Problems about the Jónsson distributivity spectrum

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To the memory of Bjarni Jónsson

Abstract. It follows from results by B. Jónsson that if a variety $\mathcal{V}$ is congruence distributive, then, for every m, there is some k such that the congruence inclusion

$$\alpha(\beta \circ \gamma \circ \beta \dots) \subseteq \alpha\beta \circ \alpha\gamma \circ \alpha\beta \dots \tag{1}$$

holds in $\mathcal{V}$, with m occurrences of $\circ$ on the left and k occurrences of $\circ$ on the right.

Let $J_{\mathcal{V}}(m)$ denote the smallest value of k for which the above formula holds in $\mathcal{V}$. We study the problem of which functions can be represented as $J_{\mathcal{V}}$, for $\mathcal{V}$ a congruence distributive variety. The set of such functions is closed under pointwise maximum. Moreover, we show that the value of $J_{\mathcal{V}}(m)$ puts some restrictive bounds on the values of $J_{\mathcal{V}}(m')$, for $m' > m$.

Recall that an *algebra*, short for *algebraic structure* or *algebraic system*, is a set endowed with a certain number of operations[1].

A *variety* $\mathcal{V}$ is a class of algebras of the same kind which is closed under taking products, substructures and homomorphic images; equivalently, by the celebrated Birkhoff Theorem, a class which can be defined by equations. While the general study of algebras *per se* leads to a field so ample that significant results are hard to find (exceptions exist!), on the other hand many unexpected and deep results hold for varieties. For example, the assumption that every congruence lattice[2] of algebras in a variety $\mathcal{V}$ satisfies a certain property often leads to tight characterizations of the structure of algebras in $\mathcal{V}$.[3] See, e. g., [FM, G, HM, KK, L1].

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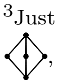
[1]In the present context, operations are assumed to be total and finitary, but no bound on arities is imposed.

[2]Recall that a *congruence* on an algebra is a binary relation which is the kernel of some homomorphism and that the set of congruences on some algebra has naturally a lattice structure.

[3]Just to present an elementary example, if the five element modular lattice $\mathbf{M}_3$, drawn as , is a sublattice (with minimum and maximum preserved) of the lattice of normal subgroups of a group $\mathbf{G}$, then $\mathbf{G}$ is abelian. This is related to congruences, since, for groups, normal subgroups are in a one-to-one correspondence with congruences. For general algebras a similar result might fail [W], but it holds in a quite broad context, see [L1]. For example, the result holds in any variety (such as the variety of groups!) in which every algebra has modular lattice of congruences. Of course, to be precise, one should have given

More generally, many theorems in universal algebra have the form "if every algebra in the variety $\mathcal{V}$ satisfies a property P, then every algebra in $\mathcal{V}$ satisfies some other property Q", where the theorem is nontrivial in the sense that there are single algebras satisfying P but not satisfying Q. For example, if every algebra in $\mathcal{V}$ has a modular lattice of congruences, then the congruence lattice of every algebra in $\mathcal{V}$ is even arguesian, where the arguesian identity is a lattice identity which can be given a clear geometrical meaning. Results of the above kind have led to the so-called field of *congruence identities*. See [CV] for an introduction and references. Recent results and further references can be found in [CHL].

In a classical paper, Jónsson [J] provided a characterization of those varieties in which every algebra has a distributive congruence lattice (from now on, for short, *congruence distributive varieties*). Jónsson result can be given an interpretation in the above sense of congruence identities. In this interpretation, Jónsson theorem states that a variety $\mathcal{V}$ is congruence distributive if and only if there is some n such that the congruence inclusion

$$\alpha(\beta \circ \gamma) \subseteq \alpha\beta \circ_n \alpha\gamma \tag{2}$$

holds in every algebra in $\mathcal{V}$. In the above inclusion, α, β, ... are intended to vary among congruences of some algebra $\mathbf{A} \in \mathcal{V}$, juxtaposition denotes intersection, $\circ$ is relational composition and $\beta \circ_m \gamma$ denotes $\beta \circ \gamma \circ \beta \dots$ with m factors, that is, with $m-1$ occurrences of $\circ$.

In general, it is trivial to show that an algebra $\mathbf{A}$ is congruence distributive if and only if, for every natural number $m \geq 2$, the congruence inclusion $\alpha(\beta \circ_m \gamma) \subseteq \alpha\beta + \alpha\gamma$ holds for all congruences of $\mathbf{A}$, where $+$ denotes join in the congruence lattice. In other words, for varieties, for every m, the congruence inclusion (2) implies the inclusion $\alpha(\beta \circ_m \gamma) \subseteq \alpha\beta + \alpha\gamma$. Recall that $\alpha + \beta = \bigcup_{i \in \mathbb{N}} \alpha \circ_i \beta$.

