COMPACT FACTORS IN FINALLY COMPACT PRODUCTS OF TOPOLOGICAL SPACES

Paolo Lipparini

We present instances of the following phenomenon: if a product of topological spaces satisfies some given compactness property then the factors satisfy a stronger compactness property, except possibly for a small number of factors.

The first known result of this kind, a consequence of a theorem by A. H. Stone, asserts that if a product is regular and Lindelöf then all but at most countably many factors are compact. We generalize this result to various forms of final compactness, and extend it to two-cardinal compactness. In addition, our results need no separation axiom.

1. Introduction

By Tychonoff Theorem, any product of compact topological spaces is compact. The converse is trivial: if a product of topological spaces is compact then all factors are compact.

The situation changes when weaker forms of compactness are taken into account. In order to present an example, recall that a topological space is said to be *Lindelöf* if and only if every open cover has a countable subcover. A product of Lindelöf spaces is not necessarily Lindelöf; actually, the square of a Lindelöf space need not be Lindelöf (see [**Go**]).

For the converse, it is trivial that if a product of topological spaces is Lindelöf then each factor is Lindelöf. What is relevant to the present paper is that if a product is Lindelöf then we can say much more about the factors: the following theorem is an immediate consequence of a classical result by A. H. Stone (see Subsection 1.1 for some history).

Theorem 1.1. *If a product of topological spaces is Lindelöf then all but at most a countable number of factors are compact.*

The classical argument seems to require some separation axiom: a minor contribution of the present paper is to provide a proof which uses no separation axiom.

PUT ON THE AUTHOR'S WEB PAGE ON SEPTEMBER 26, 2004

More importantly, we extend Theorem 1.1 to final κ-compactness. If κ is an infinite cardinal, then a topological space is said to be *finally κ-compact* if and only if every open cover has a subcover by $< \kappa$ sets. Thus, Lindelöf is the same as finally $\aleph_1$-compact. When expressed in terms of final κ-compactness, our main result reads:

Theorem 1.2. *If a product of topological spaces is finally $\aleph_{n+1}$-compact, then all but at most $\aleph_n$ factors are compact.*

Moreover, we generalize Theorem 1.1 to linearly Lindelöf spaces: a topological space is *linearly Lindelöf* if and only if every open cover which is linearly ordered by inclusion has a countable subcover (some authors use the term *chain-Lindelöf*). See [**AB**], [**K**] and [**KL**] for further information and references about linearly Lindelöf spaces. It is well-known that a space is linearly Lindelöf if and only if every uncountable subset of regular cardinality has a complete accumulation point.

There are examples of linearly Lindelöf not Lindelöf topological spaces, thus the next theorem is a proper generalization of Theorem 1.1.

Theorem 1.3. *If a product of topological spaces is linearly Lindelöf, then all but at most countably many factors are compact.*

In fact, a simultaneous generalization of Theorems 1.2 and 1.3 holds: see Theorem 7.4.

Theorems 1.2 and 1.3 have immediate consequences for powers.

Corollary 1.4. *If the $\aleph_{n+1}$-th power of the topological space X is finally $\aleph_{n+1}$-compact then X is compact.*

If the $\aleph_1$-th power of the topological space X is linearly Lindelöf then X is compact.

We have also a version of Theorem 1.2 for larger cardinals. A topological space is *countably compact* if and only if every countable open cover has a finite subcover.

Theorem 1.5. *If a product of topological spaces is finally $\aleph_\omega$-compact, then either*

(a) all factors are countably compact, or

(b) all factors are compact, except possibly for a set having cardinality less than $\aleph_\omega$.

Actually, our results are even stronger, when expressed in terms of finer notions of compactness. A topological space is said to be *initially κ-compact* if and only if every open cover by at most κ sets has a finite subcover.

If κ, λ are infinite cardinals, a topological space is said to be *$[\kappa, \lambda]$-compact* if and only if every open cover by at most λ sets has a subcover by less than κ sets.

With the above terminology, we have:

Theorem 1.6. *If a product of topological spaces is* $[\aleph_{n+1}, \aleph_{n+1}]$*-compact, then all but at most* $\aleph_n$ *factors are initially* $\aleph_{n+1}$*-compact.*

Theorem 1.7. *If a product of topological spaces is* $[\aleph_\omega, \aleph_\omega]$*-compact, then either*

(a) all factors are countably compact, or

(b) all factors are initially $\aleph_\omega$*-compact, except possibly for a set of cardinality less than* $\aleph_\omega$*.*

Notice that the notion of $[\kappa, \lambda]$-compactness encompasses both the notion of final κ-compactness and the notion of initial κ-compactness. Indeed, final κ-compactness is the same as $[\kappa, \lambda]$-compactness for all λ, and initial κ-compactness is the same as $[\omega, \kappa]$-compactness. Moreover, it appears that $[\kappa, \lambda]$-compactness is a particularly nice way of "splitting compactness into pieces": see Section 3.

The results we have stated in this section will be proved in Section 7. Actually, some more general versions will be given there.

In detail, the paper is divided as follows. After Section 2, devoted to preliminaries, in Section 3 we recall some basic properties of $[\kappa, \lambda]$-compactness. Section 4 contains the construction of a matrix very similar to the classical Ulam matrix, as well as a further construction we shall need. In Section 5 we deal with $[\lambda^+, \lambda^+]$-compact products in the case when λ is a regular cardinal, while in Section 6 we treat $[\lambda, \lambda]$-compact products in the case when λ is singular. We sum up our results in Section 7. In Section 8 we add some further remarks, and state some problems.

The results and proofs in the first part of Section 6 do not depend on Sections 4 and 5, while the proofs of Theorems 1.2 and 1.3 do not rely on Section 6.

1.1. A short historical note. A. H. Stone [**Sto**] showed that the product of uncountably many copies of N (the space of the natural numbers with the discrete topology) is not normal. As a corollary, Stone obtained that if a product of T_1 spaces is normal then all but a countable number of factors are countably compact.

Since T_3 Lindelöf spaces are normal, and since countably compact Lindelöf spaces are compact (see [**Go**]), one immediately gets that if a product of T_3 spaces is Lindelöf, then all but a countable number of factors are compact (see [**V1**]).

Apparently, the above arguments need separation axioms in an essential way.

Apparently, Theorem 1.3 cannot be obtained by the above arguments, since there are examples of linearly Lindelöf T_3 spaces which are not Lindelöf (see Section 4 of [**AB**]). In passing, let us mention that it is not known whether there exists a *normal* linearly Lindelöf not Lindelöf topological space.

Recently, X. Caicedo, using deep logical and set theoretical methods, proved results similar to Theorem 1.1 with no need of separation axioms, but only for products with arbitrarily large numbers of factors. For example, Caicedo proved that if *all* powers of a space X are Lindelöf then X is compact. More generally, he proved that if all powers of X are $[\lambda^+, \lambda^+]$-compact then X is $[\lambda, \lambda]$-compact (cf. Theorem 5.2), and similar results are obtained for families of topological spaces. The above results are explicitly stated in [**C1**], and follow easily from the results proved in [**C2**].

In [**L1**] we showed that the use of technical set-theoretical tools (in particular, regularity properties of ultrafilters) is essential in [**C1**] and [**C2**], and that the methods of [**C1**] and [**C2**] lead to set-theoretical assumptions which go beyond the commonly accepted axioms for set theory.

The methods of the present paper not only provide generalizations and strengthenings of the above mentioned results, but have the advantage of elementary proofs which need no special set-theoretical tool. We only rely on some combinatorial properties of certain matrices of sets introduced by S. Ulam already in the 30's [**U**]. In Lemma 4.2 we construct a new matrix from a version of Ulam's one.

2. Preliminaries

Our notation is fairly standard.

Space is always used as an abbreviation for *topological space*. No separation axiom is needed to prove the results of the present paper. In particular, *Lindelöf* means *exactly* that every open cover has a countable subcover (some authors incorporate some separation axiom directly in the definition of Lindelöfness).

A *product* of topological spaces is always endowed with the *Tychonoff topology*, the smallest topology under which the canonical projections are continue maps. The λ-*th power* of a topological space X is the product $\prod_{\alpha \in \lambda} X_\alpha$, where $X_\alpha = X$ for all $\alpha \in \lambda$.

$\alpha, \beta, \gamma \ldots$ denote ordinals. We assume throughout the Axiom of Choice, hence any set X is equinumerous with some ordinal. The smallest such ordinal is the *cardinality* of X, and is denoted by $|X|$. A cardinal is identified with the set of smaller ordinals (hence, for example, $\alpha \in \lambda$ and $\alpha < \lambda$ have exactly the same meaning).

