

TOLERANCE INTERSECTION PROPERTIES AND SUBALGEBRAS OF SQUARES

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Abstract. Tolerance identities can be used [5] in order to provide a fairly simple proof of a classical result by R. Freese and B. Jónsson asserting that every congruence modular variety is in fact Arguesian. The main advantage of the approach using tolerances is that stronger identities can be obtained: the higher Arguesian identities introduced by M. Haiman [12].

Discouragingly enough, however, the approach using tolerances does not appear to give a proof for the local version of Freese and Jónsson's Theorem: *If every subalgebra of $\mathbf{A} \times \mathbf{A}$ is congruence modular then \mathbf{A} is Arguesian.*

A common generalization of the results mentioned in the above two paragraphs would furnish the following (a conjecture, so far): *If every subalgebra of $\mathbf{A} \times \mathbf{A}$ is congruence modular then \mathbf{A} satisfies all Haiman's higher Arguesian identities.*

Towards a possible solution of the above conjecture, we introduce a new tolerance identity, called wTIP, and show that it holds in the algebra \mathbf{A} provided that every subalgebra of $\mathbf{A} \times \mathbf{A}$ is congruence modular. Moreover, wTIP is equivalent to the Shifting Principle introduced by H. P. Gumm. The known methods using tolerance identities apply, if we restrict ourselves to a particular class of lattice terms.

§1. From congruence varieties to tolerance identities. In the present section we briefly describe how two seemingly unrelated notions used in universal algebra came in touch in an unexpected way. We first recall some basic notions and terminology. Introductory textbooks to universal algebra are, among others, [1], [9], [26]. For a reader not familiar with universal algebra, a preliminary reading of the survey [16] would probably be of great help.

An *algebraic structure*, *algebra* for short, is a set endowed with operations and constants. Algebras are denoted by \mathbf{A} , \mathbf{B} ... A *variety* \mathcal{V} is a class of algebras of the same type which is closed under taking products,

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subalgebras and homomorphic images. Equivalently, a variety is a class of algebras which can be defined by a set of equations.

Many interesting results from *universal algebra* (better named the *general theory of algebraic systems*) have the following form:

THEOREM 1.1 (Prototype). *If every algebra in the variety \mathcal{V} satisfies Property P then every algebra in the variety \mathcal{V} satisfies Property Q ,*

where P and Q are appropriately chosen properties P and Q . In the most important cases, an implication as above is non-trivial, in the sense that it holds for varieties, not just for single algebras. This means that there exists some algebra \mathbf{A} satisfying P , and not satisfying Q .

Particularly intriguing results appear when dealing with congruences: a congruence of an algebra \mathbf{A} is the equivalence relation on A determined by some homomorphism ϕ from \mathbf{A} into some algebra \mathbf{B} . For practical purposes, an “internal” description is better suited: a *congruence* α of \mathbf{A} is a compatible equivalence relation on A , where *compatible* means that whenever f is an n -ary operation of \mathbf{A} , and $a_1\alpha b_1, \dots, a_n\alpha b_n$, then also $f(a_1, \dots, a_n)\alpha f(b_1, \dots, b_n)$. Here, $a_1\alpha b_1$ is a shorthand for “ a_1 and b_1 being α -related”, that is, $(a_1, b_1) \in \alpha$. According to notational convenience, this will also be denoted by $a_1 \equiv b_1 \pmod{\alpha}$.

The set of congruences of an algebra turns out to be endowed with a lattice structure, with a minimal congruence 0 and a maximal congruence 1 , and with meet coinciding with intersection. We shall denote the lattice operations by $+$ and \cdot (or simply juxtaposition). A lattice is *distributive* if and only if it satisfies $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$. The modular law asserts that $\alpha(\beta + \gamma) = \alpha\beta + \gamma$, provided $\gamma \leq \alpha$.

A detailed analysis of varieties, and of algebras in a variety \mathcal{V} has proved possible under the assumption that the lattices of congruences of *all* algebras in \mathcal{V} satisfy some non trivial lattice identity. Historically, the first structure theorem has been discovered for algebras in congruence distributive varieties [15]; shortly after, a characterization of congruence modularity has been found [6], and eventually a refined theory for congruence modular varieties emerged: see [28], [8], [10]. Briefly, the theory shows that congruence modular varieties share all the good properties which are in common to congruence distributive and congruence permutable varieties. Recall that an algebra is *permutable* if and only if every two congruences permute, that is, $\alpha \circ \beta = \beta \circ \alpha$. A. I. Mal’cev characterization of permutable varieties [25], in 1954, can be safely considered the starting point for modern universal algebra.

See [16] for further reference about the origins of the subject and its earlier history. A brief introduction to the latter developments (commutator theory) is given in [23]. See also [5] for further references.

Present-day research is obtaining results even for varieties satisfying any non-trivial congruence lattice equation. For further information the reader is referred to K. Kearnes and Á Szendrei's fundamental work [18], and to, among others, [13], [22] and [17]. There are also further results announced by K. Kearnes and E. Kiss. In the present note, however, we shall be dealing only with properties related to congruence modularity.

