The problem of completeness for Gromov-Hausdorff metrics on C*-algebras ⋆

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Abstract

It is proved that the family of equivalence classes of Lip-normed C*-algebras introduced by M. Rieffel, up to complete order isomorphisms preserving the Lip-seminorm, is not complete w.r.t. the matricial quantum Gromov-Hausdorff distance introduced by D. Kerr. This is shown by exhibiting a Cauchy sequence whose limit, which always exists as an operator system, is not completely order isomorphic to any C*-algebra.

Conditions ensuring the existence of a C*-structure on the limit are considered, making use of the notion of ultraproduct. More precisely, a necessary and sufficient condition is given for the existence, on the limiting operator system, of a C*-product structure inherited from the approximating C*-algebras. Such condition can be considered as a generalisation of the f-Leibniz conditions introduced by Kerr and Li. Furthermore, it is shown that our condition is not necessary for the existence of a C*-structure *tout court*, namely there are cases in which the limit is a C*-algebra, but the C*-structure is not inherited.

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1 Introduction

In this paper we study the problem of completeness of some spaces w.r.t the matricial quantum Gromov-Hausdorff metrics introduced by Kerr [8], showing

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that the space of equivalence classes of C*-algebras with Lipschitz seminorms is not complete.

As is known, Rieffel introduced and studied, in a series of papers [13–18], the notion of compact quantum metric space, and generalised the Gromov-Hausdorff distance to the quantum case. The main tool is a seminorm L on the "quantum" functions, which plays the role of the Lipschitz seminorm for functions on a compact metric space. The requirements can be formalised as follows: L should vanish exactly on the multiples of the identity element, and should induce on the states (positive normalised functionals) the weak* topology. It is not restrictive to assume that L is also lower-semicontinuous in norm, and we shall always assume it in this paper. A space endowed with such a seminorm is called Lip-normed.

Roughly speaking, the quantum Gromov-Hausdorff distance between two C*-algebras corresponds to the Gromov-Hausdorff distance between the corresponding state spaces, endowed with the Monge-Kantorovitch metric induced by the Lipschitz seminorms (for the precise definition see eq. (2.11)). However one easily realises that while two abelian Lip-normed C*-algebras having zero quantum Gromov-Hausdorff distance are isomorphic, the same is not true for noncommutative C*-algebras. The structure which is preserved by the quantum Gromov-Hausdorff distance is indeed that of order-unit space.

In fact Rieffel proved that the quantum Gromov-Hausdorff distance is indeed a distance on equivalence classes of order-unit spaces, showed that the space of equivalence classes is complete, and gave also conditions for compactness.

As mentioned above, when C*-algebras are concerned, there are non isomorphic Lip-normed C*-algebras which have zero quantum Gromov-Hausdorff distance. In order to cope with this problem, Kerr [8] introduced matricial quantum Gromov-Hausdorff distances dist_p. When p is finite, dist_p measures the distance between $p \times p$-valued state spaces, and dist_∞ corresponds to the supremum over all p.

He showed that, when $p \geq 2$, dist_p vanishes if and only if the Lip-normed C*-algebras are *-isomorphic. The question of completeness for the space of equivalence classes remained open. However Kerr and Li introduced a family of conditions [8,11], depending on a function f, related with the Leibniz property for the Lipschitz norms, showing that if all the Lip-normed C*-algebras of a Cauchy sequence satisfy the same f-Leibniz condition they converge to a C*-algebra (which satisfies the same f-Leibniz condition).

The main purpose of this paper is then to solve the completeness problem, indeed we exhibit a Cauchy sequence of Lip-normed C*-algebras which does not converge to a C*-algebra.

Following Kerr, a natural setting for matricial quantum Gromov-Hausdorff distance is that of Lip-normed operator systems. He showed that the distance dist_∞ between two Lip-normed operator systems vanishes if and only if they are completely order isomorphic and the Lipschitz norm is preserved by the isomorphism.

It is possible to show that the space of equivalence classes of operator systems with Lipschitz seminorms, endowed with dist_∞, is complete (we do so in Theorem 3.7, by making use of ultraproducts, and it was also proved independently by Kerr and Li [9] with different techniques). Therefore the problem of completeness for Lip-normed C*-algebras w.r.t. the dist_∞ metric can be rephrased as the problem of the closure of the Lip-normed C*-algebras inside the family of equivalence classes of Lip-normed operator systems.

Given a dist_∞-Cauchy sequence of Lip-normed C*-algebras, the limit always exists as an operator system S, and the question becomes to determine whether such S admits a C*-structure (is completely order isomorphic to a C*-algebra). This property can be reformulated with the aid of the notion of injective envelope $\mathcal{I}(S)$ of an operator system S due to Hamana [6]. The operator system S embeds canonically in $\mathcal{I}(S)$, and the latter admits a unique C*-product. The existence of a C*-structure on S is equivalent to the fact that S is a subalgebra of $\mathcal{I}(S)$. We use this technique in subsection 4.2.1 to show that the limit of a suitable sequence of C*-algebras with Lipschitz seminorms is not a C*-algebra.

As mentioned above, an important tool in our analysis is the notion of ultraproduct of Banach spaces, and in particular a tailored version for Banach spaces with lower semicontinuous Lipschitz norm, which we call Lip-ultraproduct.

We show that under a condition of uniform compactness on sequences of spaces, the Lip-ultraproduct is a Banach space with lower semicontinuous Lipschitz norm, and inherits some of the structures from the approximating spaces, in particular those of order-unit space and of operator system. Furthermore we show that Cauchy sequences are uniformly compact and the Lip-ultraproduct is indeed the limit.

Let us mention here that the representative of a quantum Gromov-Hausdorff limit constructed via ultraproducts is directly endowed with a lower semicontinuous Lip-seminorm.

The C*-structure is not inherited in general by the Lip-ultraproduct, however for any given free ultrafilter the Lip-ultraproduct is always a closed linear subspace of the ultraproduct, and the latter is a C*-algebra. Therefore there are cases in which the limit inherits a C*-structure, namely when the Lip-ultraproduct is a subalgebra of the ultraproduct (for a suitable free ultrafilter $\mathcal{U}$). This is a sufficient condition for the limit to be a C*-algebra, but is not necessary, namely there are cases in which the limit is a C*-algebra but the C*-structure is not inherited, cf. subsection 4.2.2.

Moreover we can completely characterize the Cauchy sequences for which the limit inherits a C*-structure in terms of a function $\varepsilon(r)$, $r \in [0, +\infty)$, associated with any Lip-normed C*-algebra, measuring how far is the set of Lipschitz elements from being an algebra. If we have a Cauchy sequence with functions $\varepsilon_n(r)$, the limit inherits the C*-structure if and only if, for a suitable subsequence n_k, $\limsup_k \varepsilon_{n_k}(r) \to 0$ for $r \to \infty$ (cf. Corollary 4.8). Therefore such condition is a maximal generalisation of the f-Leibniz condition of Kerr and Li. The fact that it is indeed more general is illustrated in subsection 4.2.3.

Let us also mention that, with the aid of the function $\varepsilon(r)$ and of the results on inherited C*-structure, we can easily manufacture a new distance on the family of equivalence classes of Lip-normed C*-algebras, for which completeness holds, cf. Corollary 4.10. The convergence condition under this new distance is clearly stronger than the convergence condition w.r.t. dist_∞, as shown by Example 2 in subsection 4.2.2. However this stronger convergence condition seems to be more natural when C*-algebras are concerned, because in this case the C*-structure is always inherited, namely the product on the limit is the limit of the approximating products, cf. equation (4.1).

As mentioned above, subsection 4.2.1 is devoted to the construction of examples of non converging Cauchy sequences w.r.t. the matricial quantum Gromov Hausdorff distance. When the limit (as an operator system) does not inherit a C*-structure, as a Lip-ultraproduct it is described by a subspace, which is not closed w.r.t. the product, of a C*-algebra (the ultraproduct). For the examples considered in subsection 4.2.1 we show that the product structure given by the immersion in the ultraproduct is the same as the product structure given by the immersion in the injective envelope, thus showing that the limit is not a C*-algebra.

Indeed the examples considered in subsection 4.2.1 depend on a C*-algebra $\mathcal{B}$, and we show that for any $\mathcal{B}$ we get a sequence $\mathcal{A}_n$ which is Cauchy w.r.t. dist_p, $p \in \mathbb{N} \cup \{\infty\}$. In the particular case in which $\mathcal{B} = \mathbb{C}I$, the sequence $\mathcal{A}_n$ consists of the constant algebra $M_2(\mathbb{C})$ of 2×2 matrices, and it is easy to show that the limit is not even positively isomorphic to a C*-algebra (cf. Remark 1). This shows that the family of equivalence classes of Lip-normed C*-algebras is not complete w.r.t. dist_p, $p \geq 2$. However, if we confine our attention to the case $\mathcal{A}_n = M_2(\mathbb{C})$, one may argue that we have simply chosen the wrong distance.

Let us recall that when Rieffel introduced the quantum Gromov-Hausdorff distance, he had to generalise to the quantum setting a distance involving spaces of points, or extremal states. Since for C*-algebras extremal states may be not closed, and even dense, as in the UHF case, he decided to consider a distance involving all states. However, when $\mathcal{A}_n = M_2(\mathbb{C})$, the replacement of the quantum Gromov-Hausdorff distance with a distance involving only extremal states, like the distance dist_q^e considered by Rieffel in [15] after Proposition 4.9, would destroy the counterexample, since the sequence is no longer Cauchy w.r.t. such distance.

This is the reason why we consider also non-trivial $\mathcal{B}$: when the C*-algebra $\mathcal{B}$ is UHF, we get a sequence made of a constant UHF algebra (with different Lip-norms), for which pure states are dense, hence matricial quantum Gromov-Hausdorff distances are the only reasonable choices. Of course in this case the proof that the limit is not completely order isomorphic to a C*-algebra is more difficult, requiring the notion of injective envelope of Hamana [6].

We conclude by mentioning a result for ultraproducts which may have an interest of its own. The dual of an ultraproduct is larger in general than the ultraproduct of the duals, the equality being attained only under a strong uniform convexity property of the sequence, which is never satisfied for infinite-dimensional C*-algebras. For the Lip-ultraproduct however, if the sequence is uniformly compact, any element in the dual can be realised as an element in the ultraproduct of the dual spaces, namely the compactness condition of the Lipschitz seminorms allows one to construct a more manageable ultraproduct, whose dual is made of equivalence classes of sequences of functionals.

This suggests the interpretation of the Lip-ultraproduct as the quantum (dualised) analogue of the ultralimit of compact metric spaces. As in the classical case, an ultralimit is a limit only if a uniform compactness condition is satisfied.

2 Order-unit spaces

This section is mainly devoted to the introduction of the Lip-ultraproduct and the study of its properties.

