MODULAR STRUCTURE OF THE CROSSED PRODUCT BY A COMPACT GROUP DUAL

Tommaso ISOLA

Dipartimento di Matematica. Università di Roma "Tor Vergata".
Via della Ricerca Scientifica. I-00133, Roma, Italy.

isola@mat.utovrm.it

Revised, October 1993

Abstract. Let M be a properly infinite von Neumann algebra, and α a dominant action of a separable compact group. Choose a faithful normal state φ_0 on the fixed-point algebra M^α and lift it to M as $\varphi := \varphi_0 \cdot \varepsilon$ by means of the canonical expectation $\varepsilon : M \to M^\alpha$. Then we express the modular objects associated with φ in terms of the modular objects associated with φ_0.

1. Introduction.

Let M be a properly infinite von Neumann algebra, and α a dominant action of a separable compact group G by automorphisms of M. In this paper we shall describe the modular structure of M, in terms of the modular structure of M^α, the fixed–point algebra of M by α.

To do this we have to use some index theory for an inclusion of von Neumann algebras with nontrivial centre. So we recall, in section 2, the definition of the index given by H. Kosaki [11] and some related results. Then, in section 3, we slightly extend a recent result of J.F. Havet's [5] on the minimal expectation between von Neumann algebras with finite dimensional centres (proposition 3.6).

In section 4 we state our problem: we start, for simplicity, from a faithful normal state φ_0 on M^α, as in case of a weight we are faced with some unsubstantial technical complications, and, denoting with $\varepsilon : M \to M^\alpha$ the normal faithful conditional expectation of M on M^α, we express the modular group associated to $\varphi := \varphi_0 \cdot \varepsilon$ in terms of the modular group associated to φ_0. To do this, we have to characterize the restriction to M^α of some conditional expectation E of M. We consider first, both for their importance and simplicity, prime actions, namely those with $M^{\alpha\prime} \wedge M = \mathbf{C}$, as in this case the characterization of $E|_{M^\alpha}$ is immediate (see proposition 4.8). Then we solve the case of finite group actions using the result on the minimal conditional expectation previously proved. The general case is solved by means of another approach, which makes use of the unitary implementing the flip to characterize the expectation E. Let us observe that we use, here and in section 6, the construction of the crossed product by the dual of G given by J.E. Roberts [20], as this approach provides us with a set of generators of M, namely M^α and the Hilbert spaces in M, in terms of which the expression of the modular objects associated to φ is particularly simple. Therefore this approach is more useful than the one based on Hopf–von Neumann algebras, for details of which we refer to [18] and [21].

Section 5 is devoted to the proof of a sufficient condition for two endomorphisms to be conjugate. This result is used in section 6 to compute the modular operators associated to φ.

Finally, we would like to mention that this structure appears, for example, in algebraic quantum field theory, where one may ask for the modular structure of the local field algebras in terms of the modular structure of the local observable algebras.

2. Preliminaries.

In this section we recall some known results on the theory of index, initiated by V. Jones [9], both for ease of reference and for fixing notations. Throughout the paper we assume that all von Neumann algebras have separable predual, and use the following notation: if $A \subset B$ are von Neumann algebras, $P(B, A)$ is the set of all normal semifinite faithful (n.s.f.) A-valued weights on B, $E(B, A)$ is the set of all normal faithful conditional expectations from B to A; if M is a von Neumann algebra, $P(M)$ is the set of all n.s.f. weights on M, and $E(M)$ is the set of all normal faithful states on M.

Let us now recall the definition of H. Kosaki's index [11] based on A. Connes' spatial theory and U. Haagerup's operator valued weights.
Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, and $\psi \in P(M')$. We use the standard notation:
$\mathcal{N}_\psi := \{x \in M' : \psi(x^*x) < \infty\}$;
$\mathcal{M}_\psi := \mathcal{N}_\psi^*\mathcal{N}_\psi$ the domain of ψ;
$\mathcal{H}_\psi :=$ the Hilbert space completion of $\mathcal{N}_\psi$ with respect to $x \to \psi(x^*x)^{1/2}$;
$\Lambda_\psi :=$ the canonical injection of $\mathcal{N}_\psi$ into $\mathcal{H}_\psi$;
$\pi_\psi :=$ the regular representation of M' on $\mathcal{H}_\psi$, that is

$$\pi_\psi(x)\Lambda_\psi(y) := \Lambda_\psi(xy), \forall x \in M', \forall y \in \mathcal{N}_\psi.$$

Define, $\forall \xi \in \mathcal{H}$, the operator $R^\psi(\xi) : \mathcal{H}_\psi \to \mathcal{H}$ by

$$\mathcal{D}(R^\psi(\xi)) := \Lambda_\psi(\mathcal{N}_\psi), \quad R^\psi(\xi)\Lambda_\psi(x) := x\xi,\ \forall x \in \mathcal{N}_\psi.$$

We say $\xi \in \mathcal{H}$ is ψ-bounded if $R^\psi(\xi)$ is a bounded operator, that is if $\exists C > 0$ s.t. $\|x\xi\| \le C\psi(x^*x)^{1/2}, \forall x \in \mathcal{N}_\psi$. The set $D(\mathcal{H};\psi) := \{\xi \in \mathcal{H} : \xi$ is ψ-bounded $\}$ is dense in $\mathcal{H}$. Let us set $\vartheta^\psi(\xi,\eta) := R^\psi(\xi)R^\psi(\eta)^*$; then $\vartheta^\psi(\xi,\xi) \in \overline{M}_+$ (the extended positive part of M), and, if $\xi,\eta \in D(\mathcal{H};\psi)$, $\vartheta^\psi(\xi,\eta) \in M$.

Let now $\varphi \in P(M)$ and extend it to $\overline{M}_+$ and set $q_\varphi : \xi \in \mathcal{H} \to q_\varphi(\xi) := \varphi(\vartheta^\psi(\xi,\xi)) \in [0,\infty]$. Then q_φ is a lower semicontinuous (hence closable) quadratic form, thus, by Friedrichs' theorem, there exists a unique positive self-adjoint operator $d\varphi/d\psi$ on $\mathcal{H}$ (called the spatial derivative of φ relative to ψ) such that $\bar{q}_\varphi(\xi) = \|(d\varphi/d\psi)^{1/2}\xi\|^2$.

Based on spatial theory one can prove that, if $A \subset B$ are von Neumann algebras, $\forall E \in P(B,A)$ there is a unique $E^{-1} \in P(A',B')$ such that $d(\varphi\cdot E)/d\psi = d\varphi/d(\psi\cdot E^{-1})$, $\forall\varphi \in P(A)$, $\forall\psi \in P(B')$. Observe that $E^{-1}(1) \in \overline{Z(B)}_+$ and does not depend on the representation of B (as the same proof of [11, th.2.2] works).

2.1 Definition. *If $A \subset B$ are von Neumann algebras and $E \in E(B,A)$, we say that E has finite index if $E^{-1}(1) \in Z(B)$, and that $Ind_E(A,B) \equiv Ind(E) := E^{-1}(1)$ is the index of E.*

Let now $A \subset B$ be von Neumann algebras and let $E \in E(B,A)$. Let $\varphi \in E(A)$, and set $\psi := \varphi \cdot E \in E(B)$; let $\Psi \in \mathcal{H} \equiv \mathcal{H}_\psi$ cyclic and separating for B and such that $\psi = (\Psi, \cdot\Psi)$, and set $e := [A\Psi] \in A'$. We call e the Jones projection of the inclusion. Then one has

2.2 Proposition. *[11]*

- (i) *$E^{-1}(e) = 1$. In particular $Ind(E) \ge 1$ and $Ind(E) = 1 \iff A = B$.*
- (ii) *$exe = E(x)e$, $\forall x \in B$.*
- (iii) *$x \in B$; $x \in A \iff [x,e] = 0$.*
- (iv) *$J_B e J_B = e$, where $J_B := J_B^\Psi$.*
- (v) *$\langle B,e\rangle = J_B A' J_B$. This algebra is called Jones basic construction.*
- (vi) *$\mathcal{A} := \{\sum_{i=1}^n a_i e b_i : a_i, b_i \in B, n \in \mathbf{N}\}$ is a dense *subalgebra of $\langle B,e\rangle$.*
- (vii) *The central support of e in $\langle B,e\rangle$ is 1.*
- $(viii)$ *$x \in A \to xe \in Ae = e\langle B,e\rangle e$ is a surjective isomorphism.*

2.3 Proposition. *Let $A \subset B$ be von Neumann algebras, $E \in E(B,A)$ with finite index, J a modular conjugation for B, and $j := adJ$. Then*

- (i) *$E_1 := Ind(E)^{-1} j \cdot E^{-1} \cdot j(\cdot) \in E(\langle B,e\rangle, B)$ and $E_1(e) = Ind(E)^{-1}$; we call E_1 the expectation dual of E.*
- (ii) *If $Ind(E) \in A$, $Ind(E_1) = Ind(E)$.*

Proof. (i) follows by direct computation.

(ii) $Ind(E_1) = E_1^{-1}(1) = j \cdot E \cdot j(Ind(E)) = j \cdot E(Ind(E)) = Ind(E)$.

$\square$

2.4 Proposition. *Let $A \subset B$ be von Neumann algebras, $E \in E(B, A)$ with finite index. Let e be Jones projection and $B_1 := \langle B, e \rangle$ the basic construction. Then $\forall x \in B_1$, $\exists!\ b \in B$ s.t. $xe = be$.*

Proof. It follows from [11, lemma 3.3] with obvious modifications.

$\square$

2.5 Proposition. *Let $A \subset B$ be properly infinite von Neumann algebras, $E \in E(B, A)$ with finite index. Then $\exists b \in B$ s.t. $x = bE(b^*x), \forall x \in B$. Moreover $beb^* = 1$, $bb^* = Ind(E)$, and $E(b^*b) = 1$.*

Proof. Let $v \in B_1$ be such that $v^*v = e$, and $vv^* = 1$ and let $b \in B$ be s.t. $v = ve = be$. Let $E_1 \in E(B_1, B)$ be the dual expectation. Then $beb^* = vv^* = 1$, $bb^* = Ind(E)bE_1(e)b^* = Ind(E)E_1(beb^*) = Ind(E)$, and $E(b^*b)e = eb^*be = v^*v = e$, so that, by uniqueness in proposition 2.4, $E(b^*b) = 1$. Finally, $\forall x \in B$, $xe = beb^*xe = bE(b^*x)e$, so that $x = bE(b^*x)$.

$\square$

3. A result on the minimal expectation of an inclusion of von Neumann algebras with nontrivial centres.

In this section we want to give one result on the minimal expectation that will be useful in the next section. To do this, we have to prove some preliminary results on the behaviour of expectations and indeces w.r.t. decompositions.

3.1 Proposition. *Let $A \subset B$ be von Neumann algebras, $E \in E(B, A)$. If $A = \int^{\oplus} A_\omega d\mu(\omega)$, $B = \int^{\oplus} B_\omega d\mu(\omega)$ are their decompositions with respect to $L^\infty(\Omega, \mu) \cong Z \subset Z(A) \wedge Z(B)$, then for almost all ω, there exists $E_\omega \in E(B_\omega, A_\omega)$ such that $E(x) = \int^{\oplus} E_\omega(x_\omega)d\mu(\omega)$, $\forall x = \int^{\oplus} x_\omega d\mu(\omega) \in B$.*

Proof. Let us choose $\varphi \in E(A)$ and set $\psi := \varphi \cdot E \in E(B)$; then $\psi = \psi \cdot E$, and, if $\sigma_t := \sigma_t^\psi$, then $\sigma_t(A) = A$, $\forall t \in \mathbf{R}$.
Let $\sigma_t(x) = \int^{\oplus} \sigma_{\omega,t}(x_\omega)d\mu(\omega)$, be its decomposition as in [22, Th.A.13] and let $\varphi = \int^{\oplus} \varphi_\omega d\mu(\omega)$, $\psi = \int^{\oplus} \psi_\omega d\mu(\omega)$ be the decompositions of φ and ψ, where $\varphi_\omega \in E(A_\omega)$, $\psi_\omega \in E(B_\omega)$ for almost every ω [23, prop.IV.8.34].
Observe that $\psi_\omega|_{A_\omega} = \varphi_\omega$, from uniqueness [23, prop.IV.8.34], therefore $\sigma_{\omega,t}(A_\omega) = A_\omega, \forall t \in \mathbf{R}$. Thus, from Takesaki's criterion, $\exists!\ E_\omega \in E(B_\omega, A_\omega)$ such that $\psi_\omega \cdot E_\omega = \psi_\omega$.