J. B. Nation [27] discovered the first non trivial result of the form given in Theorem 1.1, where both P and Q are lattice identities (intended to be satisfied by congruence lattices of algebras in \mathcal{V}): he proved that there is an identity lattice-theoretically strictly weaker than modularity which is nevertheless equivalent to modularity for congruence lattices of algebras in a given variety. Other results of this kind soon followed: see [16] for further information; see [5], [24] and references there for more recent results.

Our main concern here is a result of the above kind obtained by R. Freese and B. Jónsson [7] in the '70's; they proved that, within a variety, modularity implies a stronger identity, the *Arguesian identity*. The result is particularly significant since the Arguesian identity is an important identity with a clear geometrical meaning (see again [16] for further information).

An alternative proof of R. Freese and B. Jónsson's result has been found in [23] using commutator theory. More intriguingly, [5] provides still another proof, which uses the following tolerance identity

$$(TIP) \quad \Gamma^* \cap \Theta^* = (\Gamma \cap \Theta)^*$$

where $*$ denotes transitive closure.

A *tolerance* Θ on some algebra \mathbf{A} is a compatible symmetric and reflexive relation (in other words, a tolerance is a "not necessarily transitive" congruence). A priori, there is no evidence that tolerances should be useful in the study of congruences; however, astonishingly, tolerances have proved useful and sometimes irreplaceable (at least with present-day available techniques). A satisfying explanation of such an important role played by tolerances has still to be found, in our opinion.

We should mention that [5], using tolerances, obtained simple and short proofs for results whose previous proofs were quite tricky and complicated, let alone the fact that usually the results given in [5] are even stronger. For example, considering Freese and Jónsson's result, the main advantage of the proof given in [5] is that it furnishes even stronger identities: the *higher Arguesian identities* introduced by M. Haiman [12]. The higher Arguesian identities are interesting because they hold in all lattices of permutable equivalence relations (see [14], [11]).

However, to our disappointment, neither [23] nor [5] entirely encompass [7], since the local form in which Freese and Jónsson [7] proved their theorem is quite strong, and asserts that *if every subalgebra of $\mathbf{A} \times \mathbf{A}$ is congruence modular then \mathbf{A} is Arguesian*. Of course, it is not really fundamental

to get another proof of the local version of [7], since a proof is available. On the other hand, a new proof along the lines of [5] would probably improve [7] to *If every subalgebra of $\mathbf{A} \times \mathbf{A}$ is congruence modular then \mathbf{A} is Arguesian*. We have not yet found a proof of the above conjecture, but we suggest here a possible approach towards an affirmative solution (see also Section 6).

Let us say a few words about the proof given in [5]. [5] shows that every algebra satisfying TIP satisfies all the higher Arguesian identities. Since it is known that every congruence modular variety satisfies TIP, we immediately get that every algebra in a congruence modular variety satisfies the higher Arguesian identities. The natural way to extend the methods from [5] would be to show that if every subalgebra of $\mathbf{A} \times \mathbf{A}$ is congruence modular, then \mathbf{A} satisfies TIP; however, this is still an open problem. We only get a weaker property wTIP, and it is possible that wTIP implies congruence arguesianity, but this is open, too.

However, we show here that wTIP has some interest for itself: it implies, congruence modularity, and it is equivalent, for every algebra, to a principle introduced by H. P. Gumm in his deep study [10] of congruence modular varieties. Moreover, wTIP is strong enough for the methods of [5] to be applied, provided we restrict attention to a particular class of terms, the class of *terms without double couples* (Definition 4.1).

The paper is divided as follows. In Section 2 we introduce some further notations. In Section 3 we prove our main positive result, Theorem 3.2, asserting that *If \mathbf{A} is an algebra and every subalgebra of $\mathbf{A} \times \mathbf{A}$ is congruence modular then \mathbf{A} satisfies wTIP*. We state a related (somewhat more general) result in Theorem 3.4.

In Section 4 we introduce the notion of a term without double couples, and show that, assuming wTIP, certain joins coincide, no matter whether they are computed in the lattice of tolerances or in the lattice of congruences. As a consequence, wTIP implies congruence modularity. Some further consequences are derived in Corollaries 4.6 and 4.7.

In Section 5 we show that wTIP is equivalent to the Shifting Principle, introduced by H. P. Gumm; moreover, we introduce a stronger version of the Shifting Principle (Definition 5.4), and show that it is equivalent to TIP (Proposition 5.5). Finally, in Section 6, we recall the main problems which led to our definition of wTIP, and to the results proved in the present paper; some further problems are stated.

§2. Notations. Here are the notations we use (if not already introduced in the previous section).

R, S denote reflexive binary relations; as in the case of congruences, we shall write aRb to mean that $(a, b) \in R$. It will be notationally convenient to

use chains of the above notation: for example, $a\Theta b\alpha cRd$ means $(a, b) \in \Theta$, $(b, c) \in \alpha$ and $(c, d) \in R$.