Infinite cardinals are denoted by $\lambda, \mu, \nu, \kappa \ldots$ The smallest infinite cardinal is denoted by ω . When convenient, we denote infinite cardinals using the $\aleph$ notation: $\aleph_0 = \omega$ is the smallest infinite cardinal, $\aleph_1$ is the smallest cardinal larger than $\aleph_0$, and so on. $\aleph_\omega$ is the smallest cardinal larger than all $\aleph_n$'s (n a natural number). *Countable* means finite or denumerable, that is, having cardinality $\leq \aleph_0$.

The smallest cardinal larger than λ is called the *successor* of λ, and is denoted λ^+. Thus, if $\lambda = \aleph_\alpha$, then $\lambda^+ = \aleph_{\alpha+1}$. $\aleph_{\alpha+\omega}$ is the smallest cardinal larger than all $\aleph_{\alpha+n}$'s (n a natural number).

A cardinal λ is *singular* if and only if λ can be obtained as a union $\lambda = \bigcup_{i \in I} \lambda_i$ for some set I with $|I| < \lambda$ and where $\lambda_i < \lambda$, for all $i \in I$. The smallest cardinality of an I as above is called the *cofinality* of λ, and is denoted $\mathrm{cf}\lambda$. Thus, if λ is a singular cardinal, then $\lambda = \sup_{\alpha \in \mathrm{cf}\lambda} \lambda_\alpha$, for an appropriate choice of the λ_α's, with $\lambda_\alpha < \lambda$, for $\alpha \in \mathrm{cf}\lambda$.

A cardinal λ is *regular* if and only if it is not singular. The cofinality of a singular cardinal is always a regular cardinal. All finite cardinals and all successor cardinals are regular.

$\subseteq$ denotes inclusion, and $\subset$ denotes strict inclusion. The minus operation between sets is denoted by $\setminus$: $X \setminus Y = \{x \in X | x \notin Y\}$.

3. Properties of $[\kappa, \lambda]$-compactness

In this section we list the properties of $[\kappa, \lambda]$-compactness needed in the present paper. Most of the results in this section are due, in some form or another, to [**AU**]. We present proofs for sake of completeness.

$[\kappa, \lambda]$-compactness has been studied (in various forms and with varying terminology and notations) by many authors. See, e.g., [**Sm**], [**Ga**], [**V1**], [**V2**], [**Ste**], [**C2**], [**L1**] for further results, further references, and historical notes.

The next proposition shows that $[\kappa, \lambda]$-compactness could have been defined in terms of $[\mu, \mu]$-compactness alone, that is, we can split $[\kappa, \lambda]$-compactness into instances of $[\mu, \mu]$-compactness. If we want to show that a space is $[\kappa, \lambda]$-compact, it is enough to show that it is $[\mu, \mu]$-compact for every cardinal μ with $\kappa \leq \mu \leq \lambda$.

Proposition 3.1. *For every pair of infinite cardinals κ, λ, and every topological space X, the following are equivalent:*

(i) X is $[\kappa, \lambda]$-compact;

(ii) X is $[\mu, \mu]$-compact for every cardinal μ with $\kappa \leq \mu \leq \lambda$.

Proof. (i) $\Rightarrow$ (ii) is trivial.

Suppose that (ii) holds. We show by transfinite induction on ν, with $\kappa \leq \nu \leq \lambda$, that X is $[\kappa, \nu]$-compact.

By taking $\mu = \kappa$ in (ii) we get that X is $[\kappa, \kappa]$-compact, that is, the induction basis $\nu = \kappa$.

For the induction step, suppose that $\nu < \lambda$, and that X is $[\kappa, \nu']$-compact, for all $\nu' < \nu$: we have to show that X is $[\kappa, \nu]$-compact. Let $\mathcal{O}$ be an open covering of X with $|\mathcal{O}| \leq \nu$. By taking $\mu = \nu$ in (ii), X is $[\nu, \nu]$-compact, hence $\mathcal{O}$ has a subcover $\mathcal{O}'$ with $|\mathcal{O}'| < \nu$. If $|\mathcal{O}'| < \kappa$, then there is nothing to prove. If $|\mathcal{O}'| \geq \kappa$, let $\nu' = |\mathcal{O}'|$: by $[\kappa, \nu']$-compactness $\mathcal{O}'$ has a subcover $\mathcal{O}''$ with $|\mathcal{O}''| < \kappa$.

Thus every open covering of X of cardinality at most ν has a subcover of cardinality $< \kappa$, that is X is $[\kappa, \nu]$-compact. This completes the induction step, and the proposition is proved. □

We shall make good use of Proposition 3.1 at several points in the present paper. For example, though the statement of Theorem 1.2 mentions final $\aleph_{n+1}$-compactness only, we know just one reasonable way to prove it, that is, by splitting finally $\aleph_{n+1}$-compactness into pieces of $[\lambda, \lambda]$-compactness, as given by Proposition 3.1: see Corollary 3.3(i). Cf. also the proofs of Theorems 7.1 and 7.5.

Proposition 3.2. *If κ is a singular cardinal, and a topological space X is $[\mathrm{cf}\kappa, \mathrm{cf}\kappa]$-compact then X is $[\kappa, \kappa]$-compact.*

Proof. Let $(U_\alpha)_{\alpha \in \kappa}$ be an open cover of X. Let $(\kappa_\beta)_{\beta \in \mathrm{cf}\kappa}$ be a sequence such that $\sup_{\beta \in \mathrm{cf}\kappa} \kappa_\beta = \kappa$, and $\kappa_\beta < \kappa$ for $\beta \in \mathrm{cf}\kappa$.

For $\beta \in \mathrm{cf}\kappa$, define $V_\beta = \bigcup_{\alpha < \kappa_\beta} U_\alpha$. $(V_\beta)_{\beta \in \mathrm{cf}\kappa}$ is an open cover of X by $\mathrm{cf}\kappa$-many sets, hence there is $I \subseteq \mathrm{cf}\kappa$ such that $|I| < \mathrm{cf}\kappa$, and $(V_i)_{i \in I}$ is a cover of X.

Since $\mathrm{cf}\kappa$ is a regular cardinal, there is $\gamma < \mathrm{cf}\kappa$ such that $\sup I < \gamma$. Hence, $(V_\beta)_{\beta < \gamma}$ is a cover of X. By the definition of the V_β's then $(U_\alpha)_{\alpha < \kappa_\gamma}$ is a cover of X by less than κ sets. □

Corollary 3.3. *(i) A topological space is finally κ-compact if and only if it is $[\mu, \mu]$-compact for all $\mu \geq \kappa$.*

(ii) A topological space is initially κ-compact if and only if it is $[\mu, \mu]$-compact for all infinite $\mu \leq \kappa$, if and only if it is $[\mu, \mu]$-compact for all infinite regular $\mu \leq \kappa$.

(iii) A topological space is compact if and only if it is $[\mu, \mu]$-compact for all infinite μ, if and only if it is $[\mu, \mu]$-compact for all regular infinite μ.

Proof. Immediate from Propositions 3.1 and 3.2. □

In the particular case when κ is a regular cardinal, there are many interesting and useful characterizations of $[\kappa, \kappa]$-compactness. We list below some of them.

Recall that if Y is an infinite subset of the topological space X and $x \in X$ then x is said to be a *complete accumulation point* of Y (in X) if and only if $|U \cap Y| = |Y|$, for every neighbourhood U (in X) of x.

Proposition 3.4. *For every infinite regular cardinal κ and every topological space X, the following are equivalent.*

(i) X is $[\kappa, \kappa]$-compact.

(ii) Whenever $(U_\alpha)_{\alpha < \kappa}$ is a sequence of open sets of X, such that $U_\alpha \subseteq U_{\alpha'}$ for every $\alpha < \alpha'$, and such that $\bigcup_{\alpha < \kappa} U_\alpha = X$, then there is an $\alpha < \kappa$ such that $U_\alpha = X$.

(iii) Whenever $(C_\alpha)_{\alpha<\kappa}$ is a sequence of closed sets of X, such that $C_\alpha \supseteq C_{\alpha'}$ for every $\alpha < \alpha'$, and such that $\bigcap_{\alpha<\kappa} C_\alpha = \emptyset$, then there is an $\alpha < \kappa$ such that $C_\alpha = \emptyset$.

(iv) For every sequence $(x_\alpha)_{\alpha<\kappa}$ of elements of X, there exists $x \in X$ such that $|\{\alpha < \kappa | x_\alpha \in U\}| = \kappa$ for every neighbourhood U of x.

(v) (CAP_κ) Every subset $Y \subseteq X$ with $|Y| = \kappa$ has a complete accumulation point.

Proof. (i) $\Rightarrow$ (ii) follows from the assumption that κ is regular.

