*

Proof. The first statement is immediate from the proof of Theorem 2.2.

The second statement is proved in the same way by taking $\mathbf{B}$ to be the subalgebra of $\mathbf{A}^2$ generated by (a_0, b_0), (a_1, b_1) $\dots$ (a_m, b_m). Since $a_0 \alpha b_0$, $\dots$, $a_m \alpha \beta_m$, and since α is compatible, we have that $c \alpha d$, whenever $(c, d) \in \mathbf{B}$. □

If we do not care about the value assumed by the index, Property (X_m) characterizes m-permutability.

Proposition 2.4. *For every variety $\mathcal{V}$, the following are equivalent:*
(i) $\mathcal{V}$ is n-permutable for some n, and
(ii) $\mathcal{V}$ satisfies (X_m) for some m.

Proof. By Theorem 2.2, if $\mathcal{V}$ is n-permutable then $\mathcal{V}$ satisfies (X_n), thus (i)⇒(ii) is proved.

For (ii)⇒(i), notice that every algebra satisfying (X_m) is $2m-1$-permutable: just take $\alpha = 1$ and $\delta = 0$ in (X_m). □

Problem 2.5. Is the relationship between n and m given by the above proof optimal?

As far as small values of m and n are concerned, we know that for no m (X_m) is equivalent to permutability. Moreover, (X_2) is equivalent to 3-permutability.

Our original proof of Theorem 2.2 used Hagemann and Mitschke's terms [HM] and was valid only for varieties. The present proof is simpler and provides a result which holds locally. However, our original proof provided a stronger inclusion, which holds for arbitrary relations, not only for congruences. We state it here in the hope for further applications. See [L5] for a proof.

Proposition 2.6. *If $\mathbf{A}$ belongs to an m-permutable variety, $R_0, \dots, R_m$ are relations, $S_1, \dots, S_m, T_1, \dots, T_m$ are reflexive relations on $\mathbf{A}$, then*

$$R_0(S_1 \circ R_1(S_2 \circ R_2(S_3 \circ \dots R_{m-1}(S_m \circ R_m \circ T_m) \cdots \circ T_3) \circ T_2) \circ T_1) \subseteq$$
$$R_0(S'_1 \circ R'_1(S'_2 \circ R'_2(S'_3 \circ \dots R'_{m-1}(S'_m \circ R_m \circ T'_m) \dots \circ T'_3) \circ T'_2) \circ T'_1)$$

where we put ($\overline{X}$ denoting the least compatible relation containing X):

$$R'_i = \overline{R_{i-1} \cup R_i \cup R_{i+1}}, \qquad \textit{for } i = 1, \dots, m-1,$$
$$S'_1 = \overline{S_2}, \qquad T'_1 = \overline{T_2}, \qquad S'_m = \overline{S_{m-1}}, \qquad T'_m = \overline{T_{m-1}},$$
$$S'_i = \overline{S_{i-1}} \circ \overline{S_{i+1}}, \quad T'_i = \overline{T_{i+1}} \circ \overline{T_{i-1}}, \quad \textit{for } i = 2, \dots, m-1,$$

3. Applying commutator theory

We first recall the definitions of some commutators from [L1]. The actual definitions shall not be used in the present paper: we shall use only the properties stated in Theorem 3.2 below, as well as the trivial properties of

(monotonicity) $\qquad \alpha \leq \alpha'$ and $\beta \leq \beta'$ imply $[\alpha, \beta] \leq [\alpha', \beta']$

(submultiplicativity) $\qquad [\alpha, \beta] \leq \alpha\beta$

Definition 3.1. Let $\mathbf{A}$ be any algebra, and let $\alpha, \beta, \gamma \in \mathrm{Con}(\mathbf{A})$.