Juxtaposition denotes intersection; in particular $\alpha\beta$ denotes the meet of the congruences α and β .

R^* denotes the transitive closure of the binary relation R ; in particular, Θ^* is the smallest congruence which contains the tolerance Θ . \overline{R} denotes the smallest compatible relation containing R (where R is a binary relation on some algebra which should be clear from the context). In particular, $\overline{\Theta \cup \Gamma}$ is the smallest tolerance which contains Θ and Γ .

$R+S$ denotes $\bigcup_{n \in \mathbb{N}} \underbrace{R \circ S \circ R \circ S \dots}_{n \text{ factors}}$. Thus, $R+S = (R \cup S)^* = (R \circ S)^*$.

In particular, if α, β are congruences, $\alpha + \beta$ is the join of α and β in the lattice of congruences, while, for Θ, Γ tolerances, $\Theta + \Gamma$ is the smallest congruence which contains both Θ and Γ . Notice that $\Theta + \Gamma$ is far larger than the join of Θ and Γ in the lattice of tolerances.

R^- denotes the converse of R , that is, aR^-b if and only if bRa . In particular, $R + R^-$ is the smallest equivalence relation containing R .

§3. The weak tolerance intersection property.

DEFINITION 3.1. Let us say that an algebra \mathbf{A} satisfies wTIP when

$$\alpha \cap \Theta^* = (\alpha \cap \Theta)^*$$

holds for every congruence α and tolerance Θ of \mathbf{A} , where $*$ denotes transitive closure.

THEOREM 3.2. If \mathbf{A} is an algebra and every subalgebra of $\mathbf{A} \times \mathbf{A}$ generated by 4 elements is congruence modular then \mathbf{A} satisfies wTIP.

LEMMA 3.3. If an algebra \mathbf{A} satisfies

$$\alpha \cap (\Theta \circ \Theta) \subseteq (\alpha \cap \Theta)^*$$

for every tolerance Θ and congruence α of \mathbf{A} , then \mathbf{A} satisfies wTIP.

PROOF. The proof is a slight variation on arguments from [3] (cf. also the proof that condition (5) implies condition (3) in [4]).

The inclusion $(\alpha \cap \Theta)^* \subseteq (\alpha \cap \Theta^*)^* = \alpha \cap \Theta^*$ is trivial, since α is a congruence, hence $\alpha \cap \Theta^*$ is a congruence.

For the converse, suppose that $a, b \in A$, and $(a, b) \in \alpha \cap \Theta^*$. We have to show that $(a, b) \in (\alpha \cap \Theta)^*$.

If $(a, b) \in \alpha \cap \Theta^*$, then there is some natural number n , such that $(a, b) \in \alpha \cap \Theta^n$, where we define $\Theta^n = \Theta \circ \Theta \circ \dots \circ \Theta$ (n occurrences of Θ). Hence, it is enough to show that $\alpha \cap \Theta^n \subseteq (\alpha \cap \Theta)^*$, for every n .

We shall show by induction on $m \geq 1$ that

$$(*) \quad \alpha \cap \Theta^{2^m} \subseteq (\alpha \cap \Theta)^*$$

and this is enough, since $\alpha \cap \Theta^n \subseteq \alpha \cap \Theta^{2^m}$, provided we choose m in such a way that $n \leq 2^m$.

The basis $m = 1$ of the induction is the hypothesis of the Lemma.

Now for the inductive step. Suppose that (*) is true for some given m : then

$$\alpha \cap \Theta^{2^{m+1}} = \alpha \cap (\Theta^{2^m} \circ \Theta^{2^m}) \subseteq (\alpha \cap \Theta^{2^m})^* \subseteq (\alpha \cap \Theta)^{**} = (\alpha \cap \Theta)^*$$

where the last inclusion follows from the inductive hypothesis, and the first inclusion follows from the hypothesis of the Lemma by taking the tolerance Θ^{2^m} in place of Θ .

Thus, (*) is true for $m + 1$, and the induction is complete. \dashv

PROOF OF THEOREM 3.2. In view of Lemma 3.3, it is enough to show that $\alpha \cap (\Theta \circ \Theta) \subseteq (\alpha \cap \Theta)^*$.

So let $a, c \in A$ and $(a, c) \in \alpha \cap (\Theta \circ \Theta)$. Thus $a\alpha c$, and there is $b \in A$ such that $a\Theta b\Theta c$.

Consider the subalgebra \mathbf{B} of $\mathbf{A} \times \mathbf{A}$ generated by the four elements $(a, a), (a, b), (c, b), (c, c)$.

First, observe that if $(x, y) \in \mathbf{B}$ then $x\Theta y$, since all the generators of \mathbf{B} are in Θ , and Θ is compatible.