Let us set $F(x) := \int^{\oplus} E_\omega(x_\omega)d\mu(\omega), \forall x \in B$. Then it is easy to prove that $F \in E(B, A)$ and that $\psi \cdot F = \psi$, so that, by uniqueness, $F = E$.

$\square$

3.2 Lemma. *Let A be a von Neumann algebra, $\int^{\oplus} A_\omega d\mu(\omega)$ its decomposition w.r.t. $L^\infty(\Omega, \mu) \cong Z \subset Z(A)$, $\varphi \in P(A)$, $\varphi = \int^{\oplus} \varphi_\omega d\mu(\omega)$. Then*

(i) $\mathcal{H}_\varphi = \int^{\oplus} \mathcal{H}_{\varphi_\omega} d\mu(\omega)$;

(ii) $\mathcal{N}_\varphi \subset \int^{\oplus} \mathcal{N}_{\varphi_\omega} d\mu(\omega)$;

(iii) $D(\mathcal{H};\varphi) \subset \{\xi = \int^{\oplus} \xi_\omega d\mu(\omega) : \xi_\omega \in D(\mathcal{H}_{\varphi_\omega};\varphi_\omega)$ *a.e.*$\}$.

Proof. (i) By [13, Cor. 2.7] $\mathcal{A} := \int^{\oplus} \Lambda_{\varphi_\omega}(\mathcal{N}_{\varphi_\omega} \cap \mathcal{N}^*_{\varphi_\omega})d\mu(\omega)$ is a full left Hilbert algebra, dense in $\mathcal{H}$, and whose left von Neumann algebra is A [13, Th. 2.5]. Besides φ is the weight on A determined by $\mathcal{A}$, by [22, Th. A.6], so that $\mathcal{A} = \Lambda_\varphi(\mathcal{N}_\varphi \cap \mathcal{N}^*_\varphi)$ and the thesis follows.
(ii) Let $x = \int^{\oplus} x_\omega d\mu(\omega) \in \mathcal{N}_\varphi$; then $\int \varphi_\omega(x^*_\omega x_\omega)d\mu(\omega) = \varphi(\int^{\oplus} x^*_\omega x_\omega d\mu(\omega)) = \varphi(x^*x) < \infty$ so that $\varphi_\omega(x^*_\omega x_\omega) < \infty$ a.e., that is $x_\omega \in \mathcal{N}_{\varphi_\omega}$ a.e.
(iii) Let $\xi = \int^{\oplus} \xi_\omega d\mu(\omega) \in D(\mathcal{H};\varphi)$; then $\exists C > 0$ s.t. $\|x\xi\|^2 \leq C\varphi(x^*x),\ x \in \mathcal{N}_\varphi$, that is $\int \|x_\omega\xi_\omega\|^2 d\mu(\omega) \leq C \int \varphi_\omega(x^*_\omega x_\omega)d\mu(\omega),\ x \in \mathcal{N}_\varphi$. Then in particular $\|x_\omega\xi_\omega\|^2 \leq C\varphi_\omega(x^*_\omega x_\omega),\ x_\omega \in \mathcal{N}_{\varphi_\omega} \cap \mathcal{N}^*_{\varphi_\omega}$ a.e. Let now $\{u_{\omega i}\}_{i\in I} \subset \mathcal{M}_{\varphi_\omega +}$ be s.t. $u_{\omega i} \nearrow 1$. Then $\|\Lambda_{\varphi_\omega}(x_\omega) - \Lambda_{\varphi_\omega}(u_{\omega i}x_\omega)\| \to 0$ and $u_{\omega i}x_\omega \in \mathcal{M}_{\varphi_\omega}$ [21, 2.2] so that $\|x_\omega\xi_\omega\|^2 \leq C\varphi_\omega(x^*_\omega x_\omega),\ x_\omega \in \mathcal{N}_{\varphi_\omega}$ a.e., that is $\xi_\omega \in D(\mathcal{H}_{\varphi_\omega};\varphi_\omega)$ a.e.

$\square$

3.3 Lemma. *Let $A = \int^{\oplus} A_\omega d\mu(\omega)$ be a von Neumann algebra, $\varphi = \int^{\oplus} \varphi_\omega d\mu(\omega) \in P(A)$, $\psi = \int^{\oplus} \psi_\omega d\mu(\omega) \in P(A')$, where the decompositions are w.r.t. $L^\infty(\Omega, \mu) \cong Z \subset Z(A)$. Then $\frac{d\varphi}{d\psi} = \int^{\oplus} \frac{d\varphi_\omega}{d\psi_\omega} d\mu(\omega)$.*

Proof. Let $\xi = \int^{\oplus} \xi_\omega d\mu(\omega) \in D(\mathcal{H};\psi)$ and $x' = \int^{\oplus} x'_\omega d\mu(\omega) \in \mathcal{N}_\psi$; then

$$\int^{\oplus} R^{\psi_\omega}(\xi_\omega)d\mu(\omega)\Lambda_\psi(x') = \int^{\oplus} R^{\psi_\omega}(\xi_\omega)\Lambda_{\psi_\omega}(x'_\omega)d\mu(\omega) =$$

$$= \int^{\oplus} x'_\omega \xi_\omega d\mu(\omega) = x'\xi = R^\psi(\xi)\Lambda_\psi(x')$$

so that $\int^{\oplus} R^{\psi_\omega}(\xi_\omega)d\mu(\omega) = R^\psi(\xi),\ \xi \in D(\mathcal{H};\psi)$.
Then $\vartheta^\psi(\xi,\xi) = \int^{\oplus} \vartheta^{\psi_\omega}(\xi_\omega,\xi_\omega)d\mu(\omega),\ \xi \in D(\mathcal{H};\psi)$, whence

$$(\frac{d\varphi}{d\psi}\xi,\xi) = \varphi(\vartheta^\psi(\xi,\xi)) = \int \varphi_\omega(\vartheta^{\psi_\omega}(\xi_\omega,\xi_\omega))d\mu(\omega) =$$

$$= \int (\frac{d\varphi_\omega}{d\psi_\omega}\xi_\omega, \xi_\omega) d\mu(\omega) = (\int^\oplus \frac{d\varphi_\omega}{d\psi_\omega} d\mu(\omega)\xi, \xi)$$

for $\xi \in D(\mathcal{H};\psi)$. Therefore $\frac{d\varphi}{d\psi} \geq \int^\oplus \frac{d\varphi_\omega}{d\psi_\omega} d\mu(\omega)$.
Let now $\omega \in \Omega \to \xi_\omega \in \prod_{\omega'} D(\mathcal{H}_{\omega'};\psi_{\omega'})$ be a measurable field, and let, $\forall n \in \mathbf{N}$, $\Omega_n := \{\omega \in \Omega : \|x'_\omega \xi_\omega\|^2 \leq n\psi_\omega({x'_\omega}^* x'_\omega),\ x'_\omega \in \mathcal{N}_{\psi_\omega}\}$; then $\cup\Omega_n = \Omega$.
Let us observe that $\forall\Gamma$ Borel subset of Ω_n we get $\int^\oplus \chi_\Gamma(\omega)\xi_\omega d\mu(\omega) \in D(\mathcal{H};\psi)$ so that, from what we have proved above, we get $\int_\Gamma((\frac{d\varphi}{d\psi})_\omega \xi_\omega, \xi_\omega)d\mu(\omega) = \int_\Gamma(\frac{d\varphi_\omega}{d\psi_\omega}\xi_\omega, \xi_\omega)d\mu(\omega)$, hence $((\frac{d\varphi}{d\psi})_\omega \xi_\omega, \xi_\omega) = (\frac{d\varphi_\omega}{d\psi_\omega}\xi_\omega, \xi_\omega)$ a.e. Therefore $\frac{d\varphi_\omega}{d\psi_\omega} \geq (\frac{d\varphi}{d\psi})_\omega$ a.e., so that $\frac{d\varphi}{d\psi} \leq \int^\oplus \frac{d\varphi_\omega}{d\psi_\omega} d\mu(\omega)$ and the thesis follows.

□

3.4 Theorem. *Let $A \subset B$ be von Neumann algebras, $E \in E(B,A)$, and let $A = \int^\oplus A_\omega d\mu(\omega)$, $B = \int^\oplus B_\omega d\mu(\omega)$, $E = \int^\oplus E_\omega d\mu(\omega)$ their decompositions w.r.t. $L^\infty(\Omega,\mu) \cong Z \subset Z(A) \wedge Z(B)$. Then $E^{-1} = \int^\oplus E_\omega^{-1} d\mu(\omega)$ and in particular $Ind(E) = \int^\oplus Ind(E_\omega) d\mu(\omega)$.*

Proof. As

$$\frac{d\varphi_\omega}{d\psi_\omega \cdot E_\omega^{-1}} = \frac{d\varphi_\omega \cdot E_\omega}{d\psi_\omega} = \frac{d(\varphi \cdot E)_\omega}{d\psi_\omega} = \Big(\frac{d\varphi \cdot E}{d\psi}\Big)_\omega =$$

$$= \Big(\frac{d\varphi}{d\psi \cdot E^{-1}}\Big)_\omega = \frac{d\varphi_\omega}{d(\psi \cdot E^{-1})_\omega} = \frac{d\varphi_\omega}{d\psi_\omega \cdot (E^{-1})_\omega}$$

from uniqueness of decomposition of $\frac{d\varphi}{d\psi \cdot E^{-1}}$, we get $(E^{-1})_\omega = E_\omega^{-1}$ a.e. and the thesis follows.

□

Recall that, when the two algebras A and B have finite dimensional centres, J.F. Havet proved in [5, th. 2.9] the existence of a minimal expectation.

3.5 Theorem. *Let $A \subset B$ be von Neumann algebras with finite dimensional centres, and exists $E \in E(B,A)$ with finite index. Setting $\mu := \min\{\|Ind(F)\| : F \in E(B,A)\}$, $\exists\, !E_m \in E(B,A)$ s.t.*
(i) $\|Ind(E_m)\| = \mu$
(ii) $\forall\ F \in E(B,A)$ s.t. $\|Ind(F)\| = \mu$ there follows $Ind(E_m) \leq Ind(F)$.
Besides $Ind(E_m) \in Z(A) \cap Z(B)$ and $E_m(xy) = E_m(yx),\ x \in A' \wedge B,\ y \in B$.

□

3.6. Proposition. *Let $A \subset B$ be von Neumann algebras, $E \in E(B,A)$, $A = \int^\oplus A_\omega d\mu(\omega)$, $B = \int^\oplus B_\omega d\mu(\omega)$, $E = \int^\oplus E_\omega d\mu(\omega)$ their decompositions with respect to*

$L^\infty(\Omega,\mu) \cong Z \subset Z(A) \wedge Z(B)$. *Suppose that* $dim(Z(A_\omega)) < \infty$, $dim(Z(B_\omega)) < \infty$ a.e. *and that* E_ω *is the unique expectation of theorem 3.5. Set* $m(z) := min\{\|Ind(F)z\| : F \in E(B,A)\}$, *for all* $z \in Proj(Z)$. *Then* E *is the unique expectation in* $E(B,A)$ *s.t.*

(*i*) $\|Ind(E)z\| = m(z),\ \forall z \in Proj(Z)$

(*ii*) $\forall\ F \in E(B,A)$ *s.t.* $\|Ind(F)z\| = m(z),\ \forall z \in Proj(Z)$, *there follows* $Ind(E) \leq Ind(F)$.

Proof. (*i*) Let $F = \int^\oplus F_\omega d\mu(\omega) \in E(B,A)$ and $z \in Proj(Z)$ corresponding to the Borel subset $\Gamma \subset \Omega$; then $\|Ind(E)z\| = \text{ess sup}_\Gamma \|Ind(E_\omega)\| \leq \text{ess sup}_\Gamma \|Ind(F_\omega)\| = \|Ind(F)z\|$.
(*ii*) Let now $F \in E(B,A)$ be s.t. $\|Ind(F)z\| = m(z),\ z \in Proj(Z)$; then $\|Ind(E_\omega)\| = \|Ind(F_\omega)\|$ a.e., from standard measure theoretic arguments. Therefore, from 3.5, $Ind(E_\omega) \leq Ind(F_\omega)$ a.e. and the thesis follows.
Uniqueness follows from 3.5.

□

3.7 Definition. *We call* E *of proposition 3.6 the minimal conditional expectation in* $E(B,A)$.

4. The modular group.

The first part of this section is taken from an unpublished manuscript of R. Longo. First of all, let us review some notions and results from [20] that we will use throughout the following.

Let M be a properly infinite von Neumann algebra; then a norm closed linear subspace H of M is said to be a Hilbert space in M if $a \in H$ implies $a^*a \in \mathbf{C}$, and $x \in M$, $xa = 0$, $\forall a \in H$ implies $x = 0$. Let us denote with $\mathcal{H}(M)$ the set of all Hilbert spaces in M.