In order to clarify some features of the construction, we introduce the notion of Lip-space.

Let us recall (see [13], Thm. 1.9) that a lower semicontinuous Lipschitz seminorm L on a complete order-unit space can be characterised, besides the vanishing exactly on the multiples of the identity, by the fact that the elements whose norm and Lipschitz seminorm are bounded by a constant, form a compact set in norm. Indeed by introducing the norm $\|x\|_L = \max\{L(x), \frac{1}{R}\|x\|\}$, where R may be taken as half of the diameter of the state space w.r.t. the

Lipschitz distance, the compactness property may be reformulated as the fact that the $\| \cdot \|_L$-balls are norm compact, and the Lipschitz seminorm can be recovered as $L(x) = \inf_\lambda \|x - \lambda I\|_L$. Therefore, in contrast with the standard terminology, we shall reserve the term Lip-norm for $\| \cdot \|_L$, and shall call L a Lip-seminorm.

The observations above suggest the definition of a Lip-space as a Banach space with an extra norm $\| \cdot \|_L$ (finite on a dense subspace) such that the $\| \cdot \|_L$-balls are compact.

2.1 Lip-spaces

Definition 2.1 *We call Lip-space a triple* $(X, \| \cdot \|, \| \cdot \|_L)$ *where*

(i) $(X, \| \cdot \|)$ *is a Banach space,*

(ii) $\| \cdot \|_L : X \to [0, +\infty]$ *is finite on a dense vector subspace* X_0 *where it is a norm,*

(iii) the unit ball w.r.t. $\| \cdot \|_L$, $\{x \in X : \|x\|_L \leq 1\}$, *is compact in* $(X, \| \cdot \|)$.

We call *radius* of the Lip-space $(X, \| \cdot \|, \| \cdot \|_L)$, and denote it by R, the maximum of $\| \cdot \|$ on the unit ball w.r.t. $\| \cdot \|_L$, hence

$$\|x\| \leq R\|x\|_L, \qquad x \in X. \tag{2.1}$$

As we shall see it is the analogue of the radius of a Lip-normed order unit space introduced by Rieffel at the end of Section 2 in [15].

Proposition 2.2 *Let* $(X, \| \cdot \|, \| \cdot \|_L)$ *be a Lip-space,* $e \in X_0 \setminus \{0\}$, *and set* $L(x) := \inf_{\lambda \in \mathbb{R}} \|x - \lambda e\|_L$. *Then* L *is a lower semicontinuous densely defined seminorm and* $L(x) = 0 \iff x = \lambda e$ *for some* $\lambda \in \mathbb{R}$.

Proof. Indeed, it is easy to prove that L is a seminorm, and that $L(\lambda e) = 0, \lambda \in \mathbb{R}$. Moreover, as $\| \cdot \|_L$ is lower semicontinuous, because of Definition 2.1 (iii), and $\|x - \lambda e\|_L \geq |\lambda| \|e\|_L - \|x\|_L \to \infty, |\lambda| \to \infty$, we obtain $L(x) = \min_{\lambda \in \mathbb{R}} \|x - \lambda e\|_L$. Therefore, if $L(x) = 0$, then there is $\lambda_0 \in \mathbb{R}$ s.t. $\|x - \lambda_0 e\|_L = 0$, so that $x = \lambda_0 e$.

Finally, if $x, x_n \in X$, $\|x_n - x\| \to 0$, then, $L(x) \leq \liminf_{n\to\infty} L(x_n)$. Indeed, passing possibly to a subsequence, we may assume $\{L(x_n)\}$ converges. Let, for all $n \in \mathbb{N}$, $\lambda_n \in \mathbb{R}$ be s.t. $\|x_n - \lambda_n e\|_L = L(x_n)$. Then $\{\|x_n - \lambda_n e\|_L\}$ is bounded; so by Definition 2.1 (iii), there are $\{n_k\} \subset \mathbb{N}$, $a \in X$ s.t. $\|x_{n_k} - \lambda_{n_k} e - a\| \to 0$.

Therefore there is $\lambda_0 \in \mathbb{R}$ s.t. $\lambda_{n_k} \to \lambda_0$, and $a = x - \lambda_0 e$. Hence

$$\begin{aligned} L(x) &\le \|x - \lambda_0 e\|_L \le \liminf_{k\to\infty} \|x_{n_k} - \lambda_{n_k} e\|_L \\ &= \lim_{k\to\infty} L(x_{n_k}) = \lim_{n\to\infty} L(x_n), \end{aligned}$$

where the second inequality follows from Definition 2.1 (iii). □

Proposition 2.3 *Let* $(X, \|\cdot\|, \|\cdot\|_L)$ *be a Lip-space. Then the dual norm*

$$\|\xi\|'_L := \max_{x\in X} \frac{|\langle \xi, x\rangle|}{\|x\|_L}$$

*induces the weak** *topology on the bounded subsets of* X'*, the Banach space dual of* $(X, \|\cdot\|)$*.*

The constant R *is equal to the radius, in the* $\|\cdot\|'_L$ *norm, of the unit ball of* $(X', \|\cdot\|)$*.*

Proof. First observe that $\|\cdot\|'_L$, which is obviously a seminorm, is indeed a norm. In fact, if $\|\xi\|'_L = 0$, then ξ vanishes on X_0, which is dense, i.e. $\xi = 0$.

Now we consider the identity map ι from the closed unit ball B'_1 of X' endowed with the weak* topology to the same set endowed with the distance induced by $\|\cdot\|'_L$. Given $r > 0$, we consider a $r/2$-net $\{x_i : i = 1, \dots, n\}$ in $\{x \in X : \|x\|_L \le 1\}$. Then, if $\|\xi\|' \le 1$,

$$|\langle \xi, x\rangle| \le \max_{i=1,\dots,n} |\langle \xi, x_i\rangle| + r/2.$$

Therefore, the weak* open set in B'_1

$$U = \{\|\xi\|' \le 1 : \max_{i=1,\dots,n} |\langle \xi, x_i\rangle| < r/2\},$$

is contained in the $\|\cdot\|'_L$ open set in B'_1

$$V = \{\|\xi\|' \le 1 : \|\xi\|'_L < r\},$$

showing that ι is continuous. Since the domain is compact and the range is Hausdorff, ι is indeed a homeomorphism.

Finally, the radius of the unit ball of X' in the $\|\cdot\|'_L$ norm is

$$\sup_{\|\xi\|'\le 1} \|\xi\|'_L = \sup_{\xi\ne 0, x\ne 0} \frac{|\langle \xi, x\rangle|}{\|x\|_L \|\xi\|'} = \sup_{x\ne 0} \frac{\|x\|}{\|x\|_L} \sup_{\xi\ne 0} \frac{|\langle \xi, x\rangle|}{\|\xi\|' \|x\|} = R.$$

□

Definition 2.4 *A family $\mathcal{F}$ of Lip-spaces is called uniform if for all $\varepsilon > 0$ there is $n_\varepsilon \in \mathbb{N}$ such that, for any $(X, \|\cdot\|, \|\cdot\|_L)$ in $\mathcal{F}$, $\{x \in X : \|x\|_L \leq 1\}$ can be covered by n_ε $\|\cdot\|$-balls of radius ε.*

Lemma 2.5 *If $\mathcal{F}$ is a uniform family of Lip-spaces, there is $R > 0$ such that $\|x\| \leq R\|x\|_L$ for any $(X, \|\cdot\|, \|\cdot\|_L)$ in $\mathcal{F}$, $x \in X$.*

Proof. Let $(X, \|\cdot\|, \|\cdot\|_L)$ be a Lip-space such that $\{x \in X : \|x\|_L \leq 1\}$ can be covered by n balls of radius 1, and let $x_0 \in X$, $\|x_0\|_L = 1$. Since the set $\{tx_0 : t \in [0, 1]\}$ is contained in $\{x \in X : \|x\|_L \leq 1\}$, it is covered by at most n balls of radius 1, hence its length is majorised by $2n$, i.e. $R \leq 2n$. $\square$

Lemma 2.6 *Let $(V, \|\cdot\|)$ be an n-dimensional normed space. Then the ball of radius R can be covered by $(2R/\varepsilon)^n$ balls of radius ε.*

Proof. Let us recall that, denoting by $n_\varepsilon(\Omega)$ the minimum number of balls of radius ε covering Ω, and by $\nu_\varepsilon(\Omega)$ the maximum number of disjoint balls of radius ε contained in Ω, one gets $n_\varepsilon(\Omega) \leq \nu_{\varepsilon/2}(\Omega)$ (cf. e.g. [7], Lemma 1.3). Then, denoting by vol the Lebesgue measure and by B_r the ball of radius r w.r.t. the given norm, we get $\mathrm{vol}(B_R) \geq \nu_\varepsilon(B_R)\,\mathrm{vol}(B_\varepsilon)$, and $\mathrm{vol}(B_R) = (R/\varepsilon)^n\,\mathrm{vol}(B_\varepsilon)$, hence $n_\varepsilon(B_R) \leq (2R/\varepsilon)^n$. $\square$

Proposition 2.7 *A family $\mathcal{F}$ of Lip-spaces is uniform $\Leftrightarrow$ there exists a constant R as in Lemma 2.5, and $\forall\varepsilon > 0$, there is $N_\varepsilon \in \mathbb{N}$ such that any Lip-space X in $\mathcal{F}$ has a subspace V of dimension not greater than N_ε such that $\{x \in V : \|x\|_L \leq 1\}$ is ε-dense in $\{x \in X : \|x\|_L \leq 1\}$.*

Proof. ($\Rightarrow$) The constant R exists by Lemma 2.5; choose a covering of $\{x \in X : \|x\|_L \leq 1\}$ by n_ε $\|\cdot\|$-balls of radius ε and consider the vector space V generated by their centres. Its dimension is clearly majorised by n_ε.
($\Leftarrow$) Take $\varepsilon \leq 1$. The elements in $\{x \in V : \|x\|_L \leq 1\}$ are contained in $\{x \in V : \|x\| \leq R\}$, hence any covering of the R-normic ball of V with balls of radius ε gives a covering of the Lip-norm unit ball in X with balls of radius 2ε. By Lemma 2.6, one can realise the former covering with $(2R/\varepsilon)^{N_\varepsilon}$ balls, hence the implication is proved. $\square$

2.2 Ultraproducts

Given a sequence $(X_n, \|\cdot\|, \|\cdot\|_L)$ of Lip-spaces, we may consider the Banach space $\ell^\infty(X_n)$ of norm-bounded sequences $x_n \in X_n$ with the sup-norm. As is known [19], if $\mathcal{U}$ is a free ultrafilter on $\mathbb{N}$, the ultraproduct $\ell^\infty(X_n, \mathcal{U})$ is defined as the quotient of $\ell^\infty(X_n)$ w.r.t. the subspace of sequences such that

$\lim_{\mathcal{U}} \|x_n\| = 0$. We denote by $\pi_{\mathcal{U}}$ the projection from $\ell^\infty(X_n)$ onto $\ell^\infty(X_n, \mathcal{U})$.