In fact, from the proofs in [J] it follows that, for every m, there is some k (which depends only on m and on the n given by (2), but otherwise not on the variety) such that

$$\alpha(\beta \circ_m \gamma) \subseteq \alpha\beta \circ_k \alpha\gamma \tag{(m,k)-dist}$$

One of the main problems we address here consists in finding the best possible value of k. More generally, for every positive natural number m and every variety $\mathcal{V}$, let $J_{\mathcal{V}}(m)$ be the least k such that $\mathcal{V}$ satisfies the inclusion $(m+1, k+1)$-dist (notice the shift by 1. This will simplify statements). Then we might ask the following problem.

The Jónsson distributivity spectrum problem. *Which functions (with domain the set of positive natural numbers) can be realized as $J_{\mathcal{V}}$, for some congruence distributive variety $\mathcal{V}$?*

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the definition of what an *abelian* algebra is in general. This is possible, we refer to the references for definitions.

Considering some examples, $J_{\mathcal{V}}$ is the identity function when $\mathcal{V}$ is the variety of lattices. In the variety of *n-Boolean algebras* (see, e. g., [CV, Example 2.8]) we have $J_{\mathcal{V}}(m) = \min\{m, n\}$. In the variety of *implication algebras* [CV, Example 2.6] $J_{\mathcal{V}}(m) = 2$ constantly. For every n, varieties can be constructed such that $J_{\mathcal{V}}(m)$ is constantly n.

The set of those functions which can be represented as $J_{\mathcal{V}}$, for some congruence distributive variety $\mathcal{V}$, is closed under pointwise maximum. Hence we can combine the above examples in order to get more functions representable as $J_{\mathcal{V}}$. See [L2] for more details.

The above examples suggest that $J_{\mathcal{V}}(m)$ has little influence on the values of $J_{\mathcal{V}}(m')$, for $m' < m$, provided that the obvious monotonicity property is respected. On the other hand, the following theorem shows that $J_{\mathcal{V}}(m)$ puts some quite restrictive bounds on $J_{\mathcal{V}}(m')$, for $m' > m$.

Theorem. [L2] *Suppose that $\mathcal{V}$ is a congruence distributive variety.*

If $J_{\mathcal{V}}(m) = k$, then $J_{\mathcal{V}}(m\ell) \leq k\ell$, for every natural number ℓ.

If $J_{\mathcal{V}}(1) = 2$, that is, $\mathcal{V}$ is 3-distributive, then $J_{\mathcal{V}}(m) \leq m$, for every $m \geq 3$.

If $\mathcal{V}$ is m-modular, that is, congruence modularity of $\mathcal{V}$ is witnessed by $m+1$ Day terms, then $J_{\mathcal{V}}(2) \leq J_{\mathcal{V}}(1) + 2m^2 - 2m - 1$.

We do not know whether the above theorem gives the best possible evaluations. In particular, the Jónsson distributivity spectrum problem is not yet completely solved.

The above considerations, however, are only the tip of an iceberg. If one tries to develop similar definitions and theorems about congruence modular varieties, unexpected difficulties arise. Again expressing everything in terms of congruence inclusions (this is not the original formulation), a fundamental theorem by A. Day [D] implies that a variety $\mathcal{V}$ is congruence modular if and only there is some k such that the congruence inclusion

$$\alpha(\beta \circ \alpha\gamma \circ \beta) \subseteq \alpha\beta \circ_k \alpha\gamma \qquad (\mathrm{D}_k)$$

holds in $\mathcal{V}$.

For a congruence modular variety $\mathcal{V}$, we can define the *Day modularity function $D_{\mathcal{V}}$* as follows. For $m \geq 3$, $D_{\mathcal{V}}(m)$ is the least k such that $\alpha(\beta \circ_m \alpha\gamma) \subseteq \alpha\beta \circ_k \alpha\gamma$ holds in $\mathcal{V}$. The arguments from [D] show that $D_{\mathcal{V}}(m)$ is defined for every m and every congruence modular variety $\mathcal{V}$, but the methods from [D] do not furnish the best value. See [L3].

However, the case of congruence modularity is substantially more involved than the distributivity case treated at the beginning. Using results from [G], Tschantz [T] showed that a variety $\mathcal{V}$ is congruence modular if and only if the congruence inclusion $\alpha(\beta + \gamma) \subseteq \alpha(\gamma \circ \beta) \circ (\alpha\gamma + \alpha\beta)$ holds in $\mathcal{V}$. Hence it is also natural to introduce the *Tschantz modularity function $T_{\mathcal{V}}$* for a congruence modular variety $\mathcal{V}$ in such a way that, for $m \geq 2$, $T_{\mathcal{V}}(m)$ is the least k such that the following congruence inclusion holds in $\mathcal{V}$

$$\alpha(\beta \circ_m \gamma) \subseteq \alpha(\gamma \circ \beta) \circ (\alpha\gamma \circ_k \alpha\beta)$$

The relationships between $D_{\mathcal{V}}$ and $T_{\mathcal{V}}$ appear rather involved. Already the problem of the "minimal" case of the relationships between $D_{\mathcal{V}}(3)$ and $T_{\mathcal{V}}(2)$ seems still open, in general. See [LTT].

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