A unital normal endomorphism $\rho \in End(M)$ is said to be inner if there exists $H \in \mathcal{H}(M)$ such that, if $\{v_i : i \in I\}$ is a basis of H, $\rho(x) = \sum_{i\in I} v_i x v_i^*$, where the series is strongly summable. In this case we write $\rho = \rho_H$ and it follows that ρ_H is faithful and we have the isomorphism $M \cong \rho_H(M) \otimes (H,H)$, where $(H,H) := \{x \in M : a^*xb \in \mathbf{C}, \forall a,b \in H\} \cong \mathcal{B}(H)$.

Let now G be a separable compact group, and $\alpha : G \to Aut(M)$ be a continuous action. We recall that α is said to be dominant if (i) $M^\alpha := \{x \in M : \alpha_g(x) = x, \forall g \in G\}$ is properly infinite, and (ii) $\forall \pi \in \hat{G}$ there exists $H \in \mathcal{H}^\alpha(M)$, the set of α-invariant Hilbert spaces in M, such that $\alpha|_H \cong \pi$. Then, from [20, th. 6.5], we have $M = \langle M^\alpha, \mathcal{H}^\alpha(M)\rangle$ the von Neumann algebra generated by M^α and the set $\mathcal{H}^\alpha(M)$.

In this section we want to describe the modular structure of M, knowing that of M^α. To begin with, let $H \in \mathcal{H}^\alpha(M)$ be a Hilbert space in M, invariant with respect to α, and $\rho = \rho_H$ be the corresponding inner endomorphism. Then we have:

4.1 Lemma. *With the above notation, if $\{v_i, i \in I\}$ is a basis of H, the tensor product decomposition $M \cong \rho_H(M) \otimes (H, H)$ is given by*

$$x = \sum_{i,j \in I} \rho(x_{ij}) v_i v_j^*,$$

where $x_{ij} := v_i^* x v_j$, and the series is strongly summable.

Proof.

$$\sum_{i,j \in I} \rho(x_{ij}) v_i v_j^* = \sum_{i,j \in I} \sum_{k \in I} v_k x_{ij} v_k^* v_i v_j^* = \sum_{i,j \in I} v_i x_{ij} v_j^* = \sum_{i,j \in I} v_i v_i^* x v_j v_j^* = x.$$

$\square$

4.2 Remark. If we set $d := dimH$ and $\Phi : x \in M \to [v_i^* x v_j] \in Mat_d(M)$, then from lemma 4.1 we get

$$x = \sum_{i,j \in I} \rho(\Phi(x)_{ij}) v_i v_j^*.$$

Therefore the inverse of Φ is given by $\Phi^{-1}[x_{ij}] = \sum_{i,j \in I} v_i x_{ij} v_j^* = \sum_{i,j \in I} \rho(x_{ij}) v_i v_j^*$. Note also that $\Phi \cdot \rho(x) = x \otimes 1$, and $\Phi^{-1}(X) = \sum_{i,j \in I} \rho \otimes id(X)_{ij} v_i v_j^*$, where we have used the identification $Mat_d(M) \cong M \otimes Mat_d(\mathbf{C})$.

Denote by $\varepsilon : M \to M^\alpha$ the normal faithful conditional expectation given by $\varepsilon := \int_G \alpha_g(\cdot) dg$, and suppose we are given a normal faithful state φ_0 on M^α with modular group σ^{φ_0}; we wish to describe σ^φ, where $\varphi := \varphi_0 \cdot \varepsilon$ is a faithful normal state on M.
Let $H \in \mathcal{H}^\alpha(M)$; since σ^φ commutes with α, as φ is α-invariant, we get $\sigma^\varphi(H) \in \mathcal{H}^\alpha(M)$. Set $u_t := \sum_i v_i \sigma_t^\varphi(v_i)^*$, where $\{v_i, i \in I\}$ is a basis of H. Then $\sigma_t^\varphi(v) = u_t^* v, \forall v \in H$.

4.3 Lemma. *With the above notation, u_t is a unitary σ^φ-cocycle that does not depend on the basis of H.*

Proof.

$$u_{t+s}^* v = \sigma_{t+s}^\varphi(v) = \sigma_t^\varphi(\sigma_s^\varphi(v)) = \sigma_t^\varphi(u_s^* v) = \sigma_t^\varphi(u_s^*) u_t^* v, \forall v \in H.$$

Thus $u_{t+s}^* = \sigma_t^\varphi(u_s^*) u_t^*$ or $u_{t+s} = u_t \sigma_t^\varphi(u_s)$. Formula $\sigma_t^\varphi(v) = u_t^* v$ shows that u_t does not depend on the basis $\{v_i\}$ but only on H.

$\square$

Since u_t is a cocycle, by a result of Connes' [21] there is a positive linear functional φ' on M such that $\sigma_t^{\varphi'} = u_t \sigma_t^{\varphi}(\cdot) u_t^*$. We want to write φ' explicitely.

4.4 Proposition. *If u_t is the cocycle of lemma 4.3, then the positive linear functional φ' on M such that $\sigma_t^{\varphi'} = u_t \sigma_t^{\varphi}(\cdot) u_t^*$, is given by $\varphi'(x) := \varphi(\sum_i v_i^* x v_i)$.*

Proof. Indeed

$$\sigma_t^{\varphi'}(x) = u_t \sigma_t^{\varphi}(x) u_t^* = \sum_{i,j} v_i \sigma_t^{\varphi}(v_i^*) \sigma_t^{\varphi}(x) \sigma_t^{\varphi}(v_j) v_j^* =$$

$$= \sum_{i,j} v_i \sigma_t^{\varphi}(v_i^* x v_j) v_j^* = \sum_{i,j} v_i (\sigma_t^{\varphi} \otimes id)(\Phi(x))_{ij} v_j^* = \Phi^{-1} \cdot (\sigma_t^{\varphi} \otimes id) \cdot \Phi(x).$$

Thus $\sigma^{\varphi'} = \Phi^{-1} \cdot \sigma^{\varphi \otimes tr} \cdot \Phi$, where tr is the usual trace on $Mat_d(\mathbf{C})$, $tr[a_{ij}] := \sum_i a_{ii}$. Let us now verify that $\varphi' = (\varphi \otimes tr) \cdot \Phi$, that is

$$\varphi'(x) = (\varphi \otimes tr) \cdot \Phi(x) = \varphi \otimes tr[v_i^* x v_j] = \varphi(\sum_i v_i^* x v_i).$$

Indeed from KMS formulas we get

$$F(t) = \varphi'(u_t x) = \varphi'(\sum_i v_i \sigma_t^{\varphi}(v_i)^* x) = \sum_{i,j} \varphi(v_j^* v_i \sigma_t^{\varphi}(v_i)^* x v_j) = \sum_i \varphi(\sigma_t^{\varphi}(v_i)^* x v_i)$$

and

$$F(t+i) = \sum_i \varphi(x v_i \sigma_t^{\varphi}(v_i)^*) = \varphi(x u_t).$$

$\square$

Let us observe that φ' is α-invariant; therefore, if we set $\varphi_0' := \varphi'|_{M^\alpha}$, we get $u_t = (D\varphi' : D\varphi)_t = (D\varphi_0' : D\varphi_0)_t \in M^\alpha$. To determine u_t we need a more convenient expression for φ'.

4.5 Proposition. *The functional $\varphi' \in M_{*+}$ of proposition 4.4 is given by $\varphi' = d\varphi \cdot \rho^{-1} \cdot E_\tau$, where $E_\tau \in E(M, \rho(M))$ is given by $E_\tau(x \otimes y) = \tau(y)x$, $\forall x \in \rho(M)$, $y \in \mathcal{B}(H)$, in the isomorphism $M \cong \rho(M) \otimes \mathcal{B}(H)$ and $\tau := \frac{1}{d} tr$.*

Proof. Indeed

$$\varphi'(\rho(x)) = \varphi(\sum_i v_i^* \rho(x) v_i) = \varphi(\sum_{i,j} v_i^* v_j x v_j^* v_i) = d\varphi(x),$$

thus $\varphi'|_{\rho(M)} = d\varphi \cdot \rho^{-1}|_{\rho(M)}$. On the other hand

$$\varphi'(v_i v_j^*) = \varphi(\sum_k v_k^* v_i v_j^* v_k) = \delta_{ij},$$

that is to say $\varphi'|_{\rho(M)' \wedge M} = tr$.
So, if we denote with E_τ the normal conditional expectation from M to $\rho(M)$ given by τ, we get

$$\varphi' = d\varphi \cdot \rho^{-1} \cdot E_\tau.$$

□

4.6 Remark. As follows from lemma 4.1, $\forall x \in M$, $x = \sum_{i,j=1}^d \rho(v_i^* x v_j) v_i v_j^*$, so that $E_\tau(x) = \sum_{i,j=1}^d \rho(v_i^* x v_j) E_\tau(v_i v_j^*) = \frac{1}{d} \sum_{i=1}^d \rho(v_i^* x v_i)$.

4.7 Lemma.
(i) $\alpha_g \rho = \rho \alpha_g$, $\forall g \in G$, *so that* $\rho(M^\alpha) \subset M^\alpha$;
(ii) $\alpha_g E_\tau = E_\tau \alpha_g$, $\forall g \in G$, *so that* $E_\tau(M^\alpha) \subset M^\alpha$;
(iii) $\forall E \in E(M, \rho(M))$, *s.t.* $\alpha_g E = E \alpha_g,\ g \in G$, *one has* $E \cdot \varepsilon = \varepsilon \cdot E$.

Proof. (i)

$$\alpha_g(\rho(x)) = \alpha_g(\sum_i v_i x v_i^*) = \sum_i \alpha_g(v_i) \alpha_g(x) \alpha_g(v_i)^* = \rho(\alpha_g(x)).$$

(ii) Recall that, in the isomorphism $M \cong \rho(M) \otimes \mathcal{B}(H)$, we have $\alpha \cong \alpha \otimes ad\pi$, $E_\tau(x \otimes y) = \tau(y)x$, $\forall x \in \rho(M)$, $\forall y \in \mathcal{B}(H)$. Then

$$\alpha_g \cdot E_\tau(x \otimes y) = \tau(y)\alpha_g(x) = \tau(\pi(g) y \pi(g)^*)\alpha_g(x) =$$

$$\tau(ad\pi(g)(y))\alpha_g(x) = E_\tau(\alpha_g(x) \otimes ad\pi(g)(y)) = E_\tau \cdot (\alpha_g \otimes ad\pi(g))(x \otimes y),$$

from which the thesis follows.
(iii)

$$E \cdot \varepsilon(x) = E \int_G \alpha_g(x) dg = \int_G E \alpha_g(x) dg = \int_G \alpha_g E(x) dg = \varepsilon \cdot E(x), \forall x \in M.$$

□

As follows from proposition 4.5, to determine u_t we have to characterize $E_\tau|_{M^\alpha}$. For the time being, let us state a simple condition which allows us to uniquely determine $E_\tau|_{M^\alpha}$.

4.8 Proposition. *If $M^{\alpha\prime} \wedge M = \mathbf{C}$, then $\rho(M^\alpha)' \wedge M^\alpha = \mathbf{C}$. Then the restriction of E_τ to M^α is unique.*

Proof. Let $x \in \rho(M^\alpha)' \wedge M^\alpha$; then $x\rho(y) = \rho(y)x, \forall y \in M^\alpha$, *i.e.*

$$\sum_i x v_i y v_i^* = \sum_j v_j y v_j^* x.$$

Multiplying both sides of this equality by v_h^* on the left and by v_k on the right, we get

$$v_h^* x v_k y = y v_h^* x v_k, \forall y \in M^\alpha,$$

that is $v_h^* x v_k = \Phi(x)_{hk} \in M^{\alpha\prime} \wedge M$.
Therefore $\Phi(x) \in (M^{\alpha\prime} \wedge M) \otimes Mat_d(\mathbf{C}), \forall x \in \rho(M^\alpha)' \wedge M^\alpha$.
Denote with $\pi(g) := \alpha_g|_H$, where $H \in \mathcal{H}^\alpha(M)$ is the Hilbert space in M implementing ρ, and recall that π is irreducible; then $\Phi \cdot \alpha \cdot \Phi^{-1}|_{\mathcal{B}(H)} = ad\pi$, and $\Phi(M^\alpha) = M \otimes Mat_d(\mathbf{C})^{\alpha \otimes ad\pi}$.
Therefore

$$\Phi(\rho(M^\alpha)' \wedge M^\alpha) = (M^{\alpha\prime} \wedge M) \otimes Mat_d(\mathbf{C}) \wedge M \otimes Mat_d(\mathbf{C})^{\alpha \otimes ad\pi} =$$

$$= (M^{\alpha\prime} \wedge M) \otimes Mat_d(\mathbf{C})^{\alpha \otimes ad\pi}.$$

Now, as $M^{\alpha\prime} \wedge M = \mathbf{C}$, we get $\rho(M^\alpha)' \wedge M^\alpha = \mathbf{C}$.