We have that $((a, a), (a, b))$ and $((c, b), (c, c))$ belong to $(0 \times 1)_{|\mathbf{B}}$, $((a, b), (c, b))$ belongs to $(\alpha \times 0)_{|\mathbf{B}}$ and $((a, a), (c, c))$ belongs to $(\alpha \times \alpha)_{|\mathbf{B}}$.

The above relations imply that $((a, a), (c, c))$ belongs to

$$(\alpha \times \alpha)_{|\mathbf{B}} \cap \left((0 \times 1)_{|\mathbf{B}} + (\alpha \times 0)_{|\mathbf{B}} \right)$$

Since $(\alpha \times 0)_{|\mathbf{B}} \leq (\alpha \times \alpha)_{|\mathbf{B}}$, and \mathbf{B} is congruence modular,

$$\begin{aligned} (\alpha \times \alpha)_{|\mathbf{B}} \cap \left((0 \times 1)_{|\mathbf{B}} + (\alpha \times 0)_{|\mathbf{B}} \right) &\leq \left((\alpha \times \alpha)_{|\mathbf{B}} \cap (0 \times 1)_{|\mathbf{B}} \right) + (\alpha \times 0)_{|\mathbf{B}} = \\ &= (0 \times \alpha)_{|\mathbf{B}} + (\alpha \times 0)_{|\mathbf{B}} \end{aligned}$$

In conclusion, $((a, a), (c, c))$ belongs to

$$(0 \times \alpha)_{|\mathbf{B}} + (\alpha \times 0)_{|\mathbf{B}}$$

This implies that there is some n , and there are pairs $(x_i, y_i) \in \mathbf{B}$ ($0 \leq i \leq n$) such that

$$(a, a) = (x_0, y_0) \quad (x_n, y_n) = (c, c)$$

$$(x_i, y_i) \equiv (x_{i+1}, y_{i+1}) \pmod{(0 \times \alpha)_{|\mathbf{B}}} \quad \text{for } i \text{ even}$$

$$(x_i, y_i) \equiv (x_{i+1}, y_{i+1}) \pmod{(\alpha \times 0)_{|\mathbf{B}}} \quad \text{for } i \text{ odd}$$

In other words,

$$\begin{aligned} a = x_0 = y_0 & \quad c = x_n = y_n \\ x_i = x_{i+1}, & \quad y_i \alpha y_{i+1} \quad \text{for } i \text{ even} \\ x_i \alpha x_{i+1}, & \quad y_i = y_{i+1} \quad \text{for } i \text{ odd} \end{aligned}$$

In particular, $a = x_0 = x_1\alpha x_2 = x_3\alpha x_4 \dots$, and $a = y_0\alpha y_1 = y_2\alpha y_3 = y_4 \dots$, hence $x_i\alpha y_j$ for all i 's and j 's, since α is a congruence, and both x_i and y_j are congruent to a modulo α .

Moreover, since $(x_i, y_i) \in \mathbf{B}$, then $x_i\Theta y_i$ for all i 's, by the remark made after the definition of \mathbf{B} .

Hence, for all i 's, $(x_i, y_i) \in \alpha \cap \Theta$, and $(y_i, x_i) \in \alpha \cap \Theta$, since Θ is symmetric. In conclusion, the sequence

$$a = x_0 = x_1 \quad y_1 = y_2 \quad x_2 = x_3 \quad y_3 = y_4 \quad \dots \quad x_n = y_n = c$$

witnesses that $(a, c) \in (\alpha \cap \Theta)^*$. -1

The proof of Theorem 3.2 can be modified to get (see [20] for details):

THEOREM 3.4. *Suppose that \mathbf{A} is an algebra such that every subalgebra of $\mathbf{A} \times \mathbf{A}$ generated by 4 elements satisfies $\beta \cap (\gamma \circ \delta \circ \gamma) \subseteq \beta\gamma + \delta$, for all congruences β, γ, δ with $\delta \leq \beta$.*

Then \mathbf{A} satisfies wTIP.

More generally, \mathbf{A} satisfies

$$\alpha(R + S) \subseteq \alpha(\overline{R \cup S^-}) + \alpha(\overline{R^- \cup S}) \quad \text{and}$$

$$\alpha(R + R^-) \subseteq \alpha(\overline{R} + \overline{R^-}) = \alpha\overline{R} + \alpha\overline{R^-} = \alpha(Cg(R))$$

for all reflexive relations R and S and every congruence α , and where $Cg(R)$ denotes the congruence generated by R .

§4. Terms without double couples. The definition of wTIP and Theorem 3.2 allow us to apply the methods of [5] to the class of *terms without double couples* in the following sense:

DEFINITION 4.1. The class of *wdc-terms* (terms without double couples) is the smallest class (of terms in the language of lattices) which

- (a) contains all variables
- (b) contains all terms of the form

$$x_i(t^1 + t^2 + \dots + t^n)$$

where $n > 0$, x_i is a variable, and t^1, t^2, \dots, t^n are wdc-terms.