Definition 2.8 *Given a sequence $(X_n, \|\cdot\|, \|\cdot\|_L)$ of Lip-spaces, we call Lip-ultraproduct, and denote it by $\ell^\infty_L(X_n, \mathcal{U})$, or simply by $X_{\mathcal{U}}$, the image under $\pi_{\mathcal{U}}$ of $\ell^\infty_L(X_n)$, the norm closure of the space of bounded sequences for which $\|\{x_n\}\|_L := \sup_{\mathbb{N}} \|x_n\|_L < +\infty$.*

The quotient norm $\|\cdot\|_{\mathcal{U}}$ of the equivalence class $x_{\mathcal{U}}$ of a sequence x_n is defined as

$$\|x_{\mathcal{U}}\|_{\mathcal{U}} = \inf_{[y_n]=x_{\mathcal{U}}} \sup_n \|y_n\|,$$

hence $\|x_{\mathcal{U}}\|_{\mathcal{U}} = \lim_{\mathcal{U}} \|x_n\|$ ([1], Chap. 2 Prop. 2.3).

Analogously, the quotient norm $\|\cdot\|_{L,\mathcal{U}}$ of $x_{\mathcal{U}}$ is defined as

$$\|x_{\mathcal{U}}\|_{L,\mathcal{U}} = \inf_{[y_n]=x_{\mathcal{U}}} \sup_n \|y_n\|_L. \tag{2.2}$$

This implies that $\|x_{\mathcal{U}}\|_{L,\mathcal{U}} \le \lim_{\mathcal{U}} \|x_n\|_L$, in fact for any $\varepsilon > 0$ there exists an element U of the free ultrafilter such that, for any $n \in U$, $\|x_n\|_L \le \lim_{\mathcal{U}} \|x_m\|_L + \varepsilon$. Then we may define $y_n = x_n$ for $n \in U$ and $y_n = 0$ for $n \notin U$. Since $[y_n] = [x_n]$, the result follows.

Lemma 2.9 *The infimum in* (2.2) *is indeed a minimum.*

Proof. Given $x_{\mathcal{U}} \in X_{\mathcal{U}}$, we may choose sequences x_n^k realising it and such that $\|x_n^k\|_L \le \|x_{\mathcal{U}}\|_{L,\mathcal{U}}(1 + \frac{1}{k})$. It is also not restrictive to ask that all the vectors x_n^k have norm bounded by $2\|x_{\mathcal{U}}\|$. Then we set

$$\begin{aligned} &V_k = \{n \ge k : \|x_n^j - x_n^i\| \le \frac{1}{i}, i \le j \le k\}, \\ &V_0 = \mathbb{N}, \end{aligned}$$

and observe that $V_k \in \mathcal{U}$, $V_{k+1} \subseteq V_k$, and $\bigcup_{k\ge 0} V_k \setminus V_{k+1} = \mathbb{N}$. Then we define

$$\tilde{x}_n = \frac{k}{k+1} x_n^k, \quad n \in V_k \setminus V_{k+1},$$

implying $\|\tilde{x}_n\|_L \le \|x_{\mathcal{U}}\|_{L,\mathcal{U}}$. Now we show that $\tilde{x}_{\mathcal{U}} = x_{\mathcal{U}}$. Indeed, if $n \in V_i$, $\exists k \ge i$ s.t. $n \in V_k \setminus V_{k+1}$, hence

$$\|\tilde{x}_n - x_n^i\| \le \|\tilde{x}_n - x_n^k\| + \|x_n^k - x_n^i\| \le \frac{1}{k+1}\|x_n^k\| + \frac{1}{i} \le (2\|x_{\mathcal{U}}\|_{\mathcal{U}} + 1)\frac{1}{i}.$$

Since n is eventually in V_i w.r.t. $\mathcal{U}$, we get

$$\|\tilde{x}_{\mathcal{U}} - x_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|\tilde{x}_n - x_n^i\| \le (2\|x_{\mathcal{U}}\|_{\mathcal{U}} + 1)\,\frac{1}{i}.$$

By the arbitrariness of i we get the result. $\square$

Choosing $\tilde{x}_n$ as in the proof above, we get

$$\|x_{\mathcal{U}}\|_{L,\mathcal{U}} = \lim_{\mathcal{U}} \|\tilde{x}_n\|_L = \sup_n \|\tilde{x}_n\|_L.$$

In particular we obtain that, for any element $x \in \ell^\infty_L(X_n, \mathcal{U})$,

$$\|x\|_{L,\mathcal{U}} = \min_{[x_n]=x} \lim_{\mathcal{U}} \|x_n\|_L. \tag{2.3}$$

Proposition 2.10 *Given a uniform sequence $(X_n, \|\cdot\|, \|\cdot\|_L)$ of Lip-spaces, the Lip-ultraproduct $\ell^\infty_L(X_n, \mathcal{U})$, endowed with the quotient norms $\|\cdot\|_{\mathcal{U}}$, $\|\cdot\|_{L,\mathcal{U}}$, is a Lip-space. Moreover, the radius R for $\ell^\infty_L(X_n, \mathcal{U})$ is equal to $\lim_{\mathcal{U}} R_n$, where R_n is the radius of X_n.*

Proof. Let us show that the closed Lip-norm unit ball in $X_{\mathcal{U}}$ is totally bounded in norm. Indeed, since given $\varepsilon > 0$, the closed Lip-norm unit ball in X_n is covered by n_ε balls of radius ε, we may choose points $x_{n,1}, \dots, x_{n,n_\varepsilon}$ in X_n such that the closed Lip-norm ball of radius 2 in X_n is covered by

$$\bigcup_{i=1}^{n_\varepsilon} B(x_{n,i}, 2\varepsilon).$$

Now, given any sequence $\{x_n\}_{n\in\mathbb{N}}$, $x_n \in X_n$, $\|x_n\|_L \leq 2$, we get

$$\min_{i=1,\dots,n_\varepsilon} \|x_{\mathcal{U}} - x_{\mathcal{U},i}\|_{\mathcal{U}} = \min_{i=1,\dots,n_\varepsilon} \lim_{\mathcal{U}} \|x_n - x_{n,i}\| = \lim_{\mathcal{U}} \min_{i=1,\dots,n_\varepsilon} \|x_n - x_{n,i}\| \leq 2\varepsilon.$$

Since any $x_{\mathcal{U}}$ s.t $\|x_{\mathcal{U}}\|_{L,\mathcal{U}} \leq 1$ can be realized with a sequence x_n such that $\|x_n\|_L \leq 2$, we get that the closed Lip-norm unit ball in $X_{\mathcal{U}}$ is covered by n_ε balls of radius 3ε.

Then we show that the closed Lip-norm unit ball in $X_{\mathcal{U}}$ is norm complete, hence compact. In fact let $\{x^k\}_{k\in\mathbb{N}}$ be a Cauchy sequence of elements of $X_{\mathcal{U}}$, $\|x^k\|_{L,\mathcal{U}} \leq 1$, and, according to the argument above, realize them via sequences x^k_n such that $\|x^k_n\|_L \leq 1$. Let us choose a diagonal sequence as follows.

Set $\varepsilon_k = \sup_{i,j\geq k} \|x^i - x^j\|$, and observe that $\varepsilon_k \to 0$. Then we consider the sets $V_k \subset \mathbb{N}$ defined as

$$V_k = \{n \geq k : \|x^j_n - x^i_n\| \leq 2\varepsilon_i, i \leq j \leq k\}$$
$$V_0 = \mathbb{N},$$

and observe that $V_{k+1} \subseteq V_k$, $\bigcup_{k\geq 0} V_k \setminus V_{k+1} = \mathbb{N}$, and since $\lim_{\mathcal{U}} \|x^j_n - x^i_n\| = \|x^j - x^i\| \leq \varepsilon_i$, then $V_k \in \mathcal{U}$. Now we define the diagonal sequence as

$$\tilde{x}_n = x^k_n, \qquad n \in V_k \setminus V_{k+1}.$$

Then, when $n \in V_i$, and $k \geq i$ satisfies $n \in V_k \backslash V_{k+1}$, we have $\|\tilde{x}_n - x_n^i\| = \|x_n^k - x_n^i\| \leq 2\varepsilon_i$. Since n is eventually in V_i w.r.t. $\mathcal{U}$, $\|\tilde{x}_{\mathcal{U}} - x^i\| = \lim_{\mathcal{U}} \|\tilde{x}_n - x_n^i\| \leq 2\varepsilon_i$, namely $\tilde{x}_{\mathcal{U}}$ is the limit of the sequence x^k. Therefore $\|\tilde{x}_{\mathcal{U}}\|_{L,\mathcal{U}} \leq \lim_{\mathcal{U}} \|\tilde{x}_n\|_L \leq 1$, i.e. the result.

Finally we compute the constant R. Let $x_n \in X_n$ be s.t. $\|x_n\|_L = 1$, $\|x_n\| = R_n$, and consider the element $x_{\mathcal{U}} \in X_{\mathcal{U}}$. As observed above, $\|x_{\mathcal{U}}\|_L \leq 1$ and $\|x_{\mathcal{U}}\| = \lim_{\mathcal{U}} R_n$, implying $R \geq \lim_{\mathcal{U}} R_n$. Now, given $y_{\mathcal{U}} \in X_{\mathcal{U}}$ with $\|y_{\mathcal{U}}\|_L \leq 1$, realise it via a sequence y_n s.t. $\|y_n\|_L \leq 1$. By definition, $\|y_n\| \leq R_n$, therefore

$$\|y_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|y_n\| \leq \lim_{\mathcal{U}} R_n,$$

implying $R \leq \lim_{\mathcal{U}} R_n$. The thesis follows. $\square$

The rest of this subsection is devoted to the study of the relation between $\ell_L^\infty(X_n, \mathcal{U})'$ and $\ell_L^\infty(X_n', \mathcal{U})$.