(ii) $\Rightarrow$ (i) is similar to the proof of Proposition 3.2. If $(U_\alpha)_{\alpha\in\kappa}$ is an open cover of X, and $\beta \in \kappa$ define $V_\beta = \bigcup_{\alpha<\beta} U_\alpha$. Since $(U_\alpha)_{\alpha\in\kappa}$ is a cover of X, then $(V_\beta)_{\beta\in\kappa}$ is a cover of X and, moreover, $V_\beta \subseteq V_{\beta'}$ for $\beta \leq \beta'$. By (ii), there is $\beta \in \kappa$ such that $V_\beta = X$. Since $V_\beta = \bigcup_{\alpha<\beta} U_\alpha$, we get that $(U_\alpha)_{\alpha<\beta}$ is a subcover of $(U_\alpha)_{\alpha\in\kappa}$ of cardinality $< \kappa$.

The equivalence of (ii) and (iii) is immediate, by taking complements.

(iii) $\Rightarrow$ (iv). Let $(x_\alpha)_{\alpha<\kappa}$ be a sequence of elements of X.

For $\beta < \kappa$, define C_β to be the closure of the set $\{x_\alpha | \alpha > \beta\}$. $C_\beta \supseteq C_{\beta'}$ for every $\beta < \beta'$, and $C_\beta \neq \emptyset$ for every $\beta < \kappa$, since $x_{\beta+1} \in C_\beta$.

Hence, by (iii), $\bigcap_{\beta<\kappa} C_\beta \neq \emptyset$, say $x \in \bigcap_{\beta<\kappa} C_\beta$.

We claim that x satisfies the property stated in (iv). If not, there is an open set U containing x and such that $|\{\alpha < \kappa | x_\alpha \in U\}| < \kappa$. Let $A = \{\alpha < \kappa | x_\alpha \in U\}$. Since κ is regular, $\sup A < \kappa$; hence $x_\alpha \notin U$, for every $\alpha > \sup A$.

But this contradicts $x \in C_{\sup A}$, since $C_{\sup A}$ is the closure of $\{x_\alpha | \alpha > \sup A\}$.

(iv) $\Rightarrow$ (iii). Suppose that (iv) holds, and suppose by contradiction that $(C_\alpha)_{\alpha<\kappa}$ is a sequence of closed sets of X, such that $C_\alpha \supseteq C_{\alpha'}$ for every $\alpha < \alpha'$, $\bigcap_{\alpha<\kappa} C_\alpha = \emptyset$, but $C_\alpha \neq \emptyset$, for every $\alpha < \kappa$.

For every $\alpha < \kappa$, choose $x_\alpha \in C_\alpha$. By (iv), there exists $x \in X$ such that $|\{\alpha < \kappa | x_\alpha \in U\}| = \kappa$ for every neighbourhood U of x. Thus, for every neighbourhood U of x and for every $\alpha < \kappa$ there is $\alpha' > \alpha$ such that $x_{\alpha'} \in U$.

Since $C_\alpha \supseteq C_{\alpha'}$ for every $\alpha < \alpha'$, every neighbourhood U of x intersects every C_α, that is, x belongs to every C_α, since they are closed sets. Thus, $x \in \bigcap_{\alpha<\kappa} C_\alpha$, a contradiction.

(iv) $\Rightarrow$ (v) is trivial: just arrange the elements of Y into a sequence of length κ.

Conversely, suppose that (v) holds, and that $(x_\alpha)_{\alpha<\kappa}$ is a sequence of elements of X.

If there exists $\beta < \kappa$ such that $|\{\alpha < \kappa | x_\alpha = x_\beta\}| = \kappa$, then $x = x_\beta$ satisfies the conclusion of (iv) (with no use of (v)).

Otherwise, for every $\beta < \kappa$, $|\{\alpha < \kappa | x_\alpha = x_\beta\}| < \kappa$. Hence, the *set* $Y = \{x_\alpha | \alpha < \kappa\}$ has cardinality κ, since κ is a regular cardinal.

By applying (v) to Y, one easily gets (iv). □

Of course, there is a more general version of Proposition 3.4 which deals with $[\kappa,\lambda]$-compactness: just combine Proposition 3.4 and Proposition 3.1; however, here we shall not need the more general version.

The assumption that κ is regular is necessary in Proposition 3.4: see [**V2**] for various counterexamples.

Using the methods in the proofs of Propositions 3.4 and 3.2 one can easily show that a space X is linearly Lindelöf if and only if X is $[\kappa,\kappa]$-compact for every regular infinite cardinal $\kappa > \omega$, if and only if every subset of X of regular uncountable cardinality has a complete accumulation point.

The following is trivial (but useful!).

Proposition 3.5. *If $f : X \to Y$ is surjective and continuous, and X is $[\kappa,\lambda]$-compact, then Y is $[\kappa,\lambda]$-compact, too.*

In particular, if $\prod_{i\in I} X_i$ is $[\kappa,\lambda]$-compact, and $J \subseteq I$, then $\prod_{i\in J} X_i$ is $[\kappa,\lambda]$-compact.

In particular, all factors of a $[\kappa,\lambda]$-compact product are themselves $[\kappa,\lambda]$-compact.

4. Two Ulam-like matrices

The next Lemma is a variation on a classical result by S. Ulam, as employed by K. Prikri, and G.V. Čhudnovskiĭ and D. V. Čhudnovskiĭ: see Lemmata 8.33 and 8.34 of [**CN**]. We give the proof for the reader's convenience. See [**EU**] and [**CN**] for historical notes.

Lemma 4.1. *For every infinite cardinal λ there is a family $(A_{\alpha,\beta})_{\alpha<\lambda,\beta<\lambda^+}$ of subsets of λ^+ such that:*

(i) For every $\beta < \lambda^+$, $|\lambda^+ \setminus \bigcup_{\alpha<\lambda} A_{\alpha,\beta}| \leq \lambda$;

(ii) For every $\beta < \lambda^+$ and $\alpha \leq \alpha' < \lambda$, $A_{\alpha,\beta} \subseteq A_{\alpha',\beta}$;

(iii) Whenever $\alpha < \lambda$ and $C \subseteq \lambda^+$ is such that $|C| > |\alpha|$ then $\bigcap_{\beta\in C} A_{\alpha,\beta} = \emptyset$.

Proof. For every $\gamma < \lambda^+$, $|\gamma| \leq \lambda$, hence we can choose an injective function $\phi_\gamma : \gamma \to \lambda$.

Define $A_{\alpha,\beta} = \{\gamma < \lambda^+ | \beta < \gamma \text{ and } \phi_\gamma(\beta) < \alpha\}$.

(i) is easy, since $\lambda^+ \setminus \bigcup_{\alpha<\lambda} A_{\alpha,\beta} \subseteq \beta \cup \{\beta\}$

(ii) is trivial.

Let α, C be as in the hypothesis of (iii). Suppose by contradiction that there is $\gamma \in \bigcap_{\beta\in C} A_{\alpha,\beta}$: then, by the definition of $A_{\alpha,\beta}$, $\phi_\gamma(\beta) < \alpha$, for every $\beta \in C$, thus ϕ_γ, restricted to C, would be injective from C to α, and this contradicts $|C| > |\alpha|$. □

It is convenient to visualize $(A_{\alpha,\beta})_{\alpha<\lambda,\beta<\lambda^+}$ as an infinite matrix with λ rows and λ^+ columns: each column is an increasing sequence of subsets of λ^+

whose union is the whole of λ^+, except perhaps for a subset of cardinality λ; Condition (iii) in Lemma 4.1 asserts that if we take more than $|\alpha|$ elements from the α^{th} row, then their intersection is empty. Actually, in what follows, we shall need only the particular case $|C| = \lambda$ of Condition (iii) in Lemma 4.1.

From the matrix given by 4.1 we shall construct another matrix, the one which shall be used in order to obtain our results on compact factors in products. This matrix, too, has λ rows and λ^+ columns; it satisfies property (ii) of Lemma 4.1, and a property stronger than (i), but the main point is that property (iii) is changed to: for every possible choice of one element from each column, there is a pair of the chosen elements whose intersection has cardinality $\leq \lambda$. We know no reference for this consequence of Ulam's construction.

Lemma 4.2. *If λ is an infinite regular cardinal, there is a set H with $|H| = \lambda^+$, and there is a family $(B_{\alpha,h})_{\alpha<\lambda, h\in H}$ of subsets of λ^+ such that:*

(i) For every $h \in H$, $\bigcup_{\alpha<\lambda} B_{\alpha,h} = \lambda^+$;

(ii) For every $h \in H$ and $\alpha \leq \alpha' < \lambda$, $B_{\alpha,h} \subseteq B_{\alpha',h}$;

(iii) For every function $f : H \to \lambda$ there exists a subset $F \subseteq H$ with $|F| = 2$ and such that $|\bigcap_{h\in F} B_{f(h),h}| \leq \lambda$.

Proof. Let $H = \lambda^+ \cup \{(\gamma, \beta) | \gamma < \lambda, \beta < \lambda^+, |\beta| = \lambda\}$. Clearly, $|H| = \lambda^+$. For every $\beta < \lambda^+$ with $|\beta| = \lambda$, fix a bijection $\psi_\beta : \lambda \to \beta$.