Essentially, a wdc-term is a term with no subterm of the form $(t^1 + t^2)(t^3 + t^4)$. Wdc-terms had been called *simple terms* in [21].

DEFINITION 4.2. If t is a term for the language of lattices $\cdot, +$, define by induction the term t_2 for the language \cap, \circ as follows:

- If $t = x_i$ is a variable, let $t_2 = t$;
- if $t = r \cdot s$, let $t_2 = r \cap s$, and
- if $t = r + s$, let $t_2 = r \circ s$.

Thus, t_2 is obtained from t simply by replacing \cdot by \cap and $+$ by \circ .

In the present paper, we are mainly concerned with lattices of congruences. Tolerances, too, form a lattice. In both cases, meet is intersection. Congruences are (particular) tolerances but usually the join of two congruences in the lattice of tolerances is far smaller than their join in the lattice of congruences. Since we need consider both cases simultaneously, given a lattice term t , we shall write t_{con} , t_{tol} , respectively, for the case when join is computed in the lattice of congruences, tolerances, respectively. $+$ will always denote join in the lattice of congruences. Recall that the join of two tolerances Γ , Θ (in the lattice of tolerances) is $\overline{\Gamma \cup \Theta}$, and that juxtaposition denotes intersection.

PROPOSITION 4.3. *If \mathbf{A} is an algebra which satisfies wTIP, then $t_{tol}(\bar{\alpha})^* = t_2(\bar{\alpha})^* = t_{con}(\bar{\alpha})$, for every wdc-term t , and for every n -tuple $\bar{\alpha}$ of congruences of \mathbf{A} .*

PROOF. The proof is essentially the same as the proof of Lemma 1 in [5]. wTIP allows us to apply the methods of [5], provided we consider only wdc-terms ([5] can be applied to *every* term, but using the stronger assumption TIP).

We shall first prove $t_{con}(\bar{\alpha}) = t_{tol}(\bar{\alpha})^*$ by induction on the complexity of the wdc-term t .

If $t = x_i$ is a variable, then $t_{con}(\bar{\alpha}) = \alpha_i = t_{tol}(\bar{\alpha})^*$, since α is supposed to be a congruence.

Now let $t = x_i(t^1 + t^2 + \dots + t^n)$, where the t^j 's are wdc-terms. Thus,

$$\begin{aligned} t_{con}(\bar{\alpha}) &= \alpha_i(t_{con}^1(\bar{\alpha}) + t_{con}^2(\bar{\alpha}) + \dots + t_{con}^n(\bar{\alpha})) = \\ \alpha_i(t_{tol}^1(\bar{\alpha})^* + t_{tol}^2(\bar{\alpha})^* + \dots + t_{tol}^n(\bar{\alpha})^*) &= \alpha_i(\overline{t_{tol}^1(\bar{\alpha}) \cup t_{tol}^2(\bar{\alpha}) \cup \dots \cup t_{tol}^n(\bar{\alpha})})^* = \\ &= \left(\alpha_i(\overline{t_{tol}^1(\bar{\alpha}) \cup t_{tol}^2(\bar{\alpha}) \cup \dots \cup t_{tol}^n(\bar{\alpha})}) \right)^* = t_{tol}(\bar{\alpha})^* \end{aligned}$$

by the inductive hypothesis applied to t^1, t^2, \dots, t^n , and by wTIP applied to the congruence α_i and to the tolerance $\overline{t_{tol}^1(\bar{\alpha}) \cup t_{tol}^2(\bar{\alpha}) \cup \dots \cup t_{tol}^n(\bar{\alpha})}$. The induction step is thus complete, hence we have proved $t_{con}(\bar{\alpha}) = t_{tol}(\bar{\alpha})^*$.

Since $t_{tol}(\bar{\alpha}) \subseteq t_2(\bar{\alpha}) \subseteq t_{con}(\bar{\alpha})$ trivially, and since $t_{con}(\bar{\alpha})$ is a congruence, hence $t_{con}(\bar{\alpha})^* = t_{con}(\bar{\alpha})$, by the closure properties of $*$ we get $t_{tol}(\bar{\alpha})^* \subseteq t_2(\bar{\alpha})^* \subseteq t_{con}(\bar{\alpha})$.

Since we have already proved $t_{con}(\bar{\alpha}) = t_{tol}(\bar{\alpha})^*$, we get $t_{tol}(\bar{\alpha})^* = t_2(\bar{\alpha})^* = t_{con}(\bar{\alpha})$. \dashv

COROLLARY 4.4. *If \mathbf{A} satisfies wTIP, s, t are lattice terms, and t is wdc, then for every sequence of congruences $\bar{\alpha}$ of \mathbf{A} the following are equivalent:*

- (i) $t_{tol}(\bar{\alpha}) \subseteq s_{con}(\bar{\alpha})$;
- (ii) $t_2(\bar{\alpha}) \subseteq s_{con}(\bar{\alpha})$;
- (iii) $t_{con}(\bar{\alpha}) \subseteq s_{con}(\bar{\alpha})$.