Proposition 2.11 *Let $\{\sigma_n \in X_n'\}$ be uniformly bounded, and denote by $\sigma_{\mathcal{U}}(x_{\mathcal{U}}) := \lim_{\mathcal{U}} \sigma_n(x_n)$, $[x_n] = x_{\mathcal{U}} \in \ell_L^\infty(X_n, \mathcal{U})$. Then $\sigma_{\mathcal{U}}$ is well-defined, $\sigma_{\mathcal{U}} \in \ell_L^\infty(X_n, \mathcal{U})'$, and*

$$\|\sigma_{\mathcal{U}}\|'_{L,\mathcal{U}} = \lim_{\mathcal{U}} \|\sigma_n\|'_L.$$

Proof. Let $M > 0$ be s.t. $\|\sigma_n\|' \leq M$, $n \in \mathbb{N}$. We first prove that $\sigma_{\mathcal{U}}$ is well defined and bounded. Indeed, if $[x_n'] = [x_n] \in \ell_L^\infty(X_n, \mathcal{U})$, then $\lim_{\mathcal{U}} |\sigma_n(x_n') - \sigma_n(x_n)| \leq M \lim_{\mathcal{U}} \|x_n' - x_n\| = 0$. Moreover $|\sigma_{\mathcal{U}}(x_{\mathcal{U}})| \leq M \lim_{\mathcal{U}} \|x_n\| = M\|x_{\mathcal{U}}\|$, so that $\|\sigma_{\mathcal{U}}\|'_{\mathcal{U}} \leq M$. Finally

$$\begin{aligned}
\lim_{\mathcal{U}} \|\sigma_n\|'_L &= \lim_{\mathcal{U}} \sup_{x_n \in X_n} \frac{|\sigma_n(x_n)|}{\|x_n\|_L} = \sup_{\{x_n\} \in \ell_L^\infty(X_n)} \lim_{\mathcal{U}} \frac{|\sigma_n(x_n)|}{\|x_n\|_L} \\
&= \sup_{\{x_n\} \in \ell_L^\infty(X_n)} \frac{\lim_{\mathcal{U}} |\sigma_n(x_n)|}{\lim_{\mathcal{U}} \|x_n\|_L} = \sup_{x_{\mathcal{U}} \in \ell_L^\infty(X_n, \mathcal{U})} \sup_{[x_n] = x_{\mathcal{U}}} \frac{|\sigma_{\mathcal{U}}(x_{\mathcal{U}})|}{\lim_{\mathcal{U}} \|x_n\|_L} \\
&= \sup_{x_{\mathcal{U}} \in \ell_L^\infty(X_n, \mathcal{U})} \frac{|\sigma_{\mathcal{U}}(x_{\mathcal{U}})|}{\|x_{\mathcal{U}}\|_{L,\mathcal{U}}} = \|\sigma_{\mathcal{U}}\|'_{L,\mathcal{U}},
\end{aligned}$$

where in the last but one equality we used (2.3). Note also that, in that equality, the set of allowed elements in the supremum on the right is tacitly assumed not to contain $x_{\mathcal{U}} = 0$, while the set of allowed elements in the supremum on the left might also contain $x_{\mathcal{U}} = 0$, since in some examples one may find sequences $\{x_n\}$ such that $[x_n] = 0$ but $\lim_{\mathcal{U}} \|x_n\|_L > 0$. However for such sequences the numerator $|\sigma_{\mathcal{U}}(x_{\mathcal{U}})|$ is zero, therefore the supremum does not change. $\square$

Theorem 2.12 *Given a uniform sequence $(X_n, \|\cdot\|, \|\cdot\|_L)$ of Lip-spaces, the ultraproduct $\ell^\infty(X_n', \mathcal{U})$ of the dual spaces projects on the dual $\ell_L^\infty(X_n, \mathcal{U})'$*

of the Lip-ultraproduct. Moreover, given a uniformly bounded sequence σ_n of elements in X'_n, the element $\sigma_{\mathcal{U}}$ in $\ell^\infty(X'_n, \mathcal{U})$ gives the null functional on $\ell^\infty_L(X_n, \mathcal{U})$ if and only if $\lim_{\mathcal{U}} \|\sigma_n\|'_L = 0$.

Proof. We already observed that an element in $\ell^\infty(X'_n, \mathcal{U})$ gives rise to a functional on $X_{\mathcal{U}}$, and since $\|\cdot\|_{L,\mathcal{U}}$ is a norm on $(X_{\mathcal{U}})'$, the last statement follows from Proposition 2.11.

It is known ([19], Lemma 1, p. 77), and easy to show, that the pairing between $\ell^\infty(X'_n)$ and $\ell^\infty(X_n)$ given by $\langle\{\varphi_n\}, \{x_n\}\rangle = \lim_{\mathcal{U}} \varphi_n(x_n)$ gives rise to a pairing between $\ell^\infty(X'_n, \mathcal{U})$ and $\ell^\infty(X_n, \mathcal{U})$, hence to an isometric map $\ell^\infty(X'_n, \mathcal{U}) \to \ell^\infty(X_n, \mathcal{U})'$. We are interested in the contraction $\pi : \ell^\infty(X'_n, \mathcal{U}) \to \ell^\infty_L(X_n, \mathcal{U})'$ obtained by composing the previous isometric map with the quotient map from $\ell^\infty(X_n, \mathcal{U})'$ to $\ell^\infty_L(X_n, \mathcal{U})'$. We have to show that π is surjective.

Given $\varepsilon > 0$, let us choose the subspaces $V_n \subset X_n$ as in Proposition 2.7; we may also assume that all vectors in V_n have finite Lip-norm, hence the V_n form a uniform sequence of Lip-spaces with dimension bounded by N_ε. Clearly the Lip-ultraproduct $V_{\mathcal{U}}$ can be seen as a subspace of $X_{\mathcal{U}}$ of dimension at most N_ε, and the Lip-norm unit ball of $V_{\mathcal{U}}$ is ε-dense in the Lip-norm unit ball of $X_{\mathcal{U}}$.

Since the V_n have uniformly bounded dimension, $\ell^\infty(V_n, \mathcal{U})' \equiv \ell^\infty(V'_n, \mathcal{U})$ (cf. [19], Theorem 2, p. 78). Now take any norm-one element $\varphi \in (X_{\mathcal{U}})'$, restrict it to $V_{\mathcal{U}}$ and then extend it by Hahn-Banach theorem to an element $\tilde{\varphi}$ acting on $\ell^\infty(V_n, \mathcal{U})$. $\tilde{\varphi}$ can then be identified with an element of $\ell^\infty(V'_n, \mathcal{U})$, namely we may find elements $\tilde{\varphi}_n \in V'_n$ such that $\tilde{\varphi} = [\tilde{\varphi}_n]$, $\|\tilde{\varphi}\|' = \lim_{\mathcal{U}} \|\tilde{\varphi}_n\|'$. Extend then $\tilde{\varphi}_n$ to an element $\varphi'_n \in X'_n$, and set $\varphi' := [\varphi'_n] \in \ell^\infty(X'_n, \mathcal{U})$. Clearly $\|\varphi'\| \leq 1$, hence $\pi(\varphi') \equiv \varphi'_{\mathcal{U}}$, has norm less than 1, and observe that, by construction, φ and $\pi(\varphi')$ coincide on $V_{\mathcal{U}}$.

For any element x in $X_{\mathcal{U}}$ with $\|x\|_L \leq 1$ we may find $x_\varepsilon \in V_{\mathcal{U}}$ such that $\|x - x_\varepsilon\| \leq \varepsilon$, therefore

$$|\varphi(x) - \pi(\varphi')(x)| \leq |\varphi(x) - \varphi(x_\varepsilon)| + |\pi(\varphi')(x_\varepsilon) - \pi(\varphi')(x)| \leq 2\varepsilon,$$

As a consequence,

$$\|\pi(\varphi') - \varphi\|'_{L,\mathcal{U}} = \sup_{\|x\|_L \leq 1} |\varphi(x) - \pi(\varphi')(x)| \leq 2\varepsilon.$$

Choosing $\varepsilon = 1/2k$, we may then construct sequences $\varphi^k_n \in X'_n$ such that $\|\varphi^k_n\| \leq 1$ and, setting $\varphi^k = [\varphi^k_n]$, $\|\pi(\varphi^k) - \varphi\|'_{L,\mathcal{U}} \leq 1/k$. Then we construct a diagonal sequence as in the proof of Proposition 2.10.

Consider the sets $V_k \subset \mathbb{N}$ defined as

$$V_k = \{n \geq k : \|\varphi_n^j - \varphi_n^i\|'_L \leq \frac{3}{i}, i \leq j \leq k\},$$

and observe that the V_k's are non-increasing and belong to $\mathcal{U}$. Now we define the diagonal sequence as

$$\tilde{\varphi}_n = \varphi_n^k, \qquad n \in V_k \setminus V_{k+1}.$$

Then, when $n \in V_k$, and $k' \geq k$ satisfies $n \in V_{k'} \setminus V_{k'+1}$, we have $\|\tilde{\varphi}_n - \varphi_n^k\|'_L = \|\varphi_n^{k'} - \varphi_n^k\|'_L \leq 3/k$ hence, denoting by $\tilde{\varphi}$ the element in $\ell^\infty(X', \mathcal{U})$ corresponding to the sequence $\tilde{\varphi}_n$, we get $\|\pi(\tilde{\varphi}) - \pi(\varphi^k)\|'_{L,\mathcal{U}} = \lim_{\mathcal{U}} \|\tilde{\varphi}_n - \varphi_n^k\|'_L \leq 3/k$, hence $\|\pi(\tilde{\varphi}) - \varphi\|'_{L,\mathcal{U}} \leq \|\pi(\tilde{\varphi}) - \pi(\varphi^k)\|'_{L,\mathcal{U}} + \|\varphi - \pi(\varphi^k)\|'_{L,\mathcal{U}} \leq 4/k$. By the arbitrariness of k we get $\pi(\tilde{\varphi}) = \varphi$. $\square$

2.3 Order-unit spaces

In this subsection the results obtained thus far are used to prove that a Cauchy sequence of Lip-normed order-unit spaces converges to the Lip-ultraproduct for any free ultrafilter, thereby providing a different proof of a result already established by Rieffel [15], namely the completeness of the space of equivalence classes of Lip-normed order-unit spaces w.r.t. the quantum Gromov Hausdorff distance. In section 3 the same approach, suitably modified, will prove the completeness of the space of equivalence classes of Lip-normed operator systems w.r.t. d_∞, a result recently proved by Kerr and Li (though with different methods).

We recall now the definition of order-unit space, referring to [2] for more details.

An *order-unit space* is a real partially ordered vector space, X, with a distinguished element e (the order unit) satisfying:
1) (Order unit property) For each $a \in X$ there is an $r \in \mathbb{R}$ such that $a \leq re$;
2) (Archimedean property) For $a \in X$, $a \leq re$ for all $r > 0 \Rightarrow a \leq 0$.

On an order-unit space (X, e), we can define a norm as

$$\|a\| = \inf\{r \in \mathbb{R} : -re \leq a \leq re\}.$$

Then X becomes a normed vector space and we can consider its dual, X', consisting of the bounded linear functionals, equipped with the dual norm $\|\cdot\|'$.

By a *state* of an order-unit space (X, e), we mean a $\omega \in X'$ such that $\omega(e) = \|\omega\|' = 1$. States are automatically positive. Denote the set of all states of X by $S(X)$. It is a compact convex subset of X' under the weak$^*-$topology. Kadison's basic representation theorem [2] says that the natural pairing between X and $S(X)$ induces an isometric order isomorphism of X onto a dense subspace of the space $Af_{\mathbb{R}}(S(X))$ of all affine $\mathbb{R}-$valued continuous functions on $S(X)$, equipped with the supremum norm and the usual order on functions. We denote by $\hat{a}(\omega) := \omega(a)$, $\omega \in S(X)$, the affine function corresponding to $a \in X$.