□

To determine $E_\tau|_{M^\alpha}$ in the general case we need some preparation.

Let $A \subset B$ be properly infinite von Neumann algebras, G a separable compact group, $\alpha : G \to Aut(B, A)$ a dominant action.

4.9 Lemma. *With the previous notation, $\forall \mathcal{E} \in E(B^\alpha, A^\alpha)$, $\exists E \in E(B, A)$ such that $E\alpha_g = \alpha_g E,\ g \in G$, and $E|_{B^\alpha} = \mathcal{E}$.*

Proof. From [20, th. 6.5] and [18, th. IV.4.8] there exists an isomorphism $\psi : \{B, \alpha\} \to \{B^\alpha \times_\delta G, \hat{\delta}\}$, where δ is a dual coaction. If we identify $B = B^\alpha \times_\delta G$, $\alpha = \hat{\delta}$, then $A = A^\alpha \times_\delta G$, and, [8, §5], there exists $E := \mathcal{E} \otimes id_{\mathcal{B}(L^2(G))}|_B \in E(B, A)$, such that $E\alpha_g = \alpha_g E,\ g \in G$, and $E|_{B^\alpha} = \mathcal{E}$.

□

4.10 Definition. *We call E the canonical lifting of $\mathcal{E}$ [8]. Besides we call any $E \in E(B, A)$ such that $E\alpha_g = \alpha_g E,\ g \in G$, and $E|_{B^\alpha} = \mathcal{E}$, a lifting of $\mathcal{E}$.*

4.11 Lemma. *Let*

$$\begin{array}{ccc} A & \subset & B \\ \cup & & \cup \\ A_0 & \subset & B_0 \end{array}$$

be von Neumann algebras, $E \in E(B,A)$, $\mathcal{E} := E|_{B_0} \in E(B_0, A_0)$, $Ind(E) \in \overline{Z(B_0)}_+$. Then $Ind(E) \geq Ind(\mathcal{E})$.

Proof. Let $\varphi \in E(A)$, $\psi \in E(B_0')$, then we have

$$\frac{d\varphi}{d\psi|_{B'} \cdot E^{-1}} = \frac{d\varphi \cdot E}{d\psi|_{B'}} \leq \frac{d(\varphi \cdot E)|_{B_0}}{d\psi} = \frac{d\varphi|_{A_0} \cdot \mathcal{E}}{d\psi} = \frac{d\varphi|_{A_0}}{d\psi \cdot \mathcal{E}^{-1}}$$

where the inequality follows from [7, lemma 1.8]. Then $\forall \xi \in D(\mathcal{H}; \varphi) \subset D(\mathcal{H}; \varphi|_{A_0})$ we have $\psi \cdot \mathcal{E}^{-1}(\theta^{\varphi|_{A_0}}(\xi,\xi)) \leq \psi|_{B'} \cdot E^{-1}(\theta^{\varphi}(\xi,\xi))$. By normality we get $\psi \cdot \mathcal{E}^{-1}|_{A'} \leq \psi|_{B'} \cdot E^{-1}$, so that, in particular, $\psi(Ind(E)) \geq \psi(Ind(\mathcal{E}))$, $\forall \psi \in E(Z(B_0))$, and, as $Ind(E)$, $Ind(\mathcal{E}) \in \overline{Z(B_0)}_+$, we obtain $Ind(E) \geq Ind(\mathcal{E})$ in $\overline{Z(B_0)}_+$.

□

4.12 Proposition. *Let $\mathcal{E} \in E(B^\alpha, A^\alpha)$, $E \in E(B,A)$ a lifting of $\mathcal{E}$, and suppose E has finite index. If $\alpha|_A$ is dominant, then $Ind(E) = Ind(\mathcal{E})$.*

Proof. Let $\varphi_0 \in E(A^\alpha)$ and $\varphi := \varphi_0 \cdot E \cdot \varepsilon \in E(B)$ (since E and ε commute, $E \cdot \varepsilon \in E(B^\alpha, A^\alpha)$). The modular group σ^φ of φ leaves A, A^α and B^α globally invariant. As $E = \alpha_g \cdot E \cdot \alpha_g^{-1}$ we get $Ind(E) \in Z(B)^\alpha \subset Z(B^\alpha)$. Indeed, if $\alpha_g = adu_g|_B$, let us set $\bar{\alpha}_g := adu_g \in Aut(\mathcal{B}(\mathcal{H}))$. Then by [10, lemma 1.6], $E^{-1} = (\alpha_g \cdot E \cdot \alpha_g^{-1})^{-1} = \bar{\alpha}_g \cdot E^{-1} \cdot \bar{\alpha}_g^{-1}$ so that $Ind(E) = \bar{\alpha}_g(Ind(E)) = \alpha_g(Ind(E))$.
As follows from previous lemma, $Ind(\mathcal{E})$ is finite, so that from proposition 2.5 we get that $\exists b \in B^\alpha$ s.t. $x = b\mathcal{E}(b^* x)$, $\forall x \in B^\alpha$. As $\alpha|_A$ is dominant, $B = \langle B^\alpha, \mathcal{H}^\alpha(A)\rangle$, $A = \langle A^\alpha, \mathcal{H}^\alpha(A)\rangle$. Then $\{b\}$ is a basis for B w.r.t. E, as $\forall H \in \mathcal{H}^\alpha(A)$, $\forall v \in H$ we get $bE(b^* v) = bE(b^*)v = v$. So we obtain $Ind(E) = bb^* = Ind(\mathcal{E})$.

□

4.13 Remark. Let $E \in E(B,A)$; then E is uniquely determined by its action on B^α as $B = \langle B^\alpha, \mathcal{H}^\alpha(A)\rangle$ and $E(v) = v$, $\forall v \in H$, $\forall H \in \mathcal{H}^\alpha(A)$. Hence $\forall \mathcal{E} \in E(B^\alpha, A^\alpha)$ there is a unique lifting $E \in E(B,A)$.

We can finally come to the characterization of the restriction, to the fixed-point algebra, of $E_\tau \in E(M, \rho_H(M))$, the conditional expectation given by the trace on $\rho_H(M)' \wedge M$ at least in the case $dim Z(M^\alpha) < \infty$.

Notice that $\rho_H(M)$ and M have the same centre, and that E_τ is the minimal expectation in $E(M, \rho_H(M))$, as defined in 3.6, and has scalar index. Besides $\alpha|_{\rho_H(M)}$ is dominant as ρ_H is an equivariant isomorphism between $\{M, \alpha\}$ and $\{\rho_H(M), \alpha|_{\rho_H(M)}\}$.

4.14 Proposition. *Suppose that $dim Z(M^\alpha) < \infty$ and let E_τ be the minimal expectation in $E(M, \rho(M))$ and $\mathcal{E}_m$ the minimal expectation in $E(M^\alpha, \rho(M^\alpha))$. Then $\mathcal{E}_\tau := E_\tau|_{M^\alpha} = \mathcal{E}_m$.*

Proof. Let E_m be the canonical lifting of $\mathcal{E}_m$ to M. Then from proposition 4.12 we get $\|Ind(E_m)\| = \|Ind(\mathcal{E}_m)\| \le \|Ind(\mathcal{E}_\tau)\| = Ind(\mathcal{E}_\tau) = Ind(E_\tau)$, as E_τ is a lifting of $\mathcal{E}_\tau$, as is easily verified.
As $Ind(E_\tau) \le \|Ind(E_m)\|$ we get $\|Ind(E_m)\| = \|Ind(\mathcal{E}_m)\| = Ind(\mathcal{E}_\tau) = Ind(E_\tau)$. Therefore from 3.5 and 4.12 we get $Ind(E_m) = Ind(\mathcal{E}_m) \le Ind(\mathcal{E}_\tau) = Ind(E_\tau) \le Ind(E_m)$. Hence $E_m \equiv E_\tau$, as E_τ is the unique minimal expectation in $E(M, \rho(M))$. Then $E_\tau|_{M^\alpha} = E_m|_{M^\alpha} = \mathcal{E}_m$.

□

Gathering together what we have found thus far we can state the following

4.15 Theorem. *Let M be a properly infinite von Neumann algebra, α a dominant action of a separable compact group s.t. $dimZ(M^\alpha) < \infty$, $\varepsilon : M \to M^\alpha$ the normal faithful conditional expectation from M to M^α, φ_0 a faithful normal state on M^α and set $\varphi := \varphi_0 \cdot \varepsilon$. Then $\forall H \in \mathcal{H}^\alpha(M)$, α-invariant Hilbert space in M, we get*

$$\sigma_t^\varphi(x) = \sigma_t^{\varphi_0}(x), \quad \forall x \in M^\alpha,$$
$$\sigma_t^\varphi(v) = u_{H,t}^* v, \quad \forall v \in H,$$

where $u_{H,t} := (D\varphi_H : D\varphi_0)_t \in M^\alpha$, $\varphi_H := dim(H)\varphi_0 \cdot \rho_H^{-1} \cdot \mathcal{E}_H$, $\mathcal{E}_H : M^\alpha \to \rho_H(M^\alpha)$ the minimal normal conditional expectation.

4.16 Proposition. *Let M be a properly infinite von Neumann algebra, α a dominant action of a finite group, $\varepsilon : M \to M^\alpha$ the normal faithful conditional expectation from M to M^α, φ_0 a faithful normal state on M^α and set $\varphi := \varphi_0 \cdot \varepsilon$. Then $\forall H \in \mathcal{H}^\alpha(M)$, α-invariant Hilbert space in M, we get*

$$\sigma_t^\varphi(x) = \sigma_t^{\varphi_0}(x), \quad \forall x \in M^\alpha,$$
$$\sigma_t^\varphi(v) = u_{H,t}^* v, \quad \forall v \in H,$$

where $u_{H,t} := (D\varphi_H : D\varphi_0)_t \in M^\alpha$, $\varphi_H := dim(H)\varphi_0 \cdot \rho_H^{-1} \cdot \mathcal{E}_H$, and *$\mathcal{E}_H : M^\alpha \to \rho_H(M^\alpha)$ is the minimal expectation given in proposition 3.6.*

Proof. Let $M = \int^\oplus M_\omega d\mu(\omega)$ be the decomposition of M w.r.t. $Z(M)^\alpha$. Then, as $Z(M)^\alpha = \int^\oplus Z(M_\omega)^{\alpha_\omega} d\mu(\omega)$, we get $Z(M_\omega)^{\alpha_\omega} = \mathbf{C}$ a.e. As $Z(M_\omega)^{\alpha_\omega} \subset Z(M_\omega)$ has finite index because the group is finite, from [1] $dimZ(M_\omega) < \infty$. As the inclusion $M_\omega^{\alpha_\omega} \subset M_\omega$ has finite index again because the group is finite, from [1] $dimZ(M_\omega^{\alpha_\omega}) < \infty$. Let us denote with $\mathcal{E}_{H,\omega} \in E(M_\omega, \rho_{H,\omega}(M_\omega))$ the minimal expectation, and with $\mathcal{E}_H := \int^\oplus \mathcal{E}_{H,\omega} d\mu(\omega)$ the minimal expectation given in proposition 3.6. Then it is easy to see, using arguments similar to those in proposition 4.14, that $\mathcal{E}_H$ is the restriction to M^α of the minimal expectation E_H of $E(M, \rho_H(M))$ and this completes the proof.

$\square$

The lack of a definition of minimal conditional expectation in full generality prevents us to solve the problem in the general case of a dominant action of a separable compact group, so we use another approach, based on [16]. We hope to return to the approach based on the minimal expectation somewhere else.

Let $H \in \mathcal{H}^\alpha(M)$, and $\{v_i : i = 1, \dots, d\}$ be a basis of H; then

$$\vartheta_H := \sum_{i,j=1}^{d} v_i v_j v_i^* v_j^*$$

is independent of the chosen basis; besides $\vartheta_H \in M^\alpha$ is unitary and $v_i^* \vartheta_H v_j = v_j v_i^*$. So that, from remark 4.6, we get $E_\tau(\vartheta_H) = \frac{1}{d}\sum_{i=1}^{d} \rho(v_i^* \vartheta_H v_i) = \frac{1}{d}\sum_{i=1}^{d} \rho(v_i v_i^*) = \frac{1}{d}$.