$M(\alpha, \beta)$ is the set of all *matrices* of the form

$$\begin{vmatrix} t(\overline{a}, \overline{b}) & t(\overline{a}, \overline{b}\,') \\ t(\overline{a}\,', \overline{b}) & t(\overline{a}\,', \overline{b}\,') \end{vmatrix}$$

where $\overline{a}, \overline{a}\,' \in A^n$, $\overline{b}, \overline{b}\,' \in A^m$, for some $m, n \geq 0$, t is an $m+n$-ary term operation of $\mathbf{A}$, and $\overline{a}\alpha\overline{a}\,'$, $\overline{b}\beta\overline{b}\,'$. Further, we set

$$K(\alpha, \beta; \gamma) = \left\{ (z, w) | \begin{vmatrix} x & y \\ z & w \end{vmatrix} \in M(\alpha, \beta), \text{ for some } x\gamma y \right\},$$

$$[\alpha, \beta|0] = 0_{\mathbf{A}}, \qquad [\alpha, \beta|n+1] = Cg(K(\alpha, \beta; [\alpha, \beta|n])),$$

where Cg means "the congruence generated by".

In the results that follow, $[\frac{m-1}{2}]$ denotes the *integer part* of $\frac{m-1}{2}$.

Theorem 3.2. *(i) If β, γ m-permute then for every n $[\beta+\gamma, \alpha|n] \leq \alpha\beta_{mn}$.*

(ii) If $\mathcal{V}$ is an m-permutable variety then there is a ternary term d such that

$$d(b, b, a) \equiv a \equiv d(a, b, b) \pmod{[\alpha, \alpha|n]}$$

for every algebra $\mathbf{A} \in \mathcal{V}$, every congruence $\alpha \in \mathrm{Con}\,\mathbf{A}$ and elements $a\alpha b \in A$, and where $n = [\frac{m-1}{2}]$.

Clause (i) in Theorem 3.2 is from [L1, Lemma 1(i)]. Condition (ii) is an easy corollary of the proof of [T, Theorem 2], as noticed in [L1, Lemma 2, and Remark (c) on p. 162]. Full details are given in the proof of [L2, Theorem 1.2(c)]. Replace n, s, t, m there by, respectively, $m-2$, $[\frac{m-2}{2}]$, $[\frac{m-1}{2}]$, 1.

Proposition 3.3. *Let $m \geq 3$ and $\mathcal{V}$ be an m-permutable variety, and put $k = m[\frac{m-1}{2}]$. Then for all $j \geq k-1$ all algebras in $\mathcal{V}$ satisfy*

$$\beta_{j+1} = \beta_k \circ \alpha\gamma_j \circ \beta_k$$

Proof. By Theorem 3.2(ii) and by [L4, Lemma 3.1(iii)] with $F(\delta) = [\delta, \delta|n]$, and $n = [\frac{m-1}{2}]$, we get

$$(*) \qquad \delta + \varepsilon = \big([\delta, \delta|n] + [\varepsilon, \varepsilon|n]\big) \circ \delta \circ \varepsilon \circ \big([\delta, \delta|n] + [\varepsilon, \varepsilon|n]\big)$$

for every pair of congruences δ and ε in every algebra in $\mathcal{V}$.

By Theorem 3.2(i), $[\beta+\gamma, \alpha|n] \leq \alpha\beta_{mn}$. Hence, by monotonicity, and since, for all j, $\alpha\gamma_j \leq \alpha(\beta+\gamma)$, we have $[\alpha\gamma_j, \alpha\gamma_j|n] \leq [\alpha(\beta+\gamma), \alpha(\beta+\gamma)|n] \leq [\beta+\gamma, \alpha|n] \leq \alpha\beta_k \leq \beta_k$, for all j.