Suppose that we have a family of matrices $(A_{\alpha,\beta})_{\alpha<\lambda,\beta<\lambda^+}$ as given by Lemma 4.1.

Let $\alpha < \lambda$. We now define $B_{\alpha,h}$ for $h \in H$. We need to consider the two cases $h \in \lambda^+$ and $h \notin \lambda^+$.

Suppose that $h \in H$, and $h \in \lambda^+$, thus $h = \beta$, for some $\beta < \lambda^+$; then let $B_{\alpha,h} = A_{\alpha,\beta} \cup \left(\lambda^+ \setminus \bigcup_{\gamma<\lambda} A_{\gamma,\beta}\right)$.

Suppose that $h \in H$ and $h \notin \lambda^+$, that is $h = (\gamma, \beta)$, for some $\gamma < \lambda, \beta < \lambda^+$, with $|\beta| = \lambda$. In this case, put $B_{\alpha,h} = \lambda^+ \setminus \bigcup_{\alpha<\varepsilon<\lambda} A_{\gamma,\psi_\beta(\varepsilon)}$.

Condition (i) trivially holds when $h \in \lambda^+$. Hence suppose $h \notin \lambda^+$, say $h = (\gamma, \beta)$. We want to show that $\bigcup_{\alpha<\lambda} B_{\alpha,h} = \lambda^+$, so let δ be any element of λ^+. We have to show that there is $\alpha < \lambda$ such that $\delta \in B_{\alpha,h}$. Consider the set $C = \{\varepsilon < \lambda | \delta \in A_{\gamma,\psi_\beta(\varepsilon)}\}$. Thus, $\delta \in \bigcap_{\varepsilon\in C} A_{\gamma,\psi_\beta(\varepsilon)}$, hence $\bigcap_{\varepsilon\in C} A_{\gamma,\psi_\beta(\varepsilon)} \neq \emptyset$. Since ψ_β is injective, Condition (iii) in Lemma 4.1 implies that $|C| \leq |\gamma|$, thus $|C| \leq |\gamma| < \lambda$. Choose α such that $\lambda > \alpha \geq \sup C$ (this is possible, since λ is supposed to be a regular cardinal, and since $|C| < \lambda$). By the very definition of C, for every $\varepsilon > \alpha$, $\delta \notin A_{\gamma,\psi_\beta(\varepsilon)}$, that is, $\delta \notin \bigcup_{\alpha<\varepsilon<\lambda} A_{\gamma,\psi_\beta(\varepsilon)}$, that is, $\delta \in B_{\alpha,h} = \lambda^+ \setminus \bigcup_{\alpha<\varepsilon<\lambda} A_{\gamma,\psi_\beta(\varepsilon)}$. We have showed that Condition (i) holds.

In the case $h \notin \lambda^+$, Condition (ii) is trivial. In the case $h \in \lambda^+$, Condition (ii) follows immediately from Condition (ii) in Lemma 4.1.

Let us now show that Condition (iii) holds, so let $f : H \to \lambda$. There is $\gamma < \lambda$ such that $|\{\beta < \lambda^+ | f(\beta) = \gamma\}| = \lambda^+$, since otherwise λ^+ would be the union of λ sets each of cardinality $\leq \lambda$. Choose such a γ, and choose $\beta < \lambda^+$ in such a way that $|\{\beta' < \beta | f(\beta') = \gamma\}| = \lambda$. Notice that necessarily $|\beta| = \lambda$. Consider $h = (\gamma, \beta)$, and choose some $\beta' < \beta$ such that $f(\beta') = \gamma$ and $\beta' \notin \{\psi_\beta(\varepsilon) | \varepsilon \leq f(h)\}$. Such a β' exists, since the latter set has cardinality $< \lambda$ (since $f(h) < \lambda$, hence $|f(h)| < \lambda$), while $|\{\beta' < \beta | f(\beta') = \gamma\}|$ has been chosen to have cardinality λ. Since ψ_β is surjective, $\beta' = \psi_\beta(\varepsilon)$, for some $\varepsilon > f(h)$.

We claim that $F = \{\beta', h\}$ is a subset of H which satisfies the conclusion of Condition (iii). Indeed,

$$B_{f(h),h} = \lambda^+ \setminus \bigcup_{f(h) < \varepsilon < \lambda} A_{\gamma, \psi_\beta(\varepsilon)} = \bigcap_{f(h) < \varepsilon < \lambda} (\lambda^+ \setminus A_{\gamma, \psi_\beta(\varepsilon)})$$

Since $\beta' = \psi_\beta(\varepsilon)$, for some $\varepsilon > f(h)$, we get $B_{f(h),h} \cap A_{\gamma,\beta'} = \emptyset$.

Since $B_{\gamma,\beta'} = A_{\gamma,\beta'} \cup \left(\lambda^+ \setminus \bigcup_{\alpha<\lambda} A_{\alpha,\beta'}\right)$, we get

$$B_{f(h),h} \cap B_{\gamma,\beta'} = B_{f(h),h} \cap \left(A_{\gamma,\beta'} \cup \left(\lambda^+ \setminus \bigcup_{\alpha<\lambda} A_{\alpha,\beta'} \right) \right) =$$

$$\left(B_{f(h),h} \cap A_{\gamma,\beta'}\right) \cup \left(B_{f(h),h} \cap \left(\lambda^+ \setminus \bigcup_{\alpha<\lambda} A_{\alpha,\beta'} \right) \right) =$$

$$\left(B_{f(h),h} \cap \left(\lambda^+ \setminus \bigcup_{\alpha<\lambda} A_{\alpha,\beta'} \right) \right) \subseteq \left(\lambda^+ \setminus \bigcup_{\alpha<\lambda} A_{\alpha,\beta'} \right)$$

By Condition (i) in Lemma 4.1, the last set in the above chain of inclusions has cardinality $\leq \lambda$, hence we have $|B_{f(h),h} \cap B_{\gamma,\beta'}| \leq \lambda$, which is the desired conclusion, since $\gamma = f(\beta')$. □

5. $[\lambda, \lambda]$-compact factors in $[\lambda^+, \lambda^+]$-compact products

Proposition 5.1. *Suppose that λ is an infinite regular cardinal. If $X = \prod_{j \in J} X_j$, $|J| = \lambda^+$, and no X_j is $[\lambda, \lambda]$-compact, then X is not $[\lambda^+, \lambda^+]$-compact.*

Proof. Let $X, (X_j)_{j \in J}$ be as in the statement of the proposition. Suppose that $(B_{\alpha,h})_{\alpha<\lambda, h \in H}$ is a set of matrices as given by Lemma 4.2. Since $|H| = |J|$, by fixing a bijection from I onto H, we can rearrange the indices in such a way that $X = \prod_{h \in H} X_h$.

Since no X_h is $[\lambda, \lambda]$-compact, and since λ is regular, by Condition (iv) in Proposition 3.4, for every $h \in H$ there is a sequence $\{x_{\alpha,h} | \alpha < \lambda\}$ such that every $x \in X_h$ has a neighbourhood U in X_h such that $|\{\alpha < \lambda | x_{\alpha,h} \in U\}| < \lambda$.

We shall define a sequence $(y_\beta)_{\beta<\lambda^+}$ of elements of X such that for every $y \in X$ there is a neighbourhood U in X of y such that $|\{\beta < \lambda^+ | y_\beta \in U\}| < \lambda^+$, thus X is not $[\lambda^+, \lambda^+]$-compact, again by Condition (iv) in Proposition 3.4, and since successor cardinals are always regular.

For $\beta < \lambda^+$, let $y_\beta = ((y_\beta)_h)_{h\in H} \in \prod_{h\in H} X_h$ be defined by: $(y_\beta)_h = x_{\alpha,h}$, where α is the first ordinal such that $\beta \in B_{\alpha,h}$ (such an ordinal exists by Condition (i) in Lemma 4.2).

Suppose by contradiction that there is $y \in X$ such that for every neighbourhood U in X of y $|\{\beta < \lambda^+ | y_\beta \in U\}| = \lambda^+$.

Consider the components $(y_h)_{h\in H}$ of $y \in X = \prod_{h\in H} X_h$. Because of the way we have chosen the $x_{\alpha,h}$'s, for each $h \in H$, y_h has a neighbourhood U_h in X_h such that $|\{\alpha | x_{\alpha,h} \in U_h\}| < \lambda$. For every $h \in H$, fix some U_h as above. For each $h \in H$, choose $f(h)$ in such a way that $\lambda > f(h) > \sup\{\alpha | x_{\alpha,h} \in U_h\}$ (this is possible since λ is regular, and $|\{\alpha | x_{\alpha,h} \in U_h\}| < \lambda$).