PROOF. Since $s_{con}(\bar{\alpha})$ is a congruence, then $t_{tol}(\bar{\alpha}) \subseteq s_{con}(\bar{\alpha})$ if and only if $t_{tol}(\bar{\alpha})^* \subseteq s_{con}(\bar{\alpha})$. Similarly, $t_2(\bar{\alpha}) \subseteq s_{con}(\bar{\alpha})$ if and only if $t_2(\bar{\alpha})^* \subseteq s_{con}(\bar{\alpha})$.

The conclusion is now immediate from Proposition 4.3. \dashv

COROLLARY 4.5. *If \mathbf{A} satisfies wTIP then \mathbf{A} is congruence modular.*

PROOF. Apply Corollary 4.4 to the terms $t(x, y, z) = x(y + (zx))$ and $s(x, y, z) = (xy) + (zx)$.

Since $\alpha \cap (\beta \circ (\gamma \cap \alpha)) = (\alpha \cap \beta) \circ (\gamma \cap \alpha) \subseteq \alpha\beta + \gamma\alpha$, we get, by (ii) \Rightarrow (iii) in Corollary 4.4, $\alpha(\beta + (\gamma\alpha)) \leq \alpha\beta + \gamma\alpha$, that is, congruence modularity. \dashv

Indeed, Proposition 4.3 gives more: for example, if \mathbf{A} satisfies wTIP then $\alpha(\beta + \gamma) = (\alpha \cap (\beta \circ \gamma))^*$, for all congruences α, β, γ of \mathbf{A} .

COROLLARY 4.6. *Suppose that \mathbf{A} is an algebra such that every subalgebra of $\mathbf{A} \times \mathbf{A}$ generated by 4 elements satisfies $\beta(\gamma \circ \delta \circ \gamma) \subseteq \beta\gamma + \delta$, for all congruences β, γ, δ with $\delta \leq \beta$.*

Then \mathbf{A} is congruence modular.

PROOF. By Theorem 3.4 \mathbf{A} satisfies wTIP, hence \mathbf{A} is congruence modular by Corollary 4.5. \dashv

COROLLARY 4.7. *If \mathbf{A} satisfies wTIP then the following are equivalent:*

- (a) \mathbf{A} satisfies $\alpha(\overline{\beta \cup \gamma}) \subseteq \alpha\beta + \alpha\gamma$, for all congruences α, β and γ ;
- (b) \mathbf{A} satisfies $\alpha(\overline{\beta \circ \gamma}) \subseteq \alpha\beta + \alpha\gamma$, for all congruences α, β and γ ;
- (c) \mathbf{A} is congruence distributive.

PROOF. Apply Corollary 4.4 to the terms $t(x, y, z) = x(y + z)$ and $s(x, y, z) = (xy) + (zx)$. \dashv

The proofs of Proposition 4.3 and of Corollary 4.4 furnish the following slight strengthening of some results in [5]:

PROPOSITION 4.8. *Suppose that the algebra \mathbf{A} satisfies TIP, s, t are lattice terms, and $\bar{\alpha}$ is an n -uple of congruences of \mathbf{A} .*

Then $t_{tol}(\bar{\alpha})^ = t_2(\bar{\alpha})^* = t_{con}(\bar{\alpha})$.*

Moreover, the following are equivalent:

- (i) $t_{tol}(\bar{\alpha}) \subseteq s_{con}(\bar{\alpha})$;
- (ii) $t_2(\bar{\alpha}) \subseteq s_{con}(\bar{\alpha})$;
- (iii) $t_{con}(\bar{\alpha}) \subseteq s_{con}(\bar{\alpha})$.

$t_{tol}(\bar{\alpha})^* = t_{con}(\bar{\alpha})$ in Proposition 4.8 can be obtained also as a consequence of the known result stating that TIP implies that $*$ is a homomorphism from the lattice of tolerances of \mathbf{A} to the lattice of congruences of \mathbf{A} (see e.g. [2, Lemma 3.1]).

§5. Gumm's Shifting Principle. The Shifting Principle has been introduced by H. P. Gumm in connection with his geometrical interpretation of commutator theory in the study of congruence modular varieties. We show here that the Shifting Principle is equivalent to wTIP. Moreover, we introduce a stronger version of the Shifting Principle, and we show that this stronger version is equivalent to TIP.

DEFINITION 5.1. [10, p. 14] An algebra \mathbf{A} satisfies *Gumm's Shifting Principle* if and only if whenever α, γ are congruences, and Λ is a tolerance of \mathbf{A} such that $\alpha\Lambda \leq \gamma \leq \alpha$ then $\alpha(\Lambda \circ \gamma \circ \Lambda) \subseteq \gamma$ (recall that juxtaposition denotes intersection).

See [10] for a picture, and for geometrical interpretations and consequences.

Recall that, for tolerances Λ, Ψ , $\Lambda + \Psi = \overline{\Lambda \cup \Psi}^*$ = the smallest congruence containing both Λ and Ψ .