For an order-unit space (X, e), we say that a densely defined seminorm L is a *Lip-seminorm* (cf. [15, Definition 2.1], where it is called Lip-norm) if:
1) For $a \in X$, we have $L(a) = 0$ if and only if $a \in \mathbb{R}e$.
2) The topology on $S(X)$ induced by the metric ρ_L

$$\rho_L(\omega_1, \omega_2) = \sup_{L(a)\leq 1} |\omega_1(a) - \omega_2(a)| \tag{2.4}$$

is the weak$^*-$topology.

We shall call Lip-normed order-unit space a complete order-unit space endowed with a lower semicontinuous Lip-seminorm.

Let us recall that the *radius* R of a Lip normed order-unit space is defined as half of the diameter of $(S(X), \rho_L)$. We now endow X with the norm

$$\|a\|_L := \max\{\frac{\|a\|}{R}, L(a)\}.$$

In the following Proposition we prove, for the sake of completeness, some results which are needed in the sequel, even though some of them are already known.

Proposition 2.13 *Let (X, e, L) be a Lip-normed order-unit space. Then*

(i) $\|a\|_0 := \inf_{\lambda\in\mathbb{R}} \|a - \lambda e\| = \frac{1}{2}(\max \hat{a} - \min \hat{a})$,

(ii) $\|a\|_{L,0} := \inf_{\lambda\in\mathbb{R}} \|a - \lambda e\|_L = L(a) = \min_{\lambda\in\mathbb{R}} \|a - \lambda e\|_L$,

(iii) $R = \sup_{L(a)\neq 0} \frac{\|a\|_0}{L(a)}$,

(iv) $R = \sup\limits_{\omega\in S(\mathcal{A})} \|\omega\|'_L = \sup\limits_{\varphi\in\mathcal{A}^*, \|\varphi\|=1} \|\varphi\|'_L = \sup\limits_{a\neq 0} \dfrac{\|a\|}{\|a\|_L}$.

Proof. (*i*)

$$\begin{aligned}\|a\|_0 &= \inf_{\lambda\in\mathbb{R}} \|a-\lambda e\| = \inf_{\lambda\in\mathbb{R}} \sup_{\omega\in S(X)} |\hat{a}(\omega)-\lambda| \\ &= \inf_{\lambda\in\mathbb{R}} \max\{|\max\hat{a}-\lambda|, |\min\hat{a}-\lambda|\} = \frac{\max\hat{a}-\min\hat{a}}{2}.\end{aligned}$$

(*ii*)

$$\begin{aligned}\|a\|_{L,0} :&= \inf_{\lambda\in\mathbb{R}} \|a-\lambda e\|_L = \inf_{\lambda\in\mathbb{R}} \max\left\{\frac{\|a-\lambda e\|}{R}, L(a-\lambda e)\right\} \\ &= \max\left\{\inf_{\lambda\in\mathbb{R}} \frac{\|a-\lambda e\|}{R}, L(a)\right\} = \max\left\{\frac{\|a\|_0}{R}, L(a)\right\} = L(a).\end{aligned}$$

Because $\|\cdot\|_L$ is lower semicontinuous, the last equality follows.
(*iii*)

$$\begin{aligned}\operatorname{diam} S(X) :&= \sup_{\omega_1,\omega_2\in S(X)} \rho_L(\omega_1,\omega_2) = \sup_{\omega_1,\omega_2\in S(X)} \sup_{L(a)\neq 0} \frac{|\omega_1(a)-\omega_2(a)|}{L(a)} \\ &= \sup_{L(a)\neq 0} \frac{\sup_{\omega_1,\omega_2\in S(X)} |\omega_1(a)-\omega_2(a)|}{L(a)} = \sup_{L(a)\neq 0} \frac{\max_{\omega\in S(X)} \hat{a}(\omega) - \min_{\omega\in S(X)} \hat{a}(\omega)}{L(a)} \\ &= \sup_{L(a)\neq 0} \frac{2\|a\|_0}{L(a)}.\end{aligned}$$

(*iv*) Let us observe that $\|e\|_L = R^{-1}$, therefore

$$R = \sup_{\omega\in S(X)} \frac{\omega(e)}{\|e\|_L} \leq \sup_{\omega\in S(X)} \|\omega\|'_L.$$

Conversely,

$$\sup_{\omega\in S(X)} \|\omega\|'_L = \sup_{\omega\in S(X)} \sup_{a\in X} \frac{\omega(a)}{\|a\|_L} \leq \sup_{\omega\in S(X)} \sup_{a\in X} R\frac{\omega(a)}{\|a\|} = R,$$

proving the first equality.

As for the second, let $\varphi \in X'$, $\|\varphi\| = 1$. Then, from [2] II.1.14, there are $\rho, \sigma \in S(X)$, $\lambda, \mu \in [0,1]$, $\lambda+\mu = 1$, s.t. $\varphi = \lambda\rho - \mu\sigma$. Therefore

$$\begin{aligned}\|\varphi\|'_L &= \sup_{\|a\|_L\leq 1} |\varphi(a)| \leq \sup_{\|a\|_L\leq 1} (\lambda|\rho(a)| + \mu|\sigma(a)|) \\ &\leq \lambda\|\rho\|'_L + \mu\|\sigma\|'_L \\ &= \sup_{\omega\in S(X)} \|\omega\|'_L,\end{aligned}$$

giving the result.

Finally,

$$\sup_{\omega \in S(X)} \|\omega\|'_L = \sup_{\omega \in S(X)} \sup_{a \in X} \frac{\omega(a)}{\|a\|_L} = \sup_{a \in X} \sup_{\omega \in S(X)} \frac{\omega(a)}{\|a\|_L} = \sup_{a \in X} \frac{\|a\|}{\|a\|_L}.$$

$\square$

Theorem 2.14 *Let (X, e, L) be a Lip-normed order-unit space of radius R, and define $\|a\|_L := \max\{\frac{\|a\|}{R}, L(a)\}$, $a \in X$. Then $(X, \|\cdot\|, \|\cdot\|_L)$ becomes a Lip-space whose radius as a Lip-space coincides with its radius as a Lip-normed order unit space.*

Proof. As $\{a \in X : \|a\|_L \leq 1\} = \{a \in X : \|a\| \leq R, L(a) \leq 1\}$ is compact ([13], Thm. 1.9), we get a Lip-space. The equality between the radii follows from Proposition 2.13 (iv). $\square$

Proposition 2.15 *Let $\{(X_n, e_n)\}$ be complete order-unit spaces, $\mathcal{U}$ a free ultrafilter. Then the ultraproduct $(\ell^\infty(X_n, \mathcal{U}), e_\mathcal{U})$ is a complete order-unit space.*

Proof. Let us recall that $\ell^\infty(X_n, \mathcal{U}) := \ell^\infty(X_n)/\mathcal{I}_\mathcal{U}$, where $\ell^\infty(X_n) := \{\{a_n\} : a_n \in X_n, \|\{a_n\}\| := \sup_n \|a_n\| < \infty\}$, and $\mathcal{I}_\mathcal{U} := \{\{a_n\} \in \ell^\infty(X_n) : \lim_\mathcal{U} \|a_n\| = 0\}$.

Observe that $\mathcal{I}_\mathcal{U}$ is a positively generated order ideal, because for any $\{a_n\} \in \mathcal{I}_\mathcal{U}$, there are $a_{n+}, a_{n-} \in X_{n,+}$ s.t. $a_n = a_{n+} - a_{n-}$ and $\|a_{n\pm}\| \leq \|a_n\|$, see [2] II.1.2.

Therefore, by [2] II.1.6, we only have to check the Archimedean property for $\ell^\infty(X_n, \mathcal{U})$. Assume $a_\mathcal{U} \leq \varepsilon e_\mathcal{U}$, for all $\varepsilon > 0$. Then $\varepsilon e_\mathcal{U} - a_\mathcal{U} \geq 0$, for all $\varepsilon > 0$, that is there is $U_\varepsilon \in \mathcal{U}$ s.t. $\varepsilon e_n - a_n \geq -\varepsilon e_n$, for all $n \in U_\varepsilon$, which implies that, for all $\varepsilon > 0$, $\{n \in \mathbb{N} : a_n < \varepsilon e_n\} \in \mathcal{U}$. Hence, because $\mathcal{U}$ is free, $U_k := \{n \geq k : a_n < \frac{1}{k} e_n\} \in \mathcal{U}$. Clearly $U_{k+1} \subset U_k$, $k \in \mathbb{N}$, and $\cap_{k \in \mathbb{N}} U_k = \emptyset$. Set $G_0 := \mathbb{N} \setminus U_1$, $G_k := U_k \setminus U_{k+1}$, $k \in \mathbb{N}$, and

$$b_n := \begin{cases} \|a_n\| e_n & n \in G_0 \\ \frac{1}{k} e_n & n \in G_k. \end{cases}$$

This implies $\lim_\mathcal{U} \|b_n\| = 0$, and $a_n - b_n \leq 0$, $n \in \mathbb{N}$, that is $a_\mathcal{U} = \lim_\mathcal{U} a_n = \lim_\mathcal{U}(a_n - b_n) \leq 0$, which is the thesis. $\square$

Proposition 2.16 *Let $\{(X_n, e_n, L_n)\}$ be a uniform sequence of Lip-normed order-unit spaces, $\mathcal{U}$ a free ultrafilter. Then the Lip-ultraproduct $(\ell^\infty_L(X, \mathcal{U}), e_\mathcal{U})$ is a Lip-normed order-unit space.*

Proof. It follows from Theorem 2.14 and Proposition 2.10 that $(X_\mathcal{U}, e_\mathcal{U})$

is a Lip-space, with $\|a_{\mathfrak{U}}\|_{L,\mathfrak{U}} := \inf_{\{y_n\}\equiv\{a_n\}} \sup_n \|y_n\|_L$. Then $(X_{\mathfrak{U}}, e_{\mathfrak{U}})$ is an order-unit space. Indeed, $\ell^\infty_L(X_n)$ is a closed subspace of $\ell^\infty(X_n)$, containing $e := \{e_n \in X_n\}_{n\in\mathbb{N}}$. So $\ell^\infty_L(X_n) \cap \mathfrak{I}_{\mathfrak{U}}$ is a positively generated order-ideal of $\ell^\infty_L(X_n)$, and arguing as in the previous Proposition, $X_{\mathfrak{U}} = \pi_{\mathfrak{U}}(\ell^\infty_L(X_n)) = \ell^\infty_L(X_n)/\ell^\infty_L(X_n) \cap \mathfrak{I}_{\mathfrak{U}}$ is Archimedean, therefore an order-unit space.