By Condition (iii) in Lemma 4.2, there is $F \subseteq H$ such that $|F| = 2$ and $|\bigcap_{h\in F} B_{f(h),h}| \leq \lambda$. Let $V = \prod_{h\in H} V_h$, where $V_h = X_h$ if $h \notin F$, and $V_h = U_h$ if $h \in F$. V is a neighbourhood of y in X, since F is finite.

For every $\beta < \lambda^+$ and $h \in H$, by definition, $(y_\beta)_h = x_{\alpha,h}$, for some α such that $\beta \in B_{\alpha,h}$. By the definition of f, if $(y_\beta)_h = x_{\alpha,h} \in U_h$ then $f(h) > \alpha$, thus $\beta \in B_{\alpha,h} \subseteq B_{f(h),h}$, by Condition (ii) in Lemma 4.2. We have proved that, for every $h \in H$, $\{\beta < \lambda^+ | (y_\beta)_h \in U_h\} \subseteq B_{f(h),h}$.

Thus, by the definition of V, we have $\{\beta < \lambda^+ | y_\beta \in V\} = \bigcap_{h\in F}\{\beta < \lambda^+ | (y_\beta)_h \in U_h\} \subseteq \bigcap_{h\in F} B_{f(h),h}$. Hence $|\{\beta < \lambda^+ | y_\beta \in V\}| \leq |\bigcap_{h\in F} B_{f(h),h}| \leq \lambda$. This is a contradiction, since we have supposed that $|\{\beta < \lambda^+ | y_\beta \in V\}| = \lambda^+$, for every neighbourhood V of y. □

Notice that, in the proof of Proposition 5.1, we only used the fact that F, as given by Lemma 4.2, is finite: we made no particular use of the stronger conclusion $|F| = 2$.

Theorem 5.2. *Suppose that λ is an infinite regular cardinal. If a product of topological spaces is $[\lambda^+, \lambda^+]$-compact then all but at most λ factors are $[\lambda, \lambda]$-compact.*

Proof. Suppose by contradiction that some product $\prod_{i\in I} X_i$ is $[\lambda^+, \lambda^+]$-compact, but there are λ^+ factors which are not $[\lambda, \lambda]$-compact. Say, there is $J \subseteq I$ with $|J| = \lambda^+$ and such that for all $i \in J$ X_i is not $[\lambda, \lambda]$-compact.

Then Proposition 5.1 implies that $\prod_{i\in J} X_i$ is not $[\lambda^+, \lambda^+]$-compact.

$\prod_{i\in I} X_i$ is $[\lambda^+, \lambda^+]$-compact by hypothesis, hence, by Proposition 3.5, $\prod_{i\in J} X_i$ is $[\lambda^+, \lambda^+]$-compact, a contradiction. □

We can iterate a finite number of times the arguments in the proof of Proposition 5.1.

Proposition 5.3. *Suppose that $\aleph_\alpha$ is a regular cardinal, and n is a natural number. If $X = \prod_{j \in J} X_j$, $|J| = \aleph_{\alpha+n}$, and no X_j is $[\aleph_\alpha, \aleph_\alpha]$-compact, then X is not $[\aleph_{\alpha+n}, \aleph_{\alpha+n}]$-compact.*

Proof. By induction on n. The case $n = 0$ is trivial (Proposition 3.5).

Suppose $n > 0$, and that the proposition is true for $n-1$, for all topological spaces. Let X be as in the statement. Since $\aleph_{\alpha+n} \cdot \aleph_{\alpha+n-1} = \aleph_{\alpha+n}$, by standard cardinal arithmetic, and since $|J| = \aleph_{\alpha+n}$, we can partition J into $\aleph_{\alpha+n}$-many subsets, each of cardinality $\aleph_{\alpha+n-1}$. Say, $J = \bigcup_{k \in K} J_k$, where $|K| = \aleph_{\alpha+n}$ and $|J_k| = \aleph_{\alpha+n-1}$, for every $k \in K$, and, moreover, $J_k \cap J_{k'} = \emptyset$, for $k \neq k'$.

Thus, $X = \prod_{j \in J} X_j$ is (omeomorphic to) $\prod_{k \in K} \prod_{j \in J_k} X_j$. By the inductive hypothesis, for each $k \in K$, $\prod_{j \in J_k} X_j$ is not $[\aleph_{\alpha+n-1}, \aleph_{\alpha+n-1}]$-compact, since $|J_k| = \aleph_{\alpha+n-1}$, and no X_j is $[\aleph_\alpha, \aleph_\alpha]$-compact. Then, by Proposition 5.1, with $\aleph_{\alpha+n-1}$ in place of λ, K in place of J, and the $\prod_{j \in J_k} X_j$'s in place of the X_j's, we get that $X = \prod_{k \in K}(\prod_{j \in J_k} X_j)$ is not $[\aleph_{\alpha+n}, \aleph_{\alpha+n}]$-compact. □

Thus, we can generalize Theorem 5.2.

Theorem 5.4. *Suppose that $\aleph_\alpha$ is a regular cardinal, and n is a natural number. If a product of topological spaces is $[\aleph_{\alpha+n+1}, \aleph_{\alpha+n+1}]$-compact, then all but at most $\aleph_{\alpha+n}$ factors are $[\aleph_\alpha, \aleph_\alpha]$-compact.*

Proof. Same as the proof of Theorem 5.2, by using Proposition 5.3 in place of Proposition 5.1. □

6. Compact factors in $[\lambda, \lambda]$-compact products (λ singular)

We have a version of our results for singular cardinals.

The proofs of Proposition 6.1 and of Theorem 6.2 below do not rely on Sections 4 and 5.

Proposition 6.1. *Suppose that λ is a singular cardinal, and $\lambda = \sup\{\lambda_\alpha | \alpha \in \mathrm{cf}\lambda\}$, where $\lambda_\alpha < \lambda$ for all $\alpha \in \mathrm{cf}\lambda$. If $X = Y \times \prod_{\alpha \in \mathrm{cf}\lambda} Y_\alpha$ is $[\lambda, \lambda]$-compact, then either Y is $[\mathrm{cf}\lambda, \mathrm{cf}\lambda]$-compact, or there is some $\alpha \in \mathrm{cf}\lambda$ such that Y_α is $[\lambda_\alpha, \lambda]$-compact.*

Proof. Suppose by contradiction that λ, $(\lambda_\alpha)_{\alpha \in \mathrm{cf}\lambda}$ and X give a a counterexample. Thus there is a family $(U_\gamma)_{\gamma \in \mathrm{cf}\lambda}$ which is a counterexample to the $[\mathrm{cf}\lambda, \mathrm{cf}\lambda]$-compactness of Y, and, by Condition (ii) in Proposition 3.4, we can suppose that $U_\alpha \subseteq U_\beta$, for $\alpha < \beta \in \mathrm{cf}\lambda$. Moreover, for every $\alpha \in \mathrm{cf}\lambda$ there is a family $\mathcal{V}_\alpha = (V_{\alpha\beta})_{\beta \in \lambda}$ which is a counterexample to the $[\lambda_\alpha, \lambda]$-compactness of Y_α.

Since the order in which the product is taken is not relevant, we can rearrange the indices in such a way that $\lambda_\alpha \leq \lambda_{\alpha'}$, for $\alpha \leq \alpha'$.

Consider the family $\mathcal{F} = \{W_{\delta\beta\gamma} | \beta \in \lambda, \gamma < \delta \in \mathrm{cf}\lambda\}$, where $W_{\delta\beta\gamma}$ is defined as follows: $W_{\delta\beta\gamma} = U_\gamma \times \prod_{\alpha\in\mathrm{cf}\lambda} Z_\alpha$, where $Z_\alpha = V_{\alpha\beta}$ if $\alpha = \delta$, and $Z_\alpha = Y_\alpha$ if $\alpha \neq \delta$. Notice that all the $W_{\delta\beta\gamma}$'s are open sets of X.

We claim that $\mathcal{F}$ is an open cover of X by λ sets. Indeed, let $x \in X = Y \times \prod_{\alpha\in\mathrm{cf}\lambda} Y_\alpha$, say $x = (y, (y_\alpha)_{\alpha\in\mathrm{cf}\lambda})$. Since $(U_\gamma)_{\gamma\in\mathrm{cf}\lambda}$ is a cover of Y, there is $\gamma \in \mathrm{cf}\lambda$ such that $y \in U_\gamma$. Choose any $\delta \in \mathrm{cf}\lambda$ with $\delta > \gamma$. Since $\mathcal{V}_\delta = (V_{\delta\beta})_{\beta\in\lambda}$ is a cover of Y_δ, there is a $\beta \in \lambda$ such that $y_\delta \in V_{\delta\beta}$. With this choice of γ, δ, β we have that $x = (y, (y_\alpha)_{\alpha\in\mathrm{cf}\lambda}) \in W_{\delta\beta\gamma}$ (since if $\alpha \neq \delta$ then $y_\alpha \in Z_\alpha = Y_\alpha$).