PROPOSITION 5.2. *For every algebra, the following are equivalent:*

- (i) wTIP;
- (ii) $\alpha(\Lambda + \alpha\Psi) = \alpha\Lambda + \alpha\Psi$, for all tolerances Λ and Ψ , and congruence α ;
- (iii) $\alpha(\Lambda \circ \alpha\Psi \circ \Lambda) \subseteq \alpha\Lambda + \alpha\Psi$, for all tolerances Λ and Ψ , and congruence α ;
- (iv) *The Shifting Principle.*

PROOF. (i) \Rightarrow (ii). Let Θ be the tolerance $\overline{\Lambda \cup \alpha\Psi}$. By applying wTIP, and since α is a congruence, we get $\alpha(\Lambda + \alpha\Psi) = \alpha\Theta^* = (\alpha\Theta)^* \subseteq (\alpha(\Lambda \circ \alpha\Psi))^* = (\alpha\Lambda \circ \alpha\Psi)^* = \alpha\Lambda + \alpha\Psi$.

The \supseteq inclusion is trivial, since α is a congruence.

(ii) \Rightarrow (iii) is trivial, since $\Lambda \circ \alpha\Psi \circ \Lambda \subseteq \Lambda + \alpha\Psi$.

(iii) \Rightarrow (iv). Take $\Psi = \gamma$ in (iii), thus getting $\alpha(\Lambda \circ \gamma \circ \Lambda) = \alpha(\Lambda \circ \alpha\gamma \circ \Lambda) \subseteq \alpha\Lambda + \alpha\gamma = \alpha\Lambda + \gamma = \gamma$.

(iv) \Rightarrow (i). Suppose that Gumm's Shifting Principle holds, and let α and Θ be, respectively, a congruence and a tolerance of \mathbf{A} . Consider the congruence $\gamma = (\alpha\Theta)^*$, and apply the Shifting Principle with Θ in place of Λ . We get $\alpha(\Theta \circ \Theta) \subseteq \alpha(\Theta \circ \gamma \circ \Theta) \subseteq \gamma = (\alpha\Theta)^*$.

Hence wTIP holds by Lemma 3.3. \dashv

Theorem 1 in [3] showed that congruence modular varieties satisfy a certain tolerance identity (which turned out to be equivalent to TIP, see [4]), and [3, Corollary 1] showed that this tolerance identity implies the Shifting Principle.

COROLLARY 5.3. *Suppose that \mathbf{A} is an algebra such that every subalgebra of $\mathbf{A} \times \mathbf{A}$ generated by 4 elements satisfies $\beta(\gamma \circ \delta \circ \gamma) \subseteq \beta\gamma + \delta$, for all congruences β, γ, δ with $\delta \leq \beta$. Then \mathbf{A} satisfies Gumm's Shifting Principle.*

In particular, if every subalgebra of $\mathbf{A} \times \mathbf{A}$ is congruence modular then \mathbf{A} satisfies Gumm's Shifting Principle.

PROOF. Immediate from Theorem 3.4 and Proposition 5.2. \dashv

DEFINITION 5.4. Let us say that an algebra \mathbf{A} satisfies the *Strong Shifting Principle* if and only if whenever γ is a congruence, and Γ, Λ are tolerances of \mathbf{A} such that $\Gamma\Lambda \leq \gamma \leq \Gamma^*$ then $\Gamma(\Lambda \circ \gamma \circ \Lambda) \subseteq \gamma$.

Gumm's original Shifting Principle is a particular case of the Strong Shifting Principle: just let $\Gamma = \alpha$ be a congruence.

PROPOSITION 5.5. *For every algebra, TIP is equivalent to the Strong Shifting Principle.*

PROOF. Suppose that TIP holds, and that Γ, γ and Λ are as in the hypothesis of the Strong Shifting Principle.

Since γ is a congruence, $\Gamma\Lambda \subseteq \gamma$ implies $(\Gamma\Lambda)^* \subseteq \gamma$. By TIP we get $\Gamma^*\Lambda^* \subseteq \gamma$, hence $\Gamma^*\Lambda \subseteq \gamma$.

Since TIP implies wTIP, then by Proposition 5.2 we can apply Gumm's Shifting Principle with the congruence Γ^* in place of α in order to get $\Gamma(\Lambda \circ \gamma \circ \Lambda) \subseteq \Gamma^*(\Lambda \circ \gamma \circ \Lambda) \subseteq \gamma$.

Conversely, suppose that the Strong Shifting Principle holds, and let Γ, Θ be tolerances of \mathbf{A} . Consider the congruence $\gamma = (\Gamma\Theta)^*$, and apply the Strong Shifting Principle with Θ in place of Λ . We get $\Gamma(\Theta \circ \Theta) \subseteq (\Gamma\Theta)^*$, which implies TIP (see e.g. [4]). \dashv

Other conditions equivalent to TIP have been introduced in G. Czédli and E. Horváth's fundamental contribution [3]. See [2, Lemma 3.1] and [4] for the proof that such conditions are actually equivalent to TIP. Notice that the statement of [2, Lemma 3.1] includes the assumption that \mathbf{A} is congruence modular, but this assumption turns out to be unnecessary for Conditions (ii), (iii) and (iv) therein since they all imply modularity (see [5, Lemma 1(D)], or Corollary 4.5).