Thus, by submultiplicativity and $(*)$ above,

$$\beta_{j+1} = \beta + \alpha\gamma_j = \big([\beta, \beta|n] + [\alpha\gamma_j, \alpha\gamma_j|n]\big) \circ \beta \circ \alpha\gamma_j \circ \big([\beta, \beta|n] + [\alpha\gamma_j, \alpha\gamma_j|n]\big)$$
$$\subseteq \big(\beta + [\alpha\gamma_j, \alpha\gamma_j|n]\big) \circ \alpha\gamma_j \circ \big(\beta + [\alpha\gamma_j, \alpha\gamma_j|n]\big) \subseteq (\beta + \beta_k) \circ \alpha\gamma_j \circ (\beta + \beta_k) =$$
$$(\beta + \beta + \alpha\gamma_{k-1}) \circ \alpha\gamma_j \circ (\beta + \beta + \alpha\gamma_{k-1}) = \beta_k \circ \alpha\gamma_j \circ \beta_k$$

For the reverse inclusion, notice that, trivially, $\beta_{j+1} \geq \alpha\gamma_j$, and $\beta_{j+1} \geq \beta_k$, since $j \geq k-1$, thus $\beta_{j+1} \geq \beta_k + \alpha\gamma_j \supseteq \beta_k \circ \alpha\gamma_j \circ \beta_k$. □

Proposition 3.4. *Let $m \geq 3$ and $\mathcal{V}$ be an m-permutable variety, and put $k = m[\frac{m-1}{2}]$. Then for all $n > 0$ all algebras in $\mathcal{V}$ satisfy*

$$\beta_{k+n} = \beta_k \circ \alpha(\gamma_k \circ \alpha(\beta_k \circ \dots \alpha(\gamma_k^\bullet \circ \alpha(\beta_k^\bullet \circ \alpha\gamma_k^\bullet \circ \beta_k^\bullet) \circ \gamma_k^\bullet) \dots \circ \beta_k) \circ \gamma_k) \circ \beta_k$$

with exactly $n-1$ open brackets and where $\beta_k^\bullet = \beta_k$, $\gamma_k^\bullet = \gamma_k$ if n is odd, and $\beta_k^\bullet = \gamma_k$, $\gamma_k^\bullet = \beta_k$ if n is even.

Proof. By Proposition 3.3 with $j = k$, we get $\alpha\beta_{k+1} = \alpha(\beta_k \circ \alpha\gamma_k \circ \beta_k)$ and, by symmetry, $\alpha\gamma_{k+1} = \alpha(\gamma_k \circ \alpha\beta_k \circ \gamma_k)$.

By the above identity, and by taking $j = k+1$ in Proposition 3.3 we have $\beta_{k+2} = \beta_k \circ \alpha\gamma_{k+1} \circ \beta_k = \beta_k \circ \alpha(\gamma_k \circ \alpha\beta_k \circ \gamma_k) \circ \beta_k$, and $\alpha\beta_{k+2} = \alpha(\beta_k \circ \alpha(\gamma_k \circ \alpha\beta_k \circ \gamma_k) \circ \beta_k)$, as well as the symmetrical identities.

The proposition is obtained by iterating the above arguments. □

Notice that, so far, we have not used the results of Section 2.

Theorem 3.5. *For $m \geq 3$, every m-permutable variety satisfies the congruence identity $\alpha\beta_h = \alpha\gamma_h$, for $h = m[\frac{m+1}{2}] - 1$*

Proof. First notice that if $k = m[\frac{m-1}{2}]$ then $h = m[\frac{m+1}{2}] - 1 = m[\frac{m-1}{2}] + m - 1 = k + m - 1$. By Proposition 3.4 with $n = m$, and by Theorem 2.2 with β_k, γ_k and $\gamma_k^\bullet$ in place of, respectively, β, γ and δ, we have