4.17 Proposition. *Let $F \in E(M, \rho_H(M))$ be s.t. $F(\vartheta_H) = \frac{1}{d}$. Then $F = E_\tau$.*

Proof. As $M \cong \rho_H(M) \otimes \mathcal{B}(H)$, there is a bijective correspondence between $E(M, \rho_H(M))$ and the set of faithful normal states on $\mathcal{B}(H)$, so that there is a unique $A = [a_{hk}] \in Mat_d(\mathbf{C})$, positive definite, with $\tau(A) = 1$, s.t.

$$F(x) = \sum_{i,j=1}^{d} \rho(v_i^* x v_j) F(v_i v_j^*) = \sum_{i,j=1}^{d} \rho(v_i^* x v_j)\tau(A v_i v_j^*).$$

Now, as $\{v_i v_j^*\}$ is a set of matrix units in $Mat_d(\mathbf{C})$, we get

$$A v_i v_j^* = \sum_{h,k=1}^{d} a_{hk} v_h v_k^* v_i v_j^* = \sum_{h=1}^{d} a_{hi} v_h v_j^*$$

and $\tau(A v_i v_j^*) = \frac{1}{d} a_{ji}$, so that $F(x) = \frac{1}{d}\sum_{i,j=1}^{d} a_{ji} \rho(v_i^* x v_j)$ and

$$F(\vartheta_H) = \frac{1}{d}\sum_{i,j=1}^{d} a_{ij} \rho(v_i v_j^*).$$

As $F(\vartheta_H) = \frac{1}{d}$, we get $\sum_{i,j=1}^{d} a_{ij}\rho(v_i v_j^*) = 1$ that is $\sum_{i,j=1}^{d} a_{ij} v_i v_j^* = 1$, whence $a_{ij} = \delta_{ij}$ and $F = E_\tau$.

$\square$

4.18 Proposition. *The restriction $E_\tau|_{M^\alpha}$ of E_τ to M^α is uniquely determined by the condition $E_\tau|_{M^\alpha}(\vartheta_H) = \frac{1}{d}$.*

Proof. It follows from remark 4.13 and the previous proposition.

□

4.19 Theorem. *Let M be a properly infinite von Neumann algebra, α a dominant action of a separable compact group, $\varepsilon : M \to M^\alpha$ the normal faithful conditional expectation from M to M^α, φ_0 a faithful normal state on M^α and set $\varphi := \varphi_0 \cdot \varepsilon$. Then $\forall H \in \mathcal{H}^\alpha(M)$, α-invariant Hilbert space in M, we get*

$$\sigma_t^\varphi(x) = \sigma_t^{\varphi_0}(x), \quad \forall x \in M^\alpha,$$
$$\sigma_t^\varphi(v) = u_{H,t}^* v, \quad \forall v \in H,$$

where $u_{H,t} := (D\varphi_H : D\varphi_0)_t \in M^\alpha$, $\varphi_H := dim(H)\varphi_0 \cdot \rho_H^{-1} \cdot \mathcal{E}_H$, and $\mathcal{E}_H : M^\alpha \to \rho_H(M^\alpha)$ is the unique expectation s.t. $\mathcal{E}_H(\vartheta_H) = dim(H)^{-1}$.

5. A sufficient condition for the conjugate endomorphism.

Before we come to the main result of this section we recall a few notions.

Let A, B be von Neumann algebras, then $\mathcal{H}$ is said to be an $A - B$ correspondence if it is a (separable) Hilbert space where A acts on the left, B on the right, and the actions are normal: we denote with $a\xi b$, $a \in A$, $b \in B$, $\xi \in \mathcal{H}$, the relative actions.
Let $Corr(A, B)$ be the set of $A - B$ correspondences.
Let $\rho : A \to B$ be a normal homomorphism, then we let $\mathcal{H}_\rho$ be the Hilbert space $L^2(B)$ with actions $a\xi b := \rho(a)Jb^*J\xi$, $a \in A$, $b \in B$, $\xi \in L^2(B)$, where J is the modular conjugation of B. Conversely [17, prop. 2.1] if A, B are properly infinite von Neumann algebras, and $\mathcal{H} \in Corr(A, B)$, there is $\rho : A \to B$ normal homomorphism such that $\mathcal{H} \cong \mathcal{H}_\rho$.

Let $\mathcal{H} \in Corr(A, B)$, then the conjugate correspondence $\bar{\mathcal{H}} \in Corr(B, A)$ is given by the complex conjugate Hilbert space $\bar{\mathcal{H}}$ with actions $b\bar{\xi}a := \overline{a^*\xi b^*}$, $a \in A$, $b \in B$, where $\bar{\xi} \in \bar{\mathcal{H}}$ is the conjugate vector of $\xi \in \mathcal{H}$. We say $\sigma \in End(A)$ is a conjugate endomorphism of $\rho \in End(A)$ if $\mathcal{H}_\sigma \cong \bar{\mathcal{H}}_\rho$, and set $\bar{\rho}$ for a conjugate endomorphism. Then, if A is a properly infinite von Neumann algebra, $\rho, \sigma \in End(A)$ are conjugate endomorphisms, by [17, prop. 2.3], iff $\forall \xi, \eta \in L^2(A)$, $\exists \xi', \eta' \in L^2(A)$ s.t. $(\xi, \rho(x)\eta y) = (\eta', x\xi'\sigma(y))$, $\forall x, y \in A$, or, by [17, th. 3.1], iff $\exists \gamma : A \to \rho(A)$ canonical endomorphism, s.t. $\sigma = \rho^{-1} \cdot \gamma$.

Let us state and prove a sufficient condition for two endomorphisms to be conjugate.

5.1 Assumption. Let M be a properly infinite von Neumann algebra, and $\rho, \sigma \in End(M)$ and injective. Let $v, w \in M$ be isometries s.t. $\sigma\rho(x)v = vx$, $\rho\sigma(x)w = wx$, $\forall x \in M$, and $\exists \lambda \in (0, \infty)$ s.t. $w^*\rho(v) = v^*\sigma(w) = \lambda$.

Set $\chi(x) := v^*\sigma(x)v$, $\psi(x) := w^*\rho(x)w$, $\forall x \in M$; then χ, ψ are completely positive normal maps s.t. $\chi\rho(x) = v^*\sigma\rho(x)v = x$, $\psi\sigma(x) = w^*\rho\sigma(x)w = x$, $\forall x \in M$.

5.2 Lemma. *$E := \rho \cdot \chi \in E(M, \rho(M))$, $F := \sigma \cdot \psi \in E(M, \sigma(M))$.*

Proof. We have only to prove faithfulness. Let us set $G(y) := \sigma \cdot E \cdot \sigma^{-1}(y)$ and prove that $G \in E(\sigma(M), \sigma\rho(M))$ from which will follow immediately that $E \in E(M, \rho(M))$. Set $e := vv^* \in Proj(M)$; then $\sigma\rho(x)e = \sigma\rho(x)vv^* = vxv^* = vv^*\sigma\rho(x) = e\sigma\rho(x)$, $\forall x \in M$, that is $e \in \sigma\rho(M)' \wedge M$. Besides

$$e\sigma(x)e = vv^*\sigma(x)vv^* = v\chi(x)v^* = \sigma\rho\chi(x)vv^* = \sigma(E(x))e, \forall x \in M,$$

that is $G(y)e = eye$, $\forall y \in \sigma(M)$.
Now, if $y = \sigma(x) \in \sigma(M)$ and $G(y^*y) = 0$, we have $0 = G(y^*y)e = ey^*ye$, that is $0 = ye = \sigma(x)e$, that is $\sigma(x)v = 0$, that is $\rho\sigma(x)\rho(v) = 0$, which implies $0 = w^*\rho\sigma(x)\rho(v) = xw^*\rho(v) = \lambda x$, that is $x = 0$, which implies $y = 0$ and the faithfulness of G.
The proof of the faithfulness of F is analogous.

$\square$

5.3 Theorem. *Under assumption 5.1 ρ and σ are conjugate endomorphisms and $Ind_E(\rho(M), M) = Ind_F(\sigma(M), M) = \lambda^{-2}$.*

The proof follows closely that of [17, th. 4.1]. We divide it in some lemmas.
Choose $\Omega \in \mathcal{H}$ cyclic and separating for $M, \rho(M), \sigma(M)$; let U, V be the canonical unitary implementations, with respect to Ω, of ρ, and σ respectively; set $J := J_M^\Omega$, $J_\rho := J_{\rho(M)}^\Omega$, $J_\sigma := J_{\sigma(M)}^\Omega$, $\omega := (\Omega, \cdot\Omega) \in E(M)$ and $\varphi := \omega \cdot E \in E(M)$.

5.4 Lemma. *$\varphi(x) = (\Phi, x\Phi)$, where $\Phi := V^*vU^*\Omega$.*

Proof. We have, $\forall x \in M$,

$$\begin{aligned}\varphi(x) &= \omega \cdot E(x) = (\Omega, E(x)\Omega) = (\Omega, \rho(v^*\sigma(x)v)\Omega)\\ &= (\Omega, Uv^*\sigma(x)vU^*\Omega) = (\Omega, Uv^*VxV^*vU^*\Omega) = (V^*vU^*\Omega, xV^*vU^*\Omega).\end{aligned}$$

$\square$

5.5 Lemma. *Φ is cyclic and separating for M.*

Proof. Φ is separating for M because φ is faithful. We want to show that Φ is also cyclic for M. Let us set $\sigma^{-1}(x) := V^*xV$, $\forall x \in M$. Then we get $[\rho(M)\Phi] = \sigma^{-1}(e) \in$

$\rho(M)' \wedge \sigma^{-1}(M)$. Indeed, $\forall x \in M$, we get

$$\begin{aligned}
\rho(x)\Phi &= \rho(x)V^*vU^*\Omega = V^*U^*UV\rho(x)V^*U^*UvU^*\Omega \\
&= V^*U^*\rho\sigma\rho(x)\rho(v)\Omega = V^*U^*\rho(\sigma\rho(x)v)\Omega \\
&= V^*U^*\rho(vx)\Omega = V^*U^*\rho(v)\rho(x)\Omega.
\end{aligned}$$

As Ω is cyclic for $\rho(M)$ we get,

$$\begin{aligned}
[\rho(M)\Phi] &= range(V^*U^*\rho(v)) = V^*U^*\rho(v)\rho(v)^*UV \\
&= V^*U^*\rho(e)UV = V^*eV = \sigma^{-1}(e).
\end{aligned}$$

Finally, let us set $q := [M\Phi]$, and show that $q = 1$. Indeed $q = [M\Phi] \geq [\rho(M)\Phi] = \sigma^{-1}(e)$. But we have $\sigma(w)^*v = \lambda$ so that $w^*\sigma^{-1}(v) = \lambda$ that is $\sigma^{-1}(v)^*w = \lambda$, so that $w^*\sigma^{-1}(e)w = w^*\sigma^{-1}(v)\sigma^{-1}(v)^*w = \lambda^2$ and $q = qw^*w = w^*qw \geq w^*\sigma^{-1}(e)w = \lambda^2 > 0$ which implies $q = 1$, and Φ is cyclic for M.

□

Multiplying UV by a unitary in M', if necessary, we may assume that $\Phi \in L^2(M, \Omega)_+$.

Let us now set $M_1 := \langle M, \sigma^{-1}(e)\rangle$; then, as $[\rho(M)\Phi] = \sigma^{-1}(e)$, we have that $\rho(M) \subset M \subset M_1$ is Jones' basic construction. By applying σ we have $\sigma\rho(M) \subset \sigma(M) \subset \langle\sigma(M), e\rangle \subset M$.

5.6 Lemma. *$M = \langle\sigma(M), e\rangle$, that is $\sigma\rho(M) \subset \sigma(M) \subset M = \langle\sigma(M), e\rangle$ is Jones' basic construction. Besides $Ind_E(\rho(M), M) = \lambda^{-2}$.*

Proof. We want to apply [12, lemma 1]. Remember that in lemma 5.2 we already proved that, with $G(y) := \sigma \cdot E \cdot \sigma^{-1}(y)$, we have $G(y)e = eye$, $\forall y \in \sigma(M)$.
Moreover the central support, $c_M(e)$, of e in M is 1, as $c_M(e) \equiv [Me\mathcal{H}] = [Mvv^*\mathcal{H}] \geq [\sigma(w)^*vv^*v\mathcal{H}] = 1$.
Besides $F(e) = \sigma\psi(e) = \sigma(w^*\rho(e)w) = \sigma(w^*\rho(v)\rho(v)^*w) = \lambda^2$.
Finally, $\forall x \in M$,

$$\begin{aligned}
F(xe)e &= \sigma\psi(xe)e = \sigma(w^*\rho(xe)w)e \\
&= \sigma(w^*\rho(x)\rho(v)\rho(v)^*w)e = \lambda\sigma(w^*\rho(x)\rho(v))e \\
&= \lambda\sigma(w)^*\sigma\rho(x)\sigma\rho(v)vv^* = \lambda\sigma(w)^*\sigma\rho(x)vvv^* \\
&= \lambda\sigma(w)^*vxe = \lambda^2 xe.
\end{aligned}$$

Therefore, by [12, lemma 1], $M = \langle\sigma(M), e\rangle$, and $\lambda^{-2}F = adJ_\sigma \cdot G^{-1} \cdot adJ_\sigma$ so that $\lambda^{-2} = G^{-1}(1) = Ind_G(\sigma\rho(M), \sigma(M)) = Ind_E(\rho(M), M)$.