If we show that no subfamily $\mathcal{F}'$ of $\mathcal{F}$ with $< \lambda$ sets covers X, then we contradict the $[\lambda, \lambda]$-compactness of X, hence the theorem is proved.

So, let $\mathcal{F}'$ be a subfamily of $\mathcal{F}$ with $< \lambda$ sets. Thus, there is some $\varepsilon < \mathrm{cf}\lambda$ such that $\mathcal{F}'$ has $< \lambda_\varepsilon$ sets. Without loss of generality, we can choose ε in such a way that $\lambda_\varepsilon > \mathrm{cf}\lambda$. For every $\delta > \varepsilon$ let $\mathcal{V}'_\delta = \{V_{\delta\beta} \in \mathcal{V}_\delta | \beta$ is such that there is $\gamma < \delta$ such that $W_{\delta\beta\gamma}$ belongs to $\mathcal{F}'\}$. For every $\delta > \varepsilon$, $\mathcal{V}'_\delta$ contains at most $|\delta| \cdot |\mathcal{F}'| \leq \mathrm{cf}\lambda \cdot |\mathcal{F}'| < \lambda_\varepsilon$ sets, hence is not a cover of Y_δ, since $\lambda_\delta \geq \lambda_\varepsilon$, and $\mathcal{V}_\delta = (V_{\delta\beta})_{\beta\in\lambda}$ was supposed to be a counterexample to the $[\lambda_\delta, \lambda]$-compactness of Y_δ.

By taking α in place of δ in the above argument, we get that for every $\alpha > \varepsilon$ there is $y_\alpha \in Y_\alpha$ such that for no $V_{\alpha\beta} \in \mathcal{V}'_\alpha$ it happens that $y_\alpha \in V_{\alpha\beta}$. Choose such an y_α for every $\alpha > \varepsilon$, and choose y_α arbitrarily if $\alpha \leq \varepsilon$.

Choose $y \in Y$ such that $y \notin U_\varepsilon$. This is possible since $U_\varepsilon \subset Y$ (strict inclusion), because $(U_\gamma)_{\gamma\in\mathrm{cf}\lambda}$ was supposed to be a counterexample to the $[\mathrm{cf}\lambda, \mathrm{cf}\lambda]$-compactness of Y.

We show that $x = (y, (y_\alpha)_{\alpha\in\mathrm{cf}\lambda})$ belongs to no element of $\mathcal{F}'$, where y and the y_α's are chosen as above. Suppose, to the contrary, that $x \in W_{\delta\beta\gamma}$ for some δ, β, γ such that $W_{\delta\beta\gamma} \in \mathcal{F}'$, and recall that $W_{\delta\beta\gamma} = U_\gamma \times \prod_{\alpha\in\mathrm{cf}\lambda} Z_\alpha$. We consider the two cases $\delta > \varepsilon$ and $\delta \leq \varepsilon$, and derive a contradiction in each case.

If $\delta > \varepsilon$ then $x \notin W_{\delta\beta\gamma}$ since $y_\delta \notin Z_\delta = V_{\delta\beta}$, because of the way we have chosen y_δ.

If $\delta \leq \varepsilon$, and $x \in W_{\delta\beta\gamma}$ then $y \in U_\gamma$, and this implies $\gamma > \varepsilon$, since we have assumed that $U_\alpha \subseteq U_\beta$, for $\alpha < \beta$, and since, by the construction of y, $y \notin U_\varepsilon$. But this implies $\gamma > \varepsilon \geq \delta$, a contradiction, since $W_{\delta\beta\gamma}$ is defined only for $\gamma < \delta$.

Thus, $\mathcal{F}'$ is not a cover of X, and this contradicts our hypothesis that X is $[\lambda, \lambda]$-compact. □

Theorem 6.2. *Suppose that λ is a singular cardinal. If a product $X = \prod_{i\in I} X_i$ of topological spaces is $[\lambda, \lambda]$-compact then either:*

(i) all factors are $[\mathrm{cf}\lambda, \mathrm{cf}\lambda]$-compact, or

(ii) there is $\lambda' < \lambda$ such that $|\{i \in I | X_i$ is not $[\lambda', \lambda]$-compact $\}| < \mathrm{cf}\lambda$.

Proof. Suppose to the contrary that there is $\bar{i} \in I$ such that $X_{\bar{i}}$ is not $[\mathrm{cf}\lambda, \mathrm{cf}\lambda]$-compact, and that for every $\lambda' < \lambda$ there are at least $\mathrm{cf}\lambda$-many factors which are not $[\lambda', \lambda]$-compact.

Fix any sequence of cardinals $(\lambda_\alpha)_{\alpha<\mathrm{cf}\lambda}$ such that $\lambda = \sup\{\lambda_\alpha | \alpha < \mathrm{cf}\lambda\}$, and $\lambda_\alpha < \lambda$ for $\alpha \in \mathrm{cf}\lambda$. Construct a sequence $(i_\alpha)_{\alpha<\mathrm{cf}\lambda}$ of distinct elements of I as follows.

Choose $i_0 \in I$, $i_0 \neq \bar{i}$ in such a way that X_{i_0} is not $[\lambda_0, \lambda]$-compact.

Suppose that $\alpha < \mathrm{cf}\lambda$, and suppose that we have already chosen i_β for all $\beta < \alpha$. Then choose $i_\alpha \in I$ in such a way that X_{i_α} is not $[\lambda_\alpha, \lambda]$-compact, $i_\alpha \neq \bar{i}$ and, for every $\beta < \alpha$, $i_\alpha \neq i_\beta$. This is possible, since there are at least $\mathrm{cf}\lambda$ many i's such that X_i is not $[\lambda_\alpha, \lambda]$-compact, while $|\{i_\beta | \beta < \alpha\}| = |\alpha| < \mathrm{cf}\lambda$.

Now, set $Y = X_{\bar{i}}$, and $Y_\alpha = X_{i_\alpha}$, for $\alpha < \mathrm{cf}\lambda$. By Proposition 6.1, $Y \times \prod_{\alpha\in\mathrm{cf}\lambda} Y_\alpha$ is not $[\lambda, \lambda]$-compact.

If $X = \prod_{i\in I} X_i$ is $[\lambda, \lambda]$-compact, then Poposition 3.5 implies that $Y \times \prod_{\alpha\in\mathrm{cf}\lambda} Y_\alpha$ is $[\lambda, \lambda]$-compact (since $\bar{i}$ and the i_α's are all distinct elements of I). Thus, we have reached a contradiction. □

We can put together the methods of proof of Theorem 6.2 and of Proposition 5.3.

Proposition 6.3. *Suppose that $\aleph_\beta$ is a regular cardinal. If $X = Y \times \prod_{j\in J} X_j$, $|J| = \aleph_{\beta+\omega}$, Y is not countably compact, and no X_j is $[\aleph_\beta, \aleph_\beta]$-compact, then X is not $[\aleph_{\beta+\omega}, \aleph_{\beta+\omega}]$-compact.*

Proof. The proof is somewhat similar to (and relies on) the proof of Proposition 5.3.

Since $|J| = \aleph_{\beta+\omega}$, we can write $J = \bigcup_{n\in\omega} J_n$, where $J_n \cap J_m = \emptyset$, for all $n \neq m$, and $|J_n| = \aleph_{\beta+n+1}$ for all natural numbers n.

Hence, $X = Y \times \prod_{j\in J} X_j$ is (omeomorphic to) $Y \times \prod_{n\in\omega} \prod_{j\in J_n} X_j$. If we put $Y_n = \prod_{j\in J_n} X_j$, then $X = Y \times \prod_{n\in\omega} Y_n$.

For every n, by Proposition 5.3, Y_n is not $[\aleph_{\beta+n+1}, \aleph_{\beta+n+1}]$-compact, since $|J_n| = \aleph_{\beta+n+1}$, $Y_n = \prod_{j\in J_n} X_j$, and no X_j is $[\aleph_\beta, \aleph_\beta]$-compact.

By Proposition 3.1, for every n, Y_n is not $[\aleph_{\beta+n+1}, \aleph_{\beta+\omega}]$-compact.

By Proposition 6.1, $X = Y \times \prod_{n\in\omega} Y_n$ is not $[\aleph_{\beta+\omega}, \aleph_{\beta+\omega}]$-compact (notice that $\mathrm{cf}\aleph_{\beta+\omega} = \omega$). □

Theorem 6.4. *Suppose that $\aleph_\beta$ is a regular cardinal. If $X = \prod_{i\in I} X_i$ is $[\aleph_{\beta+\omega}, \aleph_{\beta+\omega}]$-compact, then either*

(i) all factors are countably compact, or

(ii) $|\{i \in I | X_i$ is not $[\aleph_\beta, \aleph_\beta]$-compact $\}| < \aleph_{\beta+\omega}$.