$$\alpha\beta_h = \alpha\beta_{k+m-1} \leq \alpha\beta_{k+m} =$$
$$\alpha(\beta_k \circ \alpha(\gamma_k \circ \alpha(\beta_k \circ \dots \alpha(\gamma_k^\bullet \circ \alpha(\beta_k^\bullet \circ \alpha\gamma_k^\bullet \circ \beta_k^\bullet) \circ \gamma_k^\bullet) \dots \circ \beta_k) \circ \gamma_k) \circ \beta_k) =$$
$$\alpha(\gamma_k \circ \alpha(\beta_k \circ \alpha(\gamma_k \circ \dots \alpha(\beta_k^\bullet \circ \alpha(\gamma_k^\bullet \circ \alpha\gamma_k^\bullet \circ \gamma_k^\bullet) \circ \beta_k^\bullet) \dots \circ \gamma_k) \circ \beta_k) \circ \gamma_k) =$$
$$\alpha(\gamma_k \circ \alpha(\beta_k \circ \alpha(\gamma_k \circ \dots \alpha(\beta_k^\bullet \quad \circ \quad \alpha\gamma_k^\bullet \quad \circ \quad \beta_k^\bullet) \dots \circ \gamma_k) \circ \beta_k) \circ \gamma_k) =$$
$$\alpha\gamma_{k+m-1} = \alpha\gamma_h$$

since the last two lines are equal because of Proposition 3.4 with $n = m-1$ and γ in place of β.

Thus, we have proved that $\alpha\beta_h \leq \alpha\gamma_h$. By symmetry $\alpha\gamma_h \leq \alpha\beta_h$, from which we reach the conclusion. □

In the particular case of locally finite m-permutable varieties, the value $n = [\frac{m-1}{2}]$ in Theorem 3.2(ii) can be improved to $n = 1$, because of [HMK, Theorems 9.8 and 9.14], and of the result stated in the last line of [L1, p. 163]. K. Kearnes [K2] has communicated us results which imply that Theorem 3.2(ii) holds with $n = 1$ for every m-permutable variety. Thus, modulo the above results, Theorem 3.5 holds for $h = 2m - 1$.

Can h be improved further?

Considering small values of m suggests that h can be actually improved. Permutable and 3-permutable varieties are congruence modular, hence they satisfy $\alpha\beta_2 = \alpha\gamma_2$. We know that an m-permutable variety $\mathcal{V}$ satisfies $\alpha\beta_m = \alpha\gamma_m$ if at least one of the following conditions is satisfied: (a) $m = 4$ or $m = 5$ (no use of commutator theory); (b) $\mathcal{V}$ is semidistributive [K2]; (c) $\mathcal{V}$ has a difference term for $[\alpha, \alpha|1]$ (that is, a term satisfying condition (ii) in Theorem 3.2 with $n = 1$ and with one "$\equiv$" replaced by "$=$").

The proof of Theorem 3.5 applies to a more general context. First, notice that, in the proof of 3.5, in place of (X_m), it is enough to assume the following weaker property $(X_m)^*$:

$$\alpha(\beta \circ \alpha(\gamma \circ \alpha(\beta \circ \dots \alpha(\gamma^\bullet \circ \alpha(\beta^\bullet \circ \alpha\delta \circ \beta^\bullet) \circ \gamma^\bullet) \dots \circ \beta) \circ \gamma) \circ \beta) \subseteq$$
$$\Big(\alpha(\gamma \circ \alpha(\beta \circ \alpha(\gamma \circ \dots \alpha(\beta^\bullet \circ \alpha(\gamma^\bullet \circ \alpha\delta \circ \gamma^\bullet) \circ \beta^\bullet) \dots \circ \gamma) \circ \beta) \circ \gamma)\Big)^*$$

with m normal-sized open parenthesis on each side, where * denotes transitive closure.

We have a long technical proof showing that if a variety satisfies $(X_m)^*$ then for some n and k the commutator $[\alpha, \beta|n]$ satisfies $[\beta + \gamma, \alpha|n] \leq \alpha\beta_k$, and there exists a term d as in condition (ii) in Theorem 3.2. Thus we get: *If a variety $\mathcal{V}$ satisfies $(X_m)^*$ for some m then there is some h (depending on $\mathcal{V}$) such that $\mathcal{V}$ satisfies $\alpha\beta_h = \alpha\gamma_h$.*

Is the converse true?

Problem 3.6. Is it true that if $\mathcal{V}$ satisfies $\alpha\beta_h = \alpha\gamma_h$ for some h then $\mathcal{V}$ satisfies $(X_m)^*$ for some m?

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