PROBLEM 5.6. Investigate the geometrical meaning of the Strong Shifting Principle.

In [19] we introduced still another variant of TIP.

$$(wTIP_2) \quad \Gamma^* \cap \Theta^* = (\Gamma \cap (\Theta \circ \Theta))^*$$

COROLLARY 5.7. *For every variety \mathcal{V} the following are equivalent:*

- (i) *Every algebra in \mathcal{V} is congruence modular;*
- (ii) *every algebra in \mathcal{V} satisfies TIP;*
- (iii) *every algebra in \mathcal{V} satisfies wTIP₂;*
- (iv) *every algebra in \mathcal{V} satisfies wTIP;*
- (v) *every algebra in \mathcal{V} satisfies Gumm's Shifting Principle.*
- (vi) *every algebra in \mathcal{V} satisfies the Strong Shifting Principle.*

PROOF. See e.g. [4] for a proof that (i) \Rightarrow (ii).

Since $\text{TIP} \Rightarrow \text{wTIP}_2 \Rightarrow \text{wTIP}$ in every algebra, (ii) \Rightarrow (iii) \Rightarrow (iv) are trivial.

(iv) \Rightarrow (v) follows from Proposition 5.2.

(v) \Rightarrow (i) is proved in [10, Lemma 3.2]. Alternatively, (v) \Rightarrow (iv) follows from Proposition 5.2, and (iv) \Rightarrow (i) follows from Corollary 4.5.

(vi) \Leftrightarrow (ii) is Proposition 5.5.

Notice that Theorem 3.2 gives another proof for (i) \Rightarrow (iv). \dashv

The equivalence of (i) and (v) is due to [10, Corollary 3.6]. The equivalence of (i) and (ii) is already known, but we are not in the position to give explicit credits.

§6. Some problems. Let us recall the results we are trying to generalize.

THEOREM 6.1. [7] *If every subalgebra of $\mathbf{A} \times \mathbf{A}$ is congruence modular then \mathbf{A} is Arguesian.*

THEOREM 6.2. [5] *A congruence modular variety satisfies all the higher Arguesian identities introduced by Haiman.*

PROBLEM 6.3. Suppose that every subalgebra of $\mathbf{A} \times \mathbf{A}$ is congruence modular. Does \mathbf{A} satisfy Haiman's higher Arguesian identities?

PROBLEM 6.4. Does wTIP imply the Arguesian identity? Does wTIP imply the higher Arguesian identities?

In view of Theorem 3.2, affirmative answers would furnish another proof of Theorem 6.1 and, respectively, an affirmative answer to Problem 6.3.

PROBLEM 6.5. [19] Suppose that every subalgebra of $\mathbf{A} \times \mathbf{A}$ is congruence modular. Does \mathbf{A} satisfy TIP?

An affirmative answer to Problem 6.5 would give an affirmative answer to Problem 6.3 since [5] shows that TIP implies all the higher Arguesian identities.

Another route to the solution of Problem 6.3 (to another proof of Theorem 6.1, respectively) would be to show that if $\mathbf{A} \times \mathbf{A}$ is congruence modular then \mathbf{A} satisfies some property stronger than wTIP, which is nevertheless strong enough in order to show that \mathbf{A} satisfies Haiman's higher Arguesian identities (the Arguesian identity, respectively).

In this connection, we have found a proof for the following version of Theorem 3.2.

THEOREM 6.6. [19] *If \mathbf{A} is an algebra and every subalgebra of \mathbf{A}^4 is congruence modular then \mathbf{A} satisfies*

$$(\text{wTIP}_2) \quad \Gamma^* \cap \Theta^* = (\Gamma \cap (\Theta \circ \Theta))^*$$

for all tolerances Γ, Θ of \mathbf{A} .

See [19] for a proof of Theorem 6.6, as well as for further problems and comments.

PROBLEM 6.7. Can the Arguesian identity (the higher Arguesian identities, respectively) be written as $p \leq q$, for some wdc-term p ?

If Problem 6.7 has an affirmative answer, then it is conceivable that Problem 6.4 has an affirmative answer, in view of Corollary 4.4. Apparently, the methods of Section 4 should work, unless the equality $p \leq q$ in Problem 6.7 has a surprisingly strange and unexpected form.

PROBLEM 6.8. Even in case Problems 6.3 and 6.5 have a negative answer, we can ask whether the conclusions hold under the stronger assumption that every subalgebra of \mathbf{A}^n is congruence modular (for some specific $n > 2$).

PROBLEM 6.9. Find (if it exists) an example of an algebra satisfying wTIP but not TIP.

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