Proof. Similar to the proof of Theorem 5.2, using Proposition 6.3 in place of Proposition 5.1. □

7. Proofs of the results stated in the introduction (and more)

By applying Proposition 3.1, we can improve Theorem 5.4 to $[\aleph_\alpha, \aleph_{\alpha+n+1}]$-compactness.

Theorem 7.1. *Suppose that $\aleph_\alpha$ is a regular cardinal, and n is a natural number. If a product of topological spaces is $[\aleph_{\alpha+n+1}, \aleph_{\alpha+n+1}]$-compact, then all but at most $\aleph_{\alpha+n}$ factors are $[\aleph_\alpha, \aleph_{\alpha+n+1}]$-compact.*

Proof. For every i with $0 \leq i \leq n$, let us apply Theorem 5.4 with $\aleph_{\alpha+i}$ in place of $\aleph_\alpha$, and $n-i$ in place of n, noticing that $\aleph_{(\alpha+i)+(n-i)+1} = \aleph_{\alpha+n+1}$.

We get that, for each i ($0 \leq i \leq n$), all but at most $\aleph_{\alpha+i+n-i} = \aleph_{\alpha+n}$ factors are $[\aleph_{\alpha+i}, \aleph_{\alpha+i}]$-compact. Discard all such factors: since a finite union of sets having cardinality $\leq \aleph_{\alpha+n}$ has cardinality $\leq \aleph_{\alpha+n}$, we have discarded at most $\aleph_{\alpha+n}$ factors. In conclusion, all but at most $\aleph_{\alpha+n}$ factors are simultaneously $[\aleph_{\alpha+i}, \aleph_{\alpha+i}]$-compact for all i, $0 \leq i \leq n$.

Trivially, all factors are $[\aleph_{\alpha+n+1}, \aleph_{\alpha+n+1}]$-compact, e.g. by Proposition 3.5.

By Proposition 3.1, all but at most $\aleph_{\alpha+n}$ factors are $[\aleph_\alpha, \aleph_{\alpha+n+1}]$-compact. □

Theorem 1.6 is the particular case $\alpha = 0$ of Theorem 7.1 (since $\aleph_0$ is a regular cardinal).

Corollary 7.2. *If the $\aleph_{n+1}$-th power of the topological space X is $[\aleph_{n+1}, \aleph_{n+1}]$-compact, then X is initially $\aleph_{n+1}$-compact.*

More generally, if $\aleph_\alpha$ is a regular cardinal, n is a natural number, and the $\aleph_{\alpha+n+1}$-th power of the topological space X is $[\aleph_{\alpha+n+1}, \aleph_{\alpha+n+1}]$-compact, then X is $[\aleph_\alpha, \aleph_{\alpha+n+1}]$-compact.

Theorem 7.3. *Suppose that $\aleph_\alpha$ is a regular cardinal. If a product of topological spaces is finally $\aleph_{\alpha+n+1}$-compact, then all but at most $\aleph_{\alpha+n}$ factors are finally $\aleph_\alpha$-compact.*

Proof. Let X be a product which is finally $\aleph_{\alpha+n+1}$-compact. By the trivial direction in Corollary 3.3(i), X is $[\aleph_{\alpha+n+1}, \aleph_{\alpha+n+1}]$-compact. By Theorem 7.1, all but at most $\aleph_{\alpha+n}$ factors are $[\aleph_\alpha, \aleph_{\alpha+n+1}]$-compact.

Since the product is finally $\aleph_{\alpha+n+1}$-compact, all factors are finally $\aleph_{\alpha+n+1}$-compact by Proposition 3.5.

In conclusion, all but at most $\aleph_{\alpha+n}$ factors are finally $\aleph_\alpha$-compact, since it is trivial that, for every $\lambda \geq \mu$, final λ-compactness and $[\mu, \lambda]$-compactness imply final μ-compactness (here, $\lambda = \aleph_{\alpha+n+1}$ and $\mu = \aleph_\alpha$). Otherwise, apply Proposition 3.1 and Corollary 3.3(i). □

Theorem 1.2 is the particular case $\alpha = 0$ of Theorem 7.3.

In the introduction we promised a common generalization of Theorems 1.2 and 1.3. Let us say that a topological space is *finally κ-linearly Lindelöf* if and only if every open cover which is linearly ordered by inclusion has a subcover of cardinality less than κ. Thus, linear Lindelöfness is the same as final $\aleph_1$-linear Lindelöfness.

The methods in the proofs of Propositions 3.4 and 3.2 show that a space X is finally κ-linearly Lindelöf if and only if X is $[\lambda, \lambda]$-compact for all regular cardinals $\lambda \geq \kappa$ (if and only if every subset of X of regular cardinality $\geq \kappa$ has a complete accumulation point). In particular, if κ is singular, final κ-linear Lindelöfness coincides with final κ^+-linear Lindelöfness

By Corollary 3.3(i), every finally κ-compact space is finally κ-linearly Lindelöf, hence the next theorem encompasses both Theorem 1.2 and Theorem 1.3.

Theorem 7.4. *If a product of topological spaces is finally $\aleph_{\alpha+n+1}$-linearly Lindelöf (that is, $[\kappa, \kappa]$-compact for all regular cardinals $\kappa \geq \aleph_{\alpha+n+1}$), then all but at most $\aleph_{\alpha+n}$ factors are finally $\aleph_\alpha$-linearly Lindelöf.*

If a product of topological spaces is finally $\aleph_{n+1}$-linearly Lindelöf, then all but at most $\aleph_n$ factors are compact.

Proof. First, suppose that $\aleph_\alpha$ is regular. Since the product is $[\aleph_{\alpha+n+1}, \aleph_{\alpha+n+1}]$-compact, then by Theorem 7.1, all but at most $\aleph_{\alpha+n}$ factors are $[\aleph_\alpha, \aleph_{\alpha+n+1}]$-compact, that is, by the trivial direction in Proposition 3.1, $[\aleph_{\alpha+i}, \aleph_{\alpha+i}]$-compact for all i with $0 \leq i \leq n+1$.

We have proved that all but at most $\aleph_{\alpha+n}$ factors are $[\kappa, \kappa]$-compact for all cardinals κ with $\aleph_\alpha \leq \kappa \leq \aleph_{\alpha+n+1}$. Moreover, all factors are $[\kappa, \kappa]$-compact for all regular cardinals $\kappa \geq \aleph_{\alpha+n+1}$, by hypothesis and Proposition 3.5.

In conclusion, all but at most $\aleph_{\alpha+n}$ factors are $[\kappa, \kappa]$-compact for all regular cardinals $\kappa \geq \aleph_\alpha$, that is, finally $\aleph_\alpha$-linearly Lindelöf.

If $\aleph_\alpha$ is singular, the above arguments show that the product is finally $\aleph_{\alpha+1}$-linearly Lindelöf. But, since $\aleph_\alpha$ is singular, final $\aleph_{\alpha+1}$-linear Lindelöfness is the same as final $\aleph_\alpha$-linear Lindelöfness, as we remarked before the statement of the theorem.

The second statement is the particular case $\alpha = 0$ of the first statement, since final $\aleph_0$-linear Lindelöfness is the same as compactness, by Corollary 3.3(iii). □

Theorem 1.3 is the particular case $n = 0$ of the second statement in Theorem 7.4.

Notice that, so far, in the present section we have not used the results proved in Section 6.

Theorem 7.5. *If $\aleph_\alpha$ is a regular cardinal, and a product of topological spaces is $[\aleph_{\alpha+\omega}, \aleph_{\alpha+\omega}]$-compact, then either*

(a) all factors are countably compact, or

(b) all factors are $[\aleph_\alpha, \aleph_{\alpha+\omega}]$-compact except possibly for a set having cardinality less than $\aleph_{\alpha+\omega}$.

Proof. Let $\lambda = \aleph_{\alpha+\omega}$, and suppose that (a) fails. By Theorem 6.2, and since $\mathrm{cf}\aleph_{\alpha+\omega} = \omega$, there is $\lambda' < \lambda$ such that $|\{i \in I | X_i$ is not $[\lambda', \lambda]$-compact $\}| < \mathrm{cf}\lambda = \omega$.

If $\lambda' \leq \aleph_\alpha$, the theorem is proved; otherwise, $\lambda' = \aleph_{\alpha+n}$, for some natural number n, since $\aleph_\alpha < \lambda' < \lambda$, and $\lambda = \aleph_{\alpha+\omega}$. Hence we have that all factors are $[\aleph_{\alpha+n}, \lambda]$-compact, except perhaps for a finite set of factors.

Now we proceed as in the proof of Theorem 7.1, by applying Theorem 6.4: for each $i < n$, we can apply Theorem 6.4, with $\aleph_{\alpha+i}$ in place of $\aleph_\beta$, in order to get that all factors are $[\aleph_{\alpha+i}, \aleph_{\alpha+i}]$-compact, except for a set of cardinality $< \aleph_{\alpha+i+\omega} = \aleph_{\alpha+\omega}$, since we are supposing that (a) fails.