Let us set $L(a_{\mathfrak{U}}) := \inf_{\lambda\in\mathbb{R}} \|a_{\mathfrak{U}} - \lambda e_{\mathfrak{U}}\|_{L,\mathfrak{U}}$. Then it follows from Proposition 2.2 that L is a lower semicontinuous Lipschitz seminorm. Finally we prove that ρ_L induces on $S(X_{\mathfrak{U}})$ the weak*-topology. Indeed, for $\omega_1, \omega_2 \in S(X_{\mathfrak{U}})$, we have

$$\begin{aligned}\rho_L(\omega_1,\omega_2) &= \sup_a \frac{|\omega_1(a)-\omega_2(a)|}{L(a)} = \sup_a \frac{|\omega_1(a)-\omega_2(a)|}{\inf_\lambda \|a_{\mathfrak{U}} - \lambda e_{\mathfrak{U}}\|_{L,\mathfrak{U}}} \\ &= \sup_{a,\lambda} \frac{|\omega_1(a-\lambda e_{\mathfrak{U}})-\omega_2(a-\lambda e_{\mathfrak{U}})|}{\|a_{\mathfrak{U}} - \lambda e_{\mathfrak{U}}\|_{L,\mathfrak{U}}} = \sup_a \frac{|\omega_1(a)-\omega_2(a)|}{\|a_{\mathfrak{U}}\|_{L,\mathfrak{U}}} \\ &= \|\omega_1-\omega_2\|'_L. \end{aligned} \tag{2.5}$$

Therefore ρ_L induces on $S(X_{\mathfrak{U}})$ the weak*-topology by Proposition 2.3, and L is a Lip-seminorm. □

The seminorm L in the previous Proposition can be obtained more directly in terms of the seminorms L_n, as the following Proposition shows.

Proposition 2.17 *Let $\{(X_n, e_n, L_n)\}$ and $\mathfrak{U}$ be as in the previous Proposition. Then*

(i) The Lip-seminorm on the Lip-ultraproduct of order-unit spaces gives back the Lip-norm on the Lip-ultraproduct of Lip-spaces, namely, for any $x_{\mathfrak{U}}$ in the ultraproduct,

$$\|x_{\mathfrak{U}}\|_{L,\mathfrak{U}} = \max\{\frac{\|x_{\mathfrak{U}}\|}{R_{\mathfrak{U}}}, L_{\mathfrak{U}}(x_{\mathfrak{U}})\}. \tag{2.6}$$

(ii) The Lip-seminorm on the Lip-ultraproduct is the quotient seminorm, namely

$$L(x_{\mathfrak{U}}) = \inf_{[x_n]=x_{\mathfrak{U}}} \sup_n L_n(x_n). \tag{2.7}$$

Proof. Let us first observe that

$$\lim_{\mathfrak{U}} R_n = R_{\mathfrak{U}} = \sup \frac{\|x_{\mathfrak{U}}\|}{\|x_{\mathfrak{U}}\|_L},$$

where we used Propositions 2.13(vi) and 2.10.

Let us now set $L_{\mathfrak{U}}(x_{\mathfrak{U}}) := \inf_{[x_n]=x_{\mathfrak{U}}} \sup_n L_n(x_n)$. We want to prove that, $\forall x_{\mathfrak{U}} \in X_{\mathfrak{U}}$, $\exists \{\tilde{x}_n\} \in \ell^\infty_L(X_n)$ s.t. $[\tilde{x}_n] = x_{\mathfrak{U}}$ and

$$\lim_{\mathfrak{U}} \|\tilde{x}_n\|_L = \|x_{\mathfrak{U}}\|_{L,\mathfrak{U}} \qquad \lim_{\mathfrak{U}} L_n(\tilde{x}_n) = L_{\mathfrak{U}}(x_{\mathfrak{U}}). \tag{2.8}$$

Let $x_{\mathcal{U}} \in X_{\mathcal{U}}$, and, for any $k \in \mathbb{N}$, choose sequences x_n^k realising it and such that

$$L_n(x_n^k) \leq (1 + \frac{1}{k})L_{\mathcal{U}}(x_{\mathcal{U}}), \quad n \in \mathbb{N} \tag{2.9}$$

As $\lim_{\mathcal{U}} \frac{\|x_n^k\|}{R_n} = \frac{\|x\|_{\mathcal{U}}}{R_{\mathcal{U}}}$, there is $U_k \in \mathcal{U}$ s.t. $\frac{\|x_n^k\|}{R_n} \leq (1 + \frac{1}{k})\frac{\|x\|_{\mathcal{U}}}{R_{\mathcal{U}}}$, $n \in U_k$. Setting, if necessary,

$$\widetilde{x_n^k} := \begin{cases} x_n^k & n \in U_k \\ 0 & n \notin U_k, \end{cases}$$

we obtain

$$\frac{\|\widetilde{x_n^k}\|}{R_n} \leq (1 + \frac{1}{k})\frac{\|x\|_{\mathcal{U}}}{R_{\mathcal{U}}}, n \in \mathbb{N}, \tag{2.10}$$

and $\{\widetilde{x_n^k}\} \equiv \{x_n^k\}$, for all $k \in \mathbb{N}$. Therefore we can assume that $\{x_n^k\}$ have been chosen in such a way that (2.9), (2.10) are satisfied.

Using (2.2) and $L_n(y_n) \leq \|y_n\|_L$, we obtain $L_{\mathcal{U}}(x_{\mathcal{U}}) \leq \|x_{\mathcal{U}}\|_L$.

Set, for all $k \in \mathbb{N}$, $V_k = \{n \geq k : \|x_n^i - x_n^j\| \leq \frac{1}{i}, i \leq j \leq k\}$, $V_0 := \mathbb{N} \setminus V_1$, and then $\tilde{x}_n = \frac{k}{k+1}x_n^k$, $n \in V_k \setminus V_{k+1}$. Then, $[\tilde{x}_n] = x_{\mathcal{U}}$. Moreover, for $k, n \in \mathbb{N}$, we have, using (2.9), (2.10),

$$\begin{aligned} \|x_n^k\|_L &= \max\left\{L_n(x_n^k), \frac{\|x_n^k\|}{R_n}\right\} \\ &\leq (1 + \frac{1}{k})\max\left\{L_{\mathcal{U}}(x_{\mathcal{U}}), \frac{\|x_{\mathcal{U}}\|_{\mathcal{U}}}{R_{\mathcal{U}}}\right\} \leq (1 + \frac{1}{k})\|x_{\mathcal{U}}\|_L, \end{aligned}$$

so that, for $k \in \mathbb{N}$, $\ell \in \mathbb{N}$, $\ell \geq k$, $n \in V_\ell \setminus V_{\ell+1}$, we get $\|\tilde{x}_n\|_L = \frac{\ell}{\ell+1}\|x_n^\ell\|_L \leq \|x_{\mathcal{U}}\|_L$, which implies $\|\tilde{x}_n\|_L \leq \|x_{\mathcal{U}}\|_L$, for $n \in V_k$, and $\lim_{\mathcal{U}} \|\tilde{x}_n\|_L \leq \|x_{\mathcal{U}}\|_L$. As the opposite inequality is always true, we obtain

$$\lim_{\mathcal{U}} \|\tilde{x}_n\|_L = \|x_{\mathcal{U}}\|_L.$$

Finally, from (2.9), for $k \in \mathbb{N}$, $\ell \in \mathbb{N}$, $\ell \geq k$, $n \in V_\ell \setminus V_{\ell+1}$, we get $L_n(\tilde{x}_n) = \frac{\ell}{\ell+1}L_n(x_n^\ell) \leq L_{\mathcal{U}}(x_{\mathcal{U}})$, which implies $L_n(\tilde{x}_n) \leq L_{\mathcal{U}}(x_{\mathcal{U}})$, $n \in V_k$, and $\lim_{\mathcal{U}} L_n(\tilde{x}_n) \leq L_{\mathcal{U}}(x_{\mathcal{U}})$. As the opposite inequality is always true, we obtain

$$\lim_{\mathcal{U}} L_n(\tilde{x}_n) = L_{\mathcal{U}}(x_{\mathcal{U}}),$$

and we have proved (2.8).

As a consequence, we get equation (2.6):

$$\begin{aligned} \|x_{\mathcal{U}}\|_{L,\mathcal{U}} &= \lim_{\mathcal{U}} \|\tilde{x}_n\|_L = \lim_{\mathcal{U}} \max\{\frac{\|\tilde{x}_n\|}{R_n}, L_n(\tilde{x}_n)\} \\ &= \max\{\lim_{\mathcal{U}} \frac{\|\tilde{x}_n\|}{R_n}, \lim_{\mathcal{U}} L_n(\tilde{x}_n)\} = \max\{\frac{\|x_{\mathcal{U}}\|}{R_{\mathcal{U}}}, L_{\mathcal{U}}(x_{\mathcal{U}})\}. \end{aligned}$$

Let us now denote by λ_n the constant for which $\|\tilde{x}_n\|_0 = \|\tilde{x} - \lambda_n e_n\|$. Since $\{\tilde{x}_n\}$ is norm bounded, $\{\lambda_n\}$ is bounded, hence $\lim_{\mathcal{U}} \lambda_n = \lambda_{\mathcal{U}} \in \mathbb{R}$. Then

$$\begin{aligned}\|x_{\mathcal{U}}\|_0 &= \inf_\lambda \|x_{\mathcal{U}} - \lambda e_{\mathcal{U}}\| \leq \|x_{\mathcal{U}} - \lambda_{\mathcal{U}} e_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|\tilde{x}_n - \lambda_n e_n\| \\ &= \lim_{\mathcal{U}} \|\tilde{x}_n\|_0 \leq \lim_{\mathcal{U}} R_n L_n(\tilde{x}_n) = R_{\mathcal{U}} L_{\mathcal{U}}(x_{\mathcal{U}}).\end{aligned}$$

Therefore, using (2.6) for the vector $x_{\mathcal{U}} - \lambda e_{\mathcal{U}}$ and the inequality above,

$$\begin{aligned}L(x_{\mathcal{U}}) &= \inf_\lambda \|x_{\mathcal{U}} - \lambda e_{\mathcal{U}}\|_{L,\mathcal{U}} = \max\{\frac{\inf_\lambda \|x_{\mathcal{U}} - \lambda e_{\mathcal{U}}\|_{\mathcal{U}}}{R_{\mathcal{U}}}, L_{\mathcal{U}}(x_{\mathcal{U}})\} \\ &= \max\{\frac{\|x_{\mathcal{U}}\|_0}{R_{\mathcal{U}}}, L_{\mathcal{U}}(x_{\mathcal{U}})\} = L_{\mathcal{U}}(x_{\mathcal{U}}),\end{aligned}$$

concluding the proof. □

Now we can prove the analogue of Theorem 2.12.