□

Note that, exchanging the roles of ρ and σ, we also get that $M = \langle \rho(M), ww^* \rangle$ and $Ind_F(\sigma(M), M) = \lambda^{-2}$.

Proof of theorem 5.3.
From the previous lemma it follows that $\sigma^{-1}(M) = \langle M, \sigma^{-1}(e) \rangle = M_1 = J\rho(M)'J$. Therefore $\sigma^{-1}(v), \sigma^{-1}(e) \in M_1 = J\rho(M)'J$, so that $v_0 := J\sigma^{-1}(v)J \in \rho(M)'$. The canonical unitary implementation of the isomorphism $y \in \rho(M) \to y\sigma^{-1}(e) \in \rho(M)\sigma^{-1}(e)$, with respect to Ω and Φ, is given by the isometry $w_0 = v_0 z$, where $z \in \rho(M)'$ is unitary. Then, from [16, prop. 3.1], we get $\Gamma_\rho := J_\rho J = w_0^* J w_0 J = z^* v_0^* J v_0 z J = z^* v_0^* J v_0 J J z J$, thus, to compute the class of $\gamma_\rho := ad\Gamma_\rho$, the canonical endomorphism of M into $\rho(M)$, we may assume $w_0 = v_0$. Then we have, $\forall x \in M$,

$$\begin{aligned}
\Gamma_\rho x \Gamma_\rho^* &= v_0^* J v_0 J x J v_0^* J v_0 \\
&= J\sigma^{-1}(v)^* J\sigma^{-1}(v) x \sigma^{-1}(v)^* J\sigma^{-1}(v) J \\
&= J\sigma^{-1}(v)^* J\sigma^{-1}(v\sigma(x)v^*) J\sigma^{-1}(v) J \\
&= J\sigma^{-1}(v)^* J\sigma^{-1}(vv^*\sigma\rho\sigma(x)) J\sigma^{-1}(v) J \\
&= J\sigma^{-1}(v)^* J\sigma^{-1}(e)\rho\sigma(x) J\sigma^{-1}(v) J \\
&= J\sigma^{-1}(v)^* J\sigma^{-1}(e) J\sigma^{-1}(v) J\rho\sigma(x) \\
&= J\sigma^{-1}(v^* e v) J\rho\sigma(x) \\
&= \rho\sigma(x),
\end{aligned}$$

because $J\sigma^{-1}(v)J \in \rho(M)'$ and $J\sigma^{-1}(e)J = \sigma^{-1}(e)$. Hence we get $[\rho\sigma] = [\gamma_\rho]$, that is ρ and σ are conjugate.

□

6. The modular operators.

Recall the notation of section 4. M is a properly infinite von Neumann algebra, α is a dominant action of a separable compact group G, $\varepsilon : M \to M^\alpha$ the canonical expectation, $\varphi_0 \in E(M^\alpha)$, $\varphi := \varphi_0 \cdot \varepsilon \in E(M)$. Let Ω be a cyclic and separating vector for M representing φ and set $U_g x\Omega := \alpha_g(x)\Omega, \forall x \in M$. Then $g \in G \to U_g \in \mathcal{U}(\mathcal{H})$ is a strongly continuous unitary representation of G on $\mathcal{H}$ and we have the following decomposition for U

$$U = \bigoplus_{\pi \in \hat{G}} n_\pi \pi$$

where $\hat{G}$ is the set of classes of irreducible unitary representations of G modulo unitary equivalence. Let $\mathcal{H} = \bigoplus_{\pi \in \hat{G}} \mathcal{H}_\pi$ be the induced decomposition of $\mathcal{H}$, that is $U|_{\mathcal{H}_\pi} = n_\pi \pi$.

6.1 Definition. *Let* $E_\pi := \int_G \overline{\chi_\pi(g)} U_g dg$, *where* $\chi_\pi(g) := d_\pi tr(\pi(g))$, $d_\pi := deg\pi$; *then* $E_\pi \in \mathcal{B}(\mathcal{H})$ *is an orthogonal projection and* $E_\pi \mathcal{H} = \mathcal{H}_\pi$.
Set $\varepsilon_\pi := \int_G \overline{\chi_\pi(g)} \alpha_g(\cdot) dg$; *then* $\varepsilon_\pi(x)\Omega = E_\pi x \Omega$, $\forall x \in M$.
Set $M_\pi := \varepsilon_\pi(M)$; *then* $\mathcal{H}_\pi = [M_\pi \Omega]$.
Finally set $E_0 := \int_G U_g dg$, *projection on the* α*-invariant vectors, and* $\mathcal{H}_0 := E_0 \mathcal{H} = [M^\alpha \Omega]$.

Now let us choose, $\forall \pi \in \hat{G}$, an α-invariant Hilbert space H_π in M, such that $\alpha|_{H_\pi} \cong \pi$, and let $\{v_{\pi k} : k = 1, \ldots, d_\pi\}$ be a basis for H_π; then

$$E_{\bar{\pi}} = d_\pi \sum_{k=1}^{d_\pi} v_{\pi k}^* E_0 v_{\pi k},$$

where $\bar{\pi} \in \hat{G}$ is a conjugate representation of π [15].
With the previous notation, we have

6.2 Lemma.
(i) $M_{\bar{\pi}} = H_\pi^* M^\alpha$ *and* $x = d_\pi \sum_{k=1}^{d_\pi} v_{\pi k}^* \varepsilon(v_{\pi k} x), \forall x \in M_{\bar{\pi}}$;
(ii) $M_{\bar{\pi}} = M^\alpha H_{\bar{\pi}}$ *and* $x = d_\pi \sum_{k=1}^{d_\pi} \varepsilon(v_{\bar{\pi} k}^* x) v_{\bar{\pi} k}^*, \forall x \in M_{\bar{\pi}}$.

Proof. (i) Let us prove first that $M_{\bar{\pi}} \subset H_\pi^* M^\alpha$. Indeed, $\forall x \in M_{\bar{\pi}}$, we have

$$x = \varepsilon_{\bar{\pi}}(x) = \int_G \chi_\pi(g) \alpha_g(x) dg = d_\pi \sum_{k=1}^{d_\pi} v_{\pi k}^* \int_G \alpha_g(v_{\pi k}) \alpha_g(x) dg = d_\pi \sum_{k=1}^{d_\pi} v_{\pi k}^* \varepsilon(v_{\pi k} x).$$

Then we prove $H_\pi^* M^\alpha \subset M_{\bar{\pi}}$; indeed, $\forall v_{\pi k}^* x_k \in H_\pi^* M^\alpha$, we have

$$\begin{aligned} \varepsilon_{\bar{\pi}}(v_{\pi k}^* x_k) &= \int_G \chi_\pi(g) \alpha_g(v_{\pi k}^* x_k) dg = d_\pi \sum_{l,m=1}^{d_\pi} \int_G \pi(g)_{ll} \alpha_g(v_{\pi k})^* v_{\pi m} v_{\pi m}^* dg x_k \\ &= d_\pi \sum_{l,m=1}^{d_\pi} \int_G \pi(g)_{ll} \overline{\pi(g)_{mk}} dg v_{\pi m}^* x_k = \sum_{l,m=1}^{d_\pi} \delta_{lm} \delta_{lk} v_{\pi m}^* x_k = v_{\pi k}^* x_k \end{aligned}$$

and, by linearity, we are through.
(ii) is analogous.

□

For the sake of completeness we report here the proof of a result, which is part of a stronger one in [4], that we will use repeatedly in the sequel.

6.3 Proposition. *For every $\pi \in \hat{G}$, there exist $\{V_{\pi k} : k = 1, \dots, d_\pi\}$ isometries from $\mathcal{H}_0$ to $\mathcal{H}_\pi$, with mutually orthogonal ranges, such that $\sum_{k=1}^{d_\pi} V_{\pi k} V_{\pi k}^* = E_\pi$.*

Proof. For the sake of brevity, let us drop suffix π from what follows.
Let $\{\xi_1, \dots, \xi_d\} \subset \mathcal{H}_\pi$ be such that $U_g \xi_i = \sum_{j=1}^d \xi_j \pi(g)_{ji}$, and let $\{\bar{v}_1, \dots, \bar{v}_d\}$ be a basis of $H_{\bar{\pi}}$ such that $\alpha_g(\bar{v}_i) = \sum_{j=1}^d \bar{v}_j \bar{\pi}(g)_{ji}$.
Then $\xi := \sum_{i=1}^d \bar{v}_i \xi_i \in \mathcal{H}_0$: indeed

$$U_g \xi = \sum_{i=1}^d U_g \bar{v}_i \xi_i = \sum_{i=1}^d U_g \bar{v}_i U_g^* U_g \xi_i = \sum_{i=1}^d \alpha_g(\bar{v}_i) U_g \xi_i =$$

$$= \sum_{i=1}^d \sum_{j=1}^d \bar{v}_j \bar{\pi}(g)_{ji} \sum_{k=1}^d \xi_k \pi(g)_{ki} = \sum_{j,k=1}^d \bar{v}_j \xi_k \sum_{i=1}^d \pi(g)_{ki} \pi(g^{-1})_{ij} = \sum_{k=1}^d \bar{v}_k \xi_k = \xi.$$

Thus $\xi_i = \bar{v}_i^* \xi \in \bar{v}_i^* \mathcal{H}_0$.
Besides $\xi \in \mathcal{H}_0 \Rightarrow \bar{v}_i^* \xi \in \mathcal{H}_\pi$ as

$$U_g \bar{v}_i^* \xi = \alpha_g(\bar{v}_i)^* \xi = \sum_{j=1}^d \overline{\bar{\pi}(g)_{ji}} \bar{v}_j^* \xi = \sum_{j=1}^d \pi(g)_{ji} \bar{v}_j^* \xi.$$

Finally $\forall \xi, \eta \in \mathcal{H}_0$ we get

$$(\bar{v}_i^* \xi, \bar{v}_j^* \eta) = (\xi, \bar{v}_i \bar{v}_j^* \eta) = (\xi, E_0 \bar{v}_i \bar{v}_j^* E_0 \eta) = (\xi, \varepsilon(\bar{v}_i \bar{v}_j^*) \eta) =$$

$$= \int_G (\xi, \alpha_g(\bar{v}_i \bar{v}_j^*) \eta) dg = \sum_{l,m=1}^d \int_G (\alpha_g(\bar{v}_i)^* \bar{v}_l \bar{v}_l^* \xi, \alpha_g(\bar{v}_j)^* \bar{v}_m \bar{v}_m^* \eta) dg =$$

$$= \sum_{l,m=1}^d \int_G \overline{\bar{\pi}(g)_{il}} \bar{\pi}(g)_{jm} (\bar{v}_l^* \xi, \bar{v}_m^* \eta) dg = d^{-1} \sum_{l,m=1}^d \delta_{ij} \delta_{lm} (\bar{v}_l^* \xi, \bar{v}_m^* \eta) =$$

$$= d^{-1} \delta_{ij} \sum_{m=1}^d (\bar{v}_m \bar{v}_m^* \xi, \eta) = d^{-1} \delta_{ij} (\xi, \eta).$$

Therefore, if we set $V_i := \sqrt{d} \bar{v}_i^* |_{\mathcal{H}_0}$, we get
(i) $V_i : \mathcal{H}_0 \to \mathcal{H}_\pi$,
(ii) $(V_i \xi, V_j \eta) = \delta_{ij} (\xi, \eta)$, that is $\{V_i\}$ are isometries with mutually orthogonal ranges, and
(iii) $\mathcal{H}_\pi = \sum_{i=1}^d V_{\pi i} \mathcal{H}_0$, so that $\sum_{k=1}^{d_\pi} V_{\pi k} V_{\pi k}^* = E_\pi$.