Since the union of a finite number of sets of cardinality $< \aleph_{\alpha+\omega}$ has still cardinality $< \aleph_{\alpha+\omega}$, we get that all factors are simultaneously $[\aleph_{\alpha+i}, \aleph_{\alpha+i}]$-compact for all $i < n$, except possibly for a set of factors having cardinality $< \aleph_{\alpha+\omega}$.

Applying Proposition 3.1, we get that all factors are $[\aleph_\alpha, \aleph_{\alpha+\omega}]$-compact, except for a set of cardinality $< \aleph_{\alpha+\omega}$. □

Theorem 1.7 is the particular case $\alpha = 0$ of Theorem 7.5.

Corollary 7.6. *If $\aleph_\alpha$ is a regular cardinal, and a product of topological spaces is finally $\aleph_{\alpha+\omega}$-compact, then either*

(a) all factors are countably compact, or

(b) all factors are finally $\aleph_\alpha$-compact, except possibly for a set having cardinality less than $\aleph_{\alpha+\omega}$.

Proof. Same as the proof of Theorem 7.3, by using Theorem 7.5 in place of Theorem 7.1. □

Theorem 1.5 is the particular case $\alpha = 0$ of Corollary 7.6.

Corollary 7.7. *If the $\aleph_\omega$-th power of the topological space X is $[\aleph_\omega, \aleph_\omega]$-compact, then X is countably compact.*

More generally, if $\aleph_\alpha$ is a regular cardinal and the $\aleph_{\alpha+\omega}$-th power of the topological space X is $[\aleph_{\alpha+\omega}, \aleph_{\alpha+\omega}]$-compact, then X is either countably compact, or $[\aleph_\alpha, \aleph_{\alpha+\omega}]$-compact.

Corollary 7.7 is an immediate consequence of Theorem 7.5.

8. Additional remarks

The proof of Lemma 4.2, and hence the proofs of most results in Sections 5 and 7, make an essential use of the assumption that λ ($\aleph_\alpha$, $\aleph_\beta$, respectively) is a regular cardinal. It is an open problem whether the assumption that λ ($\aleph_\alpha$, $\aleph_\beta$, respectively) is regular can be removed from Proposition 5.1 (hence, say, from Theorems 5.2, 5.4, 6.4, 7.1, 7.3 and 7.5).

However, we have partial results. The proofs of the next two theorems make use of variations both on the methods of [**L2**], [**C2**] and on the constructions performed in Section 4. We shall present proofs elsewhere.

Theorem 8.1. *Suppose that λ is a singular cardinal. If a product of topological spaces is $[\lambda^+, \lambda^+]$-compact then all factors are $[\lambda, \lambda]$-compact, except possibly for a set of cardinality less than 2^λ.*

We say that a topological space X is *almost $[\kappa, \lambda]$-compact* if it satisfies the following property: whenever $|I| = \lambda$, and $(U_i)_{i \in I}$ is an open cover of X such that $(U_i)_{i \in J}$ is still a cover of X whenever $J \subseteq I$ and $|J| = \lambda$, then $(U_i)_{i \in I}$ has a subcover by less than κ sets.

Theorem 8.2. *Suppose that λ is a singular cardinal. If a product of topological spaces is $[\lambda^+, \lambda^+]$-compact then all but at most λ factors are almost $[\lambda, \lambda]$-compact.*

Clearly, $[\kappa, \lambda]$-compactness implies almost $[\kappa, \lambda]$-compactness.

If κ is a regular cardinal, then $[\kappa, \kappa]$-compactness and almost $[\kappa, \kappa]$-compactness are equivalent, since almost $[\kappa, \kappa]$-compactness implies Condition (ii) in Proposition 3.4. We do not know what happens when κ is a singular cardinal.

Problem 8.3. Is it true that if $\prod_{i \in I} X_i$ is $[\lambda^+, \lambda^+]$-compact then there exists $J \subseteq I$ such that $|I \setminus J| \leq \lambda$ and $\prod_{i \in J} X_i$ is $[\lambda, \lambda]$-compact?

A version of Problem 8.3 has an affirmative answer.

Corollary 8.4. *If $\prod_{i \in I} X_i$ is finally $\aleph_{n+1}$-compact, then there is $J \subseteq I$ such that $|I \setminus J| \leq \aleph_n$, and $\prod_{i \in J} X_i$ is compact.*

Proof. By Theorem 1.2, all but at most $\aleph_n$ factors are compact. Let J be the set of compact factors. Then $|I \setminus J| \leq \aleph_n$, and, by Tychonoff Theorem, $\prod_{i \in J} X_i$ is compact. □

A sequence $(x_\alpha)_{\alpha \in \lambda}$ of elements of a topological space X *converges* to $x \in X$ if and only if for every neighbourhood U of x in X there is $\beta \in \lambda$ such that $x_\alpha \in U$ for every $\alpha \geq \beta$.

A topological space X is *sequentially λ-compact* (or *λ-chain compact*) if and only if every sequence $(x_\alpha)_{\alpha \in \lambda}$ has a converging subsequence.

Problem 8.5. Is it true that, if λ is regular and a product is sequentially λ^+-compact then all but at most λ factors are sequentially λ-compact?

References

[AU] P. Alexandroff, P. Urysohn, *Mémorie sur les éspaces topologiques compacts*, Ver. Akad. Wetensch. Amsterdam **14** (1929), 1-96.

[AB] A. V. Arhangel'skii, R. Z. Buzyakova, *On linearly Lindelöf and strongly discretely Lindelöf spaces*, Proc. Amer. Math. Soc. **127** (1999), 2449–2458.

[C1] X. Caicedo, *On productive* $[\kappa, \lambda]$*-compactness, or the Abstract Compactness Theorem revisited*, manuscript (1995).

[C2] X. Caicedo, *The Abstract Compactness Theorem Revisited*, in *Logic and Foundations of Mathematics (A. Cantini et al. editors),* Kluwer Academic Publishers (1999), 131–141.

[CN] W. Comfort, S. Negrepontis, *The Theory of Ultrafilters*, Berlin (1974).

[EU] P. Erdös, S. Ulam, *On equations with sets as unknowns*, Proc. Nat. Acad. Sci. U.S.A. **60** (1968), 1189–1195.

[Ga] L. S. Gaal, *On the theory of* (m, n)*-compact spaces*, Pacific J. Math. **8** (1958), 721–734.

[Go] C. Good, *Lindelöf spaces*, Section d-8 in [**HNV**].

[HNV] K. P. Hart, J. Nagata, J. E. Vaughan (editors), *Encyclopedia of General Topology*, Amsterdam (2003) .

[KL] M. Kojman, V. Lubitch, *Sequentially linearly Lindelöf spaces*, Topology Appl. **128** (2003), 135–144.

[K] K. Kunen, *Locally compact linearly Lindelöf spaces*, Comment. Math. Univ. Carolin. **43** (2002), 155–158.

[KV] K. Kunen and J. E. Vaughan (editors), *Handbook of Set Theoretical Topology*, Amsterdam (1984).

[L1] P. Lipparini, *Productive* $[\lambda, \mu]$*-compactness and regular ultrafilters*, Topology Proceedings **21** (1996), 161–171.

[L2] P. Lipparini, *Every* (λ^+, κ^+)*-regular ultrafilter is* (λ, κ)*-regular*, Proc. Amer. Math. Soc. **128** (1999), 605–609.

[Sm] Y. M. Smirnov, *On topological spaces compact in a given interval of powers* (russian), Akad. Nauk. SSSR Izvest. Ser. Mat. **14** (1950), 155-178.

[Ste] R. M. Stephenson, *Initially* κ*-compact and related spaces*, Chapter 13 in [**KV**], 603–632.

[Sto] A. H. Stone, *Paracompactness and product spaces*, Bull. Amer. Math. Soc. **54** (1948), 977–982.

[U] S. Ulam, *Uber die Masstheorie in der Allgemeinen Mengenlehre*, Fund. Math. **16** (1930), 140–150.

[V1] J. E. Vaughan, *Some recent results in the theory of [a,b]-compactness*, in *TOPO 72—General Topology and its Applications* (Proc. Second Pittsburg Internat. Conf., Carnegie-Mellon Univ. and Univ. Pittsburg, 1972), Lecture Notes in Mathematics **378** (1974), 534–550.

[V2] J. E. Vaughan, *Some properties related to [a,b]-compactness*, Fund. Math. **87** (1975), 251–260.

Received Received date / Revised version date

Dipartimento di Matematica,
Viale della Ricerca Scientifica,
II Università di Roma (Tor Vregata),
I-00133 ROME ITALY
E-mail address: lipparin@axp.mat.uniroma2.it

http://www.mat.uniroma2.it/~lipparin

The author has received support from MPI and GNSAGA.
The author wishes to express his gratitude to X. Caicedo for many helpful discussions.