Theorem 2.18 *Given a uniform sequence $\{(X_n, e_n, L_n)\}$ of Lip-normed order-unit spaces, the ultraproduct $\ell^\infty(X'_n, \mathcal{U})$ of the dual spaces projects on the dual $(\ell^\infty_L(X_n, \mathcal{U}))'$ of the Lip-ultraproduct. Moreover, any state on $\ell^\infty_L(X_n, \mathcal{U})$ can be represented by an element of $\ell^\infty(X'_n, \mathcal{U})$ given by sequences of states.*

Proof. Only the last part needs a proof, which is similar to that of Theorem 2.12, so that we only indicate the small difference.

Given $\varepsilon > 0$, let us choose the subspaces $V_n \subset X_n$ as in the proof of the cited Theorem, but with the further request that $e_n \in V_n$, for any $n \in \mathbb{N}$.

Now take any $\varphi \in S(X_{\mathcal{U}})$, follow the proof of the cited Theorem until you get elements $\varphi'_n \in X'_n$, and set $\varphi' := [\varphi'_n] \in \ell^\infty(X'_n, \mathcal{U})$. In this case, since $V_n \ni e_n$, $\|\varphi'_n\| = 1$, and recall that, by construction, φ and $\pi(\varphi')$ coincide on $V_{\mathcal{U}}$. Therefore $\lim_{\mathcal{U}} \varphi'_n(e_n) = \pi(\varphi')(e_{\mathcal{U}}) = 1$. Now we may decompose $\varphi'_n = \alpha_n \psi^1_n - \beta_n \psi^2_n$ where $\alpha_n + \beta_n = 1$, $\alpha_n \geq 0$, $\beta_n \geq 0$, and ψ^i_n are states ([2], II.1.14). Therefore we obtain $\alpha_n \to 1$ and $\beta_n \to 0$, implying that $[\psi^1_n] = \varphi'$, namely can be realised via sequences of states. The proof continues as in the cited Theorem. □

Let us recall that [5], given a sequence (X_n, d_n) of metric spaces with uniformly bounded radius, and $\mathcal{U}$ a free ultrafilter on $\mathbb{N}$, the ultralimit $(X_{\mathcal{U}}, d_{\mathcal{U}})$ is defined as the space of equivalence classes $[x_n]$, $x_n \in X_n$, with distance $d_{\mathcal{U}}([x_n], [x'_n]) = \lim_{\mathcal{U}} d(x_n, x'_n)$, and it follows that $[x_n] = [x'_n]$ when they have zero distance. According to Proposition 2.11, we have $\rho_{L_{\mathcal{U}}}(\varphi_{\mathcal{U}}, \psi_{\mathcal{U}}) = \lim_{\mathcal{U}} \rho_L(\varphi_n, \psi_n)$, therefore we get the following.

Corollary 2.19 *Let $\{(X_n, e_n, L_n)\}$ be a uniform sequence of Lip-normed order-*

unit spaces. The state space of the Lip-ultraproduct can be isometrically identified with the ultralimit of the approximating state spaces.

Let us now recall Rieffel's notion of quantum Gromov-Hausdorff convergence [15].

Let (X, e_X, L_X), (Y, e_Y, L_Y) be Lip-normed order-unit spaces. Denote by $\mathcal{M}(L_X, L_Y)$ the set of lower semicontinuous Lip-seminorms on $X \oplus Y$ which induce L_X and L_Y on X, Y respectively. Any $L \in \mathcal{M}(L_X, L_Y)$ gives rise to a metric ρ_L, on $S(X \oplus Y)$. Therefore, identifying $S(X)$ and $S(Y)$ with (closed, convex) subsets of $S(X \oplus Y)$, we can consider the Hausdorff distance between them w.r.t. ρ_L, namely $\rho_L^H(S(X), S(Y))$. We define the *quantum Gromov–Hausdorff distance* between X and Y by

$$\operatorname{dist}(X, Y) = \inf\{\rho_L^H(S(X), S(Y)) : L \in \mathcal{M}(L_X, L_Y)\}. \tag{2.11}$$

Theorem 2.20 *Let $\{(X_n, e_n, L_n)\}$ be a Cauchy sequence of Lip-normed order-unit spaces. Then, for any free ultrafilter $\mathcal{U}$, the Lip-normed Lip-ultraproduct $(\ell_L^\infty(X_n, \mathcal{U}), e_\mathcal{U}, L_\mathcal{U})$ is the limit of the sequence.*

Proof. Let $\varepsilon > 0$ be given, and let $n_\varepsilon \in \mathbb{N}$ be s.t. for all $m, n > n_\varepsilon$ there is $L_{mn} \in \mathcal{M}(X_m, X_n)$ s.t. $\rho^H_{L_{mn}}(S(X_n), S(X_m)) < \varepsilon$. Observe that, having fixed $n > n_\varepsilon$, the Lip-ultraproduct of the spaces $\{X_n \oplus X_i\}_{i \in \mathbb{N}}$ naturally identifies with $X_n \oplus X_\mathcal{U}$. Therefore, $X_n \oplus X_\mathcal{U}$ inherits a Lip-seminorm $L_{n\mathcal{U}}$ with respect to which $S(X_\mathcal{U}) \subset \overline{B}_\varepsilon(S(X_n))$ and $S(X_n) \subset \overline{B}_\varepsilon(S(X_\mathcal{U}))$.

Indeed, if $\omega \in S(X_n)$, then, for all $m > n_\varepsilon$, there is $\varphi_m \in S(X_m)$ s.t. $\rho_{L_{mn}}(\omega, \varphi_m) < \varepsilon$. Set $\varphi_\mathcal{U}(x_\mathcal{U}) := \lim_\mathcal{U} \varphi_m(x_m)$, $[x_m] = x_\mathcal{U}$ (see Proposition 2.11) so that $\varphi_\mathcal{U} \in S(X_\mathcal{U})$ and, by (2.5) and Proposition 2.11,

$$\rho_{L_{n\mathcal{U}}}(\omega, \varphi_\mathcal{U}) = \lim_{m \to \mathcal{U}} \rho_{L_{mn}}(\omega, \varphi_m) \leq \varepsilon.$$

Viceversa, let $\varphi \in S(X_\mathcal{U})$, and choose, by Theorem 2.18, $\varphi_m \in S(X_m)$, s.t. $\varphi_\mathcal{U}(x_\mathcal{U}) := \lim_\mathcal{U} \varphi_m(x_m)$, $[x_m] = x_\mathcal{U}$, and let, for $m > n_\varepsilon$, $\omega_m \in S(X_n)$ be s.t. $\rho_{L_{mn}}(\varphi_m, \omega_m) < \varepsilon$. Set $\omega := \lim_{m \to \mathcal{U}} \omega_m \in S(X_n)$. Then, $\rho_{L_{n\mathcal{U}}}(\omega, \varphi) = \lim_{m \to \mathcal{U}} \rho_{L_{mn}}(\omega, \varphi_m) \leq \lim_{m \to \mathcal{U}} \rho_{L_{mn}}(\omega_m, \varphi_m) + \rho_{L_n}(\omega_m, \omega) \leq \varepsilon$. $\square$

3 Operator Systems

We begin by describing our operator system framework. For references see [12].

Definition 3.1 *An operator system X is a complex vector space with a conjugate linear involution $^* : x \in X \to x^* \in X$, satisfying*

(i) X is matrix ordered, i.e.

(i′) for any $p \in \mathbb{N}$, there is a proper cone $M_p(X)_+ \subset M_p(X)_h$, where the subscript h refers to hermitian elements

(i″) for any $p, q \in \mathbb{N}$, $A \in M_{qp}(\mathbb{C})$, $A^* M_q(X)_+ A \subset M_p(X)_+$

(ii) X has a matrix order-unit, i.e. there is $e \in X_h$ s.t., with $e^p := diag(e, \dots, e) \in M_p(X)_+$, for any $x \in M_p(X)_h$, there is $r > 0$ s.t. $x + re^p \in M_p(X)_+$

(iii) the matrix order-unit e is Archimedean, i.e. if $x \in M_p(X)$ is s.t. $x + re^p \in M_p(X)_+$, for all $r > 0$, then $x \in M_p(X)_+$.

Given operator systems X and Y we say that a linear map $\varphi : X \to Y$ is *n-positive* if the map $\mathrm{id}_n \otimes \varphi : M_n \otimes X \to M_n \otimes Y$ is positive, and if $\mathrm{id}_n \otimes \varphi$ is positive for all $n \in \mathbb{N}$ then we say that φ is *completely positive*. A completely positive (resp. unital completely positive) linear map will be referred to as a *c.p.* (resp. *u.c.p.*) map. If $\varphi : X \to Y$ is a unital m-positive map with m-positive inverse for $m = 1, \dots, n$ then φ is a *unital n-order isomorphism*, and if φ is u.c.p. with c.p. inverse then φ is a *unital complete order isomorphism*.

We denote by $UCP_n(X)$ the collection of all u.c.p. maps from X into M_n (the *matrix state spaces*).

Following Kerr [8], we introduce Lip-norms and matricial distances on operator systems. By a *Lip-normed operator system* we mean a pair (X, L) where X is a complete operator system and L is a lower semicontinuous Lip-seminorm on X satisfying $L(x^*) = L(x)$. If X is a unital C^*-algebra then we will also refer to (X, L) as a *Lip-normed unital C^*-algebra.*

Definition 3.2 *Let (X, L) be a Lip-normed operator system and $p \in \mathbb{N}$. We define the metric ρ_L on $UCP_p(X)$ by*

$$\rho_L(\varphi, \psi) = \sup_{L(x) \leq 1} \|\varphi(x) - \psi(x)\|$$

for all $\varphi, \psi \in UCP_p(X)$,

Let (X, L_X) and (Y, L_Y) be Lip-normed operator systems. We denote by $\mathcal{M}(L_X, L_Y)$ the collection of lower semicontinuous Lip-seminorms on $X \oplus Y$ which induce L_X and L_Y via the quotient maps onto X and Y, respectively.

Let $L \in \mathcal{M}(L_X, L_Y)$. Since the projection map $X \oplus Y \to X$ is u.c.p., by [8], we obtain an isometry $UCP_p(X) \to UCP_p(X \oplus Y)$ with respect to ρ_{L_X} and ρ_L.

Similarly, we also have an isometry $UCP_p(Y) \to UCP_p(X \oplus Y)$. For notational simplicity we will thus identify $UCP_p(X)$ and $UCP_p(Y)$ with their respective images under these isometries.