$\square$

We now want to calculate the modular operator Δ_φ, and the modular conjugation J_φ associated to (M,Ω), in terms of those Δ_{φ_0}, J_{φ_0} associated to $(M^\alpha_{E_0},\Omega)$.
Motivated from proposition 6.3, we now introduce the following unitary operators

$$U_\pi : \sum_{k=1}^{d_\pi} e_k \otimes \xi_k \in \mathbf{C}^{d_\pi} \otimes \mathcal{H}_0 \to \sum_{k=1}^{d_\pi} V_{\pi k}\xi_k \in \mathcal{H}_\pi,$$

where $\{e_k : k=1,\dots,d_\pi\}$ is the canonical basis of $\mathbf{C}^{d_\pi}$. To state the following theorem, we have to recall some notation.
We set, $\forall\pi\in\hat{G}$, $\varphi_\pi := d_\pi\varphi_0\cdot\rho_\pi^{-1}\cdot\mathcal{E}_\pi$, where $\rho_\pi(x) := \sum_{k=1}^{d_\pi} v_{\pi k} x v_{\pi k}^*$, and $\mathcal{E}_\pi : M^\alpha \to \rho_\pi(M^\alpha)$ is the normal conditional expectation given in proposition 4.16, if G is finite, and in theorem 4.19, in the general case.
In addition we denote by $\Delta_{\varphi_0;\varphi_\pi}$ the modular operator associated with

$$x\Omega \in M^\alpha\Omega \subset \mathcal{H}_0 \to \Lambda_{\varphi_\pi}(x^*) \in \mathcal{H}_{\varphi_\pi},$$

where $\Lambda_{\varphi_\pi} : x \in M^\alpha_{E_0} \to \pi_{\varphi_\pi}(x)\Omega_{\varphi_\pi} \in \mathcal{H}_{\varphi_\pi}$, is the canonical injection in the GNS of φ_π.
Now we are ready to state and prove

6.4 Theorem. *With the above notation, $\Delta_\varphi = \sum_{\pi\in\hat{G}} U_\pi(1\otimes\Delta_{\varphi_0;\varphi_\pi})U_\pi^*$, where the series converges in strong resolvent sense.*

Proof. As $\Delta_\varphi^{it} : \mathcal{H}_\pi \to \mathcal{H}_\pi$, if we set $\Delta_\pi := U_\pi^*\Delta_\varphi U_\pi : \mathbf{C}^{d_\pi}\otimes\mathcal{H}_0 \to \mathbf{C}^{d_\pi}\otimes\mathcal{H}_0$, we get $\Delta_\pi^{it} = U_\pi^*\Delta_\varphi^{it}U_\pi$, and then, $\forall x\in M^\alpha$,

$$\Delta_\pi^{it}\sum_{k=1}^{d_\pi} e_k\otimes x_k\Omega = U_\pi^*\Delta_\varphi^{it}\sum_{k=1}^{d_\pi} V_{\pi k}x_k\Omega = \sqrt{d_\pi}\sum_{k=1}^{d_\pi} U_\pi^*\Delta_\varphi^{it}v_{\bar\pi k}^* x_k\Omega =$$

$$= \sqrt{d_\pi}\sum_{k=1}^{d_\pi} U_\pi^*\sigma_t^\varphi(v_{\bar\pi k}^* x_k)\Omega = \sqrt{d_\pi}\sum_{k=1}^{d_\pi} U_\pi^*\sigma_t^\varphi(v_{\bar\pi k})^*\sigma_t^{\varphi_0}(x_k)\Omega =$$

$$= \sqrt{d_\pi}\sum_{k=1}^{d_\pi} U_\pi^* v_{\bar\pi k}^* u_{\bar\pi,t}\sigma_t^{\varphi_0}(x_k)\Omega,$$

where $u_{\bar\pi,t} := (D\varphi_{\bar\pi} : D\varphi_0)_t = \Delta^{it}_{\varphi_0;\varphi_{\bar\pi}}\Delta^{-it}_{\varphi_0}$, as follows from [3].
Then we get

$$\Delta_\pi^{it}\sum_{k=1}^{d_\pi} e_k\otimes x_k\Omega = \sqrt{d_\pi}\sum_{k=1}^{d_\pi} U_\pi^* v_{\bar\pi k}^*\Delta^{it}_{\varphi_0;\varphi_{\bar\pi}}\Delta^{-it}_{\varphi_0}\Delta^{it}_{\varphi_0}x_k\Omega =$$

$$= U_\pi^* \sum_{k=1}^{d_\pi} V_{\pi k} \Delta^{it}_{\varphi_0;\varphi_{\bar\pi}} x_k \Omega = \sum_{k=1}^{d_\pi} e_k \otimes \Delta^{it}_{\varphi_0;\varphi_{\bar\pi}} x_k \Omega.$$

That is $\Delta_\pi^{it} = 1 \otimes \Delta^{it}_{\varphi_0;\varphi_{\bar\pi}} \Rightarrow \Delta_\pi = 1 \otimes \Delta_{\varphi_0;\varphi_{\bar\pi}}$ and $\Delta_\varphi = \sum_{\pi\in\hat G} U_\pi \Delta_\pi U_\pi^* = \sum_{\pi\in\hat G} U_\pi (1\otimes \Delta_{\varphi_0;\varphi_{\bar\pi}}) U_\pi^*$, in strong resolvent sense.

□

6.5 Remark. We can prove a little more. Set $\mathcal{F} := \{F : F$ finite subset of $\hat G\}$, ordered by inclusion, $\mathcal{D} := \sum_{\pi\in\hat G} M_\pi \Omega$, and, $\forall F \in \mathcal{F}$, $\Delta_F := \sum_{\pi\in F} U_\pi \Delta_\pi U_\pi^*$, extended with zero on $(\sum_{\pi\in F} \mathcal{H}_\pi)^\perp$.
Then $\mathcal{D}$ is a common core for $\{\Delta_F\}$ and Δ_φ, and $\Delta_F \xi \to \Delta_\varphi \xi, \forall \xi \in \mathcal{D}$.

We now come to the decomposition of J_φ. We have the following

6.6 Theorem. *$J_\varphi = \sum_{\pi\in\hat G} U_{\bar\pi}(1\otimes W_\pi J_{\varphi_0}) U_\pi^*$, where the series is strongly summable, and W_π is the canonical implementation of $\rho_\pi|_{M^\alpha}$, the restriction of ρ_π to M^α, with respect to a cyclic separating vector $\xi \in \mathcal{H}_0$ for both M^α and $\rho_\pi(M^\alpha)$.*

We divide the proof of the theorem in a series of lemmas.

6.7 Lemma. *Let $A \subset B$ be properly infinite von Neumann algebras, $\Omega \in \mathcal{H}$ cyclic and separating for B. Then $\exists \xi \in L^2(B,\Omega)_+$ cyclic and separating for $A \subset B$.*

Proof. Let us take a ξ' cyclic and separating for both A and B and consider the normal state $\omega_{\xi'} := (\xi', \cdot\xi')$ on B. From known results [2], there exists $\xi \in L^2(B,\Omega)_+$ such that $\omega_{\xi'} = (\xi, \cdot\xi)$. Then ξ is cyclic and separating for $A \subset B$. In fact, let U be the unitary operator such that $Ux\xi' = x\xi, \forall x \in B$; then $U \in B'$ and $[A\xi] = [AU\xi'] = [UA\xi'] = 1$.

□

6.8 Remark. From this lemma it follows that, $\forall H \in \mathcal{H}^\alpha(M)$, $\exists \xi \in L^2(M^\alpha,\Omega)_+$, cyclic and separating for $\rho_H(M^\alpha) \subset M^\alpha$. Then $J^\xi_{M^\alpha} = J_{\varphi_0}$.

Let now $H \in \mathcal{H}^\alpha(M)$ be such that $\alpha|_H \cong \pi$ and let $\{v_k : k = 1,\ldots,d\}$ be a basis of H. Let $K \in \mathcal{H}^\alpha(M)$ be such that $\alpha|_K \cong \bar\pi$, and denote with $\{\bar v_k : k = 1,\ldots,d\}$ the conjugate basis of K. Let us set $Z_{HK} := \sum_{k=1}^d v_k J_\varphi \bar v_k^* J_\varphi$ and, consequently, $Z_{KH} := \sum_{k=1}^d \bar v_k J_\varphi v_k^* J_\varphi$; then we get

6.9 Lemma.
(i) *Z_{HK} is a unitary operator on $\mathcal{H}$ which is independent of the chosen basis;*
(ii) *$J_\varphi Z_{HK}^* J_\varphi = Z_{KH}$;*
(iii) *$Z_{HK} x Z_{HK}^* = \rho_H(x), \forall x \in M$;*

(iv) Z_{HK} *commutes with* $U_g, \forall g \in G$.

Proof. Let us set, for the sake of brevity, $J := J_\varphi$ and $Z := Z_{HK}$.
(i) We get

$$ZZ^* = \sum_{h,k=1}^{d} v_h J\bar{v}_h^* J J \bar{v}_k J v_k^* = \sum_{h,k=1}^{d} v_h J\bar{v}_h^* \bar{v}_k J v_k^*$$
$$= \sum_{h=1}^{d} v_h J J v_h^* = \sum_{h=1}^{d} v_h v_h^* = 1.$$

And, analogously, $Z^*Z = 1$.
Let now $\{w_k\}$, $\{\bar{w}_k\}$ be new bases in H and K respectively. Then we get

$$\sum_{k=1}^{d} w_k J\bar{w}_k^* J = \sum_{i,j,k=1}^{d} v_i v_i^* w_k J\bar{w}_k^* \bar{v}_j \bar{v}_j^* J = \sum_{i,j=1}^{d} v_i J\bar{v}_j^* J \sum_{k=1}^{d} v_i^* w_k \overline{\bar{w}_k^* \bar{v}_j}$$
$$= \sum_{i,j=1}^{d} v_i J\bar{v}_j^* J \sum_{k=1}^{d} v_i^* w_k w_k^* v_j = \sum_{i,j=1}^{d} v_i J\bar{v}_j^* J v_i^* v_j = \sum_{i=1}^{d} v_i J\bar{v}_i^* J = Z_{HK}.$$

(ii) $JZ_{HK}^* J = \sum_{k=1}^{d} \bar{v}_k J v_k^* J = Z_{KH}$.
(iii) $\forall x \in M$,

$$ZxZ^* = \sum_{h,k=1}^{d} v_h J\bar{v}_h^* J x J \bar{v}_k J v_k^* = \sum_{h,k=1}^{d} v_h J J x J\bar{v}_h^* \bar{v}_k J v_k^*$$
$$= \sum_{h=1}^{d} v_h x J J v_h^* = \sum_{h=1}^{d} v_h x v_h^* = \rho_H(x).$$

(iv) As U_g commutes with J we have

$$U_g Z U_g^* = \sum_{k=1}^{d} U_g v_k U_g^* J U_g \bar{v}_k^* U_g^* J = \sum_{k=1}^{d} \alpha_g(v_k) J \alpha_g(\bar{v}_k)^* J = Z,$$

because of (i).

□

Because of 6.9(iv) we can set $W_{HK} := Z_{HK}|_{\mathcal{H}_0} \in \mathcal{U}(\mathcal{H}_0)$.

6.10 Lemma. *$W \equiv W_{HK}$ satisfies*
(i) $Wx_{E_0}W^* = \rho_H(x)_{E_0}$, *for* $x \in M^\alpha$,
(ii) $(J_{\varphi_0} W^* J_{\varphi_0}) x_{E_0} (J_{\varphi_0} W J_{\varphi_0}) = \rho_K(x)_{E_0}$, *for* $x \in M^\alpha$.

If $W' \in \mathcal{U}(\mathcal{H}_0)$ satisfies (i) and (ii) then there is a unitary operator $z \in (\rho_K(M^\alpha)' \wedge M^\alpha)_{E_0}$ s.t. $W' = WJ_{\varphi_0} z J_{\varphi_0}$.

Proof. W verifies (i) and (ii) as a consequence of lemma 6.9. Let us now set $J_0 := J_{\varphi_0}$ for the sake of brevity and let W' be another unitary operator on $\mathcal{H}_0$ satisfying (i) and (ii). Then $V := W^* W'$ is such that, for $x \in M^\alpha$,

$$Vx_{E_0}V^* = W^*W'x_{E_0}{W'}^*W = W^*\rho_H(x)_{E_0}W = x_{E_0},$$

that is $V \in (M^\alpha_{E_0})' \Leftrightarrow J_0VJ_0 \in M^\alpha_{E_0}$, and

$$(J_0V^*J_0)\rho_K(x)_{E_0}(J_0VJ_0) = (J_0{W'}^*J_0)(J_0WJ_0)\rho_K(x)_{E_0}(J_0W^*J_0)(J_0W'J_0) =$$

$$= (J_0{W'}^*J_0)x_{E_0}(J_0W'J_0) = \rho_K(x)_{E_0},$$

that is $J_0VJ_0 \in \rho_K(M^\alpha)'_{E_0}$. Therefore $J_0VJ_0 =: z \in (\rho_K(M^\alpha)' \wedge M^\alpha)_{E_0}$, that is $W' = WV = WJ_0zJ_0$.

□

Let us set $\psi : x \in M^\alpha \to x_{E_0} \in M_0$, where $M_0 := M^\alpha_{E_0}$, and recall, from proposition 2.2, that ψ is an isomorphism. Set also $\sigma_H := \rho_H|_{M^\alpha}$, for all $H \in \mathcal{H}^\alpha(M)$, $\Gamma_H := J^\xi_{\rho_H(M^\alpha)} J^\xi_{M^\alpha} \in \mathcal{U}(\mathcal{H}_0)$, $\gamma_H := \psi^{-1} \cdot ad\Gamma_H \cdot \psi$, the canonical endomorphism [14], and finally $\overline{\sigma_H} := \sigma_H^{-1} \cdot \gamma_H$. Then we have

6.11 Lemma. *For every $H \in \mathcal{H}^\alpha(M)$ such that $\alpha|_H \cong \pi$ there exists an $L \in \mathcal{H}^\alpha(M)$ such that $\alpha|_L \cong \bar{\pi}$ and $\overline{\sigma_H} = \sigma_L$.*

Proof. Let us choose a $K \in \mathcal{H}^\alpha(M)$ such that $\alpha|_K \cong \bar{\pi}$, and recall that $\sigma_H := \rho_H|_{M^\alpha}$ and $\sigma_K := \rho_K|_{M^\alpha}$. We want to show that σ_H is conjugate to σ_K.

Let $\{v_i\}$ and $\{\bar{v}_i\}$ be conjugate bases for H and K, respectively, and set

$$v := d^{-1/2} \sum_{k=1}^{d} \bar{v}_k v_k, \qquad w := d^{-1/2} \sum_{k=1}^{d} v_k \bar{v}_k.$$

Then

$$\begin{aligned}
\alpha_g(v) &= d^{-1/2}\sum_{k=1}^{d}\alpha_g(\bar v_k)\alpha_g(v_k)\\
&= d^{-1/2}\sum_{i,j,k=1}^{d}\bar v_i\bar v_i^*\alpha_g(\bar v_k)v_jv_j^*\alpha_g(v_k)\\
&= d^{-1/2}\sum_{i,j,k=1}^{d}\bar v_iv_j\bar\pi(g)_{ik}\pi(g)_{jk}\\
&= d^{-1/2}\sum_{i,j=1}^{d}\bar v_iv_j\sum_{k=1}^{d}\pi(g)_{jk}\pi(g^{-1})_{ki}\\
&= d^{-1/2}\sum_{i=1}^{d}\bar v_iv_i = v,
\end{aligned}$$

that is $v \in M^\alpha$, and analogously $w \in M^\alpha$. Besides $v^*v = d^{-1}\sum_{i,j=1}^d v_i^*\bar v_i^*\bar v_jv_j = d^{-1}\sum_{i=1}^d v_i^*v_i = 1$, and analogously $w^*w = 1$, that is $v, w \in M^\alpha$ and are isometries. Moreover, $\forall x \in M^\alpha$, we get

$$\begin{aligned}
\sigma_K\sigma_H(x)v = d^{-1/2}\sum_{h,k=1}^{d}\bar v_h\sigma_H(x)\bar v_h^*\bar v_kv_k &= d^{-1/2}\sum_{k=1}^{d}\bar v_k\sigma_H(x)v_k\\
&= d^{-1/2}\sum_{k=1}^{d}\bar v_kv_kx = vx,
\end{aligned}$$

and analogously $\sigma_H\sigma_K(x)w = wx$. Finally

$$\begin{aligned}
w^*\sigma_H(v) = d^{-1/2}\sum_{i,j=1}^{d}\bar v_i^*v_i^*v_jvv_j^* &= d^{-1/2}\sum_{i=1}^{d}\bar v_i^*vv_i^*\\
&= d^{-1}\sum_{i,j=1}^{d}\bar v_i^*\bar v_jv_jv_i^* = d^{-1}\sum_{i=1}^{d}v_iv_i^* = d^{-1},
\end{aligned}$$

and analogously $v^*\sigma_K(w) = d^{-1}$.
Then from theorem 5.3 it follows that $\sigma_K \cong \overline{\sigma_H}$, so that there exists $z = \sigma_H(u) \in \sigma_H(M^\alpha)$ unitary, such that $\sigma_H\sigma_K = ad(z)\gamma_H$, hence $\sigma_H(u^*\sigma_K(\cdot)u) = \gamma_H$.
Let us set $L := u^*K$ and $\sigma_L := \rho_L|_{M^\alpha}$, so that $\overline{\sigma_H} = \sigma_L$.

□

6.12 Lemma. *For every $H \in \mathcal{H}^\alpha(M)$ such that $\alpha|_H \cong \pi$ there exists a $K \in \mathcal{H}^\alpha(M)$ with $\alpha|_K \cong \bar{\pi}$, such that W_{HK} is the canonical implementation of $\sigma_H \equiv \rho_H|_{M^\alpha}$ with respect to the cyclic separating vector ξ of remark 6.8.*

Proof. Let us set R for the canonical implementation of σ_H; then we get $Rx_{E_0}R^* = \sigma_H(x)_{E_0}, x \in M^\alpha$. Besides, as R satisfies $RJ_0R^* = J_H$, where $J_0 := J_{\varphi_0} \equiv J^\xi_{M^\alpha}$, and $J_H := J^\xi_{\sigma_H(M^\alpha)}$, we get, for $x \in M^\alpha$,

$$\begin{aligned}(J_0R^*J_0)x_{E_0}(J_0RJ_0) &= R^*J_HJ_0x_{E_0}J_0J_HR = R^*\Gamma_Hx_{E_0}\Gamma_H^*R \\ &= (\sigma_H^{-1}\cdot\gamma_H(x))_{E_0} = \sigma_L(x)_{E_0},\end{aligned}$$

where L is given in lemma 6.11. Therefore, by lemma 6.10, there exists $z \in (\sigma_L(M^\alpha)' \wedge M^\alpha)_{E_0}$, unitary operator such that $R = W_{HL}J_0zJ_0$. Let $u \in \rho_L(M^\alpha)' \wedge M^\alpha$ be s.t. $u_{E_0} = z^*$, and set $K := uL$. Then $\alpha|_K \cong \bar{\pi}$, and, if $\{\bar{v}_k : k = 1, \ldots, d\}$ is the conjugate basis of L, so that $\{w_k : k = 1, \ldots, d\}$, where $w_k := u\bar{v}_k$, is the conjugate basis of K, we have

$$Z_{HK} = \sum_{k=1}^d v_kJ\bar{v}_k^*u^*J = \sum_{k=1}^d v_kJ\bar{v}_k^*JJu^*J,$$

and if we restrict to $\mathcal{H}_0$,

$$W_{HK} = Z_{HK}|_{\mathcal{H}_0} = \sum_{k=1}^d v_kJ\bar{v}_k^*J|_{\mathcal{H}_0}J_0u^*_{E_0}J_0 = W_{HL}J_0zJ_0 = R.$$

□

Proof of theorem 6.6. For every $\pi \in \hat{G}$ let us choose an $H_\pi \in \mathcal{H}^\alpha(M)$ such that $\alpha|_{H_\pi} \cong \pi$, and a basis $\{v_{\pi k} : k = 1, \ldots, d_\pi\}$. Take $K_\pi \in \mathcal{H}^\alpha(M)$ such that $\alpha|_{K_\pi} \cong \bar{\pi}$ as given by lemma 6.12, and let $\{w_{\pi k} : k = 1, \ldots, d_\pi\}$ be its conjugate basis.
Then from the definition of $Z_\pi \equiv Z_{H_\pi K_\pi}$ we get $v^*_{\pi k}Z_\pi = J_\varphi w^*_{\pi k}J_\varphi$ and then, $\forall x_k \in M^\alpha$,

$$\begin{aligned}J_\varphi\sum_{k=1}^{d_\pi} V_{\pi k}x_k\Omega &= \sqrt{d_\pi}\sum_{k=1}^{d_\pi} J_\varphi w^*_{\pi k}J_\varphi J_{\varphi_0}x_k\Omega = \\ &= \sqrt{d_\pi}\sum_{k=1}^{d_\pi} v^*_{\pi k}Z_\pi J_{\varphi_0}x_k\Omega = \sum_{k=1}^{d_\pi} V_{\bar{\pi}k}W_\pi J_{\varphi_0}x_k\Omega,\end{aligned}$$

therefore $U^*_{\bar{\pi}}J_\varphi U_\pi = 1 \otimes W_\pi J_{\varphi_0}$, and eventually

$$J_\varphi = \sum_{\pi\in\hat{G}} U_{\bar{\pi}}(1 \otimes W_\pi J_{\varphi_0})U^*_\pi,$$

where the series is strongly summable.

□

Acknowledgment.

We would like to thank professor R. Longo for introducing us to the problem through his manuscript, for constant encouragement, and for his invaluable suggestions during many discussions with him. Besides we would like to thank the referee for pointing out an error in the first version of this note and for many useful suggestions.

References.

[1] M. Baillet, Y. Denizeau, J.F. Havet. Indice d' une esperance conditionnelle. *Comp. Math.*, **66**, (1988), 199–236.

[2] O. Bratteli, D.W. Robinson. *Operator algebras and quantum statistical mechanics, I* . Springer, New York,1979.

[3] T. Digernes. Duality for weights on covariant systems and its applications. *Ph.D. Thesis*, UCLA, 1975.

[4] S. Doplicher, J.E. Roberts. Endomorphisms of C*-algebras, cross products and duality for compact groups. *Ann. Math.*, **130**, (1989), 75–119.

[5] J. F. Havet. Espérance conditionnelle minimale. *J. Oper. Th.*, **24**, (1990), 33–55.

[6] F. Hiai. Minimizing indeces of conditional expectations onto a subfactor. *Publ. R.I.M.S.*, **24**, (1988), 673–678.

[7] F. Hiai. Minimum index for subfactors and entropy. *J. Oper. Th.*, **24**, (1990), 301–336.

[8] F. Hiai. Minimum index for subfactors and entropy, II. *J. Math. Soc. Japan*, **43**, (1991), 347–379.

[9] V.F.R. Jones. Index for subfactors. *Invent. Math.*, **72**, (1983), 1–25.

[10] Y. Kawahigashi. Automorphisms commuting with a conditional expectation onto a subfactor with finite index. *J. Oper. Th.*, to appear.

[11] H. Kosaki. Extension of Jones' theory on index to arbitrary factors. *J. Funct. Anal.*, **66**, (1986), 123–140.

[12] H. Kosaki, R. Longo. A remark on the minimal index of subfactors. *J. Funct. Anal.*, **107**, (1992), 458–470.

[13] C. Lance. Direct integral of left Hilbert algebras. *Math. Ann.*, **216**, (1975), 11–28.

[14] R. Longo. Solution of the factorial Stone-Weierstrass conjecture. An application of the theory of standard split W*-inclusions. *Invent. Math.*, **76**, (1984), 145–155.

[15] R. Longo. Restricting a compact action to an injective subfactor. *Ergod. Th. Dynam. Sys.*, **9**, (1989), 127–135.

[16] R. Longo. Index of subfactors and statistics of quantum field, I. *Commun. Math. Phys.*, **126**, (1989), 217–247.

[17] R. Longo. Index of subfactors and statistics of quantum field, II. *Commun. Math. Phys.*, **130**, (1990), 285–309.

[18] Y. Nakagami, M. Takesaki. *Duality for crossed products of von Neumann algebras.* Lecture Notes in Math., n. 731, 1979.

[19] M. Pimsner, S. Popa. Entropy and index for subfactors. *Ann. Scient. Ec. Norm. Sup.* (4), **19**, (1986), 57–106.

[20] J.E. Roberts. Crossed products of von Neumann algebras by group dual. *Symposia Math.*, **XX**, (1976), 335–363.

[21] S. Stratila. *Modular theory in operator algebras.* Abacus Press, Turnbridge Wells, 1981.

[22] C.E. Sutherland. Cartan subalgebras, transverse measures and non-type-I Plancherel formulae. *J. Funct. Anal.*, **60**, (1985), 281–308.

[23] M. Takesaki. *Theory of operator algebras, I.* Springer, New York, 1979.