Definition 3.3 *Let (X, L_X) and (Y, L_Y) be Lip-normed operator systems. For each $p \in \mathbb{N}$ we define the p-distance*

$$\operatorname{dist}_p(X, Y) = \inf_{L \in \mathcal{M}(L_X, L_Y)} \rho_L^H(UCP_p(X), UCP_p(Y))$$

where ρ_L^H denotes Hausdorff distance with respect to the metric ρ_L. We also define the complete quantum Gromov-Hausdorff distance

$$\operatorname{dist}_\infty(X, Y) = \inf_{L \in \mathcal{M}(L_X, L_Y)} \sup_{p \in \mathbb{N}} \rho_L^H(UCP_p(X), UCP_p(Y)).$$

Proposition 3.4 *Let $\{(X_n, e_n)\}$ be operator systems, $\mathcal{U}$ a free ultrafilter. Then the ultraproduct $(\ell^\infty(X_n, \mathcal{U}), e_\mathcal{U})$ is an operator system.*

Proof. Denote $X_\mathcal{U} := \ell^\infty(X_n, \mathcal{U})$. It follows from Definition 3.1 $(i'), (ii), (iii)$, that, for any $p \in \mathbb{N}$, $(M_p(X_n), e_n^p)$ is a complete order-unit space, so that $(M_p(X_\mathcal{U}) \equiv \ell^\infty(M_p(X_n), \mathcal{U}), e_\mathcal{U}^p)$ is a complete order-unit space, by Proposition 2.15. Finally, for any $p, q \in \mathbb{N}$, $A \in M_{qp}(\mathbb{C})$, from $A^* M_q(X_n)_+ A \subset M_p(X_n)_+$ it follows $A^* M_q(X_\mathcal{U})_+ A \subset M_p(X_\mathcal{U})_+$. Therefore $(X_\mathcal{U}, e_\mathcal{U})$ is a complete operator system. □

Proposition 3.5 *Let $\{(X_n, e_n, L_n)\}$ be a uniform sequence of Lip-normed operator systems, $\mathcal{U}$ a free ultrafilter. Then the Lip-ultraproduct $(\ell_L^\infty(X_n, \mathcal{U}), e_\mathcal{U}, L_\mathcal{U})$ is a Lip-normed operator system.*

Proof. It follows from Propositions 3.4 and 2.16. □

Proposition 3.6 *Let $\{(X_n, e_n, L_n)\}$ be a uniform sequence of Lip-normed operator systems, $\mathcal{U}$ a free ultrafilter. Let $p \in \mathbb{N}$, $\{\sigma_n \in UCP_p(X_n)\}$, $\{\tau_n \in UCP_p(X_n)\}$. Define $\sigma_\mathcal{U}(a_\mathcal{U}) := \lim_\mathcal{U} \sigma_n(a_n)$, $a_\mathcal{U} = [a_n] \in \ell_L^\infty(X_n, \mathcal{U})$, and $\tau_\mathcal{U}$ analogously. Then $\sigma_\mathcal{U}, \tau_\mathcal{U}$ are well defined and belong to $UCP_p(X_\mathcal{U})$, and*

$$\rho_{L_\mathcal{U}}(\sigma_\mathcal{U}, \tau_\mathcal{U}) = \lim_\mathcal{U} \rho_{L_n}(\sigma_n, \tau_n).$$

Proof. The first part is as in Proposition 2.11. Moreover

$$\begin{aligned}
\lim_{\mathcal{U}} \rho_{L_n}(\sigma_n, \tau_n) &= \lim_{\mathcal{U}} \sup_{x_n \in X_n} \frac{\|\sigma_n(x_n) - \tau_n(x_n)\|}{L_n(x_n)} = \sup_{x \in \ell^\infty_L(X_n)} \lim_{\mathcal{U}} \frac{\|\sigma_n(x_n) - \tau_n(x_n)\|}{L_n(x_n)} \\
&= \sup_{x_{\mathcal{U}} \in X_{\mathcal{U}}} \sup_{[x_n] = x_{\mathcal{U}}} \frac{\|\sigma_{\mathcal{U}}(x_{\mathcal{U}}) - \tau_{\mathcal{U}}(x_{\mathcal{U}})\|}{\lim_{\mathcal{U}} L_n(x_n)} \\
&= \sup_{x_{\mathcal{U}} \in X_{\mathcal{U}}} \frac{\|\sigma_{\mathcal{U}}(x_{\mathcal{U}}) - \tau_{\mathcal{U}}(x_{\mathcal{U}})\|}{L_{\mathcal{U}}(x_{\mathcal{U}})} = \rho_{L_{\mathcal{U}}}(\sigma_{\mathcal{U}}, \tau_{\mathcal{U}}),
\end{aligned}$$

where in the last but one equality we used (2.8), and the consideration at the end of the proof of Proposition 2.11 applies. □

Theorem 3.7 *Let $\{(X_n, e_n, L_n)\}$ be a Cauchy sequence of Lip-normed operator systems. Then $(X_{\mathcal{U}}, e_{\mathcal{U}}, L_{\mathcal{U}})$ is its limit, for any free ultrafilter $\mathcal{U}$.*

Proof. It is similar to the proof of Theorem 2.20, by making use of Proposition 3.6, and the analogue of Theorem 2.18. □

4 C*-algebras

4.1 The problem of completeness

Let us consider the space of equivalence classes of Lip-normed unital C*-algebras, endowed with one of the pseudo-distances dist_p, $p \in \mathbb{N} \cup \{\infty\}$. Kerr showed [8] that for $p \geq 2$ it is indeed a distance, namely that if $\mathrm{dist}_p(\mathcal{A}, \mathcal{B}) = 0$ then $\mathcal{A}$ and $\mathcal{B}$ are Lip-isometric *-isomorphic C*-algebras.

Our aim is to study the completeness of the equivalence classes of C*-algebras endowed with the metrics dist_p. When dist_∞ is considered, the limit of a Cauchy sequence exists as an operator system. The result of Kerr implies that, on such a space, the C*-structure, i.e. a product w.r.t. which the norm is a C*-norm, is unique, if it exists. However, besides the mere question of existence of such a product, we are interested in products which are approximated by the products of the approximating algebras.

A first attempt in this respect has been made by David Kerr and Hanfeng Li [8,11], who introduced the concept of f-Leibniz property, showing that if all algebras in a Cauchy sequence enjoy the f-Leibniz property for the same function f, then the limit space inherits a product structure (satisfying the f-Leibniz property).

We observe however that realising the limit space as a Lip-ultraproduct allows a much more stringent characterization of the cases in which the product structure is inherited by the limit space.

Indeed, when realising the limit as a Lip-ultraproduct, one would like to set

$$[x_n]\ [y_n] = [x_n y_n]. \tag{4.1}$$

Unfortunately it is not true in general that $[x_n y_n]$ belongs to the Lip-ultraproduct, namely has finite Lip-norm or at least can be approximated in norm by elements with finite Lip-norm. In other words, while (4.1) defines a product on $\ell^\infty(\mathcal{A}_n, \mathfrak{U})$, it is not always true that $\ell^\infty_L(\mathcal{A}_n, \mathfrak{U})$ is a subalgebra of $\ell^\infty(\mathcal{A}_n, \mathfrak{U})$.

We then introduce the following

Definition 4.1 *Let $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence of Lip-normed unital C*-algebras w.r.t. the* dist_p *metrics. If $\mathfrak{U}$ is a free ultrafilter on $\mathbb{N}$, we say that the Lip-ultraproduct $\ell^\infty_L(\mathcal{A}_n, \mathfrak{U})$ inherits the C*-structure if it is a sub-algebra of $\ell^\infty(\mathcal{A}_n, \mathfrak{U})$. In general, we say that the limit inherits the C*-structure if $\ell^\infty_L(\mathcal{A}_n, \mathfrak{U})$ does, for some free ultrafilter $\mathfrak{U}$.*

Proposition 4.2 *Let $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence of Lip-normed unital C*-algebras w.r.t. the* dist_p *metrics, and suppose $\ell^\infty_L(\mathcal{A}_n, \mathfrak{U})$ inherits the C*-structure for a suitable free ultrafilter $\mathfrak{U}$. Then the sequence $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ converges to the C*-algebra $\ell^\infty_L(\mathcal{A}_n, \mathfrak{U})$.*

Proof. Cf. the proofs of Theorems 2.20, 3.7. □

As we shall see in Subsection 4.2, the general situation is as ugly as possible: there are Cauchy sequences for which the limit is not a C*-algebra, and even Cauchy sequences for which the limit can be endowed with a C*-product, but this is not inherited from the approximating C*-algebras.

Theorem 4.3

(i) The space of equivalence classes of Lip-normed unital C-algebras, endowed with the distance* dist_p*, $p \geq 2$, is not complete.*

(ii) There exist sequences $(\mathcal{A}_n, L_n)$ converging to a Lip-normed unital C-algebra $(\mathcal{A}, L)$ for which the C*-structure is not inherited.*

We are not able to characterise the Cauchy sequences for which the limit admits a C*-product, but we can characterise those for which the C*-product is inherited. Our condition may be seen as a generalisation of the Kerr-Li condition.

Definition 4.4 *We say that the pair $(\mathcal{A}, L)$ consisting of a unital C*-algebra* *and a seminorm is a Lip-normed unital C*-algebra if L is a lower semicontinuous Lip-seminorm according to Section 3.* $(\mathcal{A}, L)$ *will be called quasi Lip-normed if we drop the assumption that Lip-elements are dense, but assume that they generate* $\mathcal{A}$ *as a C*-algebra.*

Given a quasi Lip-normed unital C*-algebra, we consider the function

$$\varepsilon(r) = \sup_{\|x\|_L \le 1} \inf_{\|y\|_L \le r} \|y - x^* x\|,$$

where $\|\cdot\|_L$ denotes the Lipschitz norm defined in subsection 2.3.

Lemma 4.5 *The quasi Lip-normed unital C*-algebra* $(\mathcal{A}, L)$ *is Lip-normed if and only if*

$$\lim_{r \to \infty} \varepsilon(r) = 0. \tag{4.2}$$

Proof. Assume Lip-elements are dense. This means that, for any $\varepsilon > 0$, the open sets

$$\Omega(\varepsilon, r) = \bigcup_{\|x\|_L \le r} B(x, \varepsilon), \quad r > 0$$

give an open cover of $\mathcal{A}$. Since $\{x^* x : \|x\|_L \le 1\}$ is compact, we may extract a finite subcover, hence $\forall \varepsilon > 0\ \exists r > 0$ s.t. $\{x^* x : \|x\|_L \le 1\} \subset \Omega(\varepsilon, r)$, or, equivalently, $\forall \varepsilon > 0\ \exists r > 0$ s.t. $\varepsilon(r) < \varepsilon$, proving one implication.

Now assume $\varepsilon(r) \to 0$. This implies that for any Lip-element x, $x^* x$ can be arbitrarily well approximated (in norm) by Lip-elements. Since xy can be written as a linear combination of $x^* x$, $y^* y$, $(x+y)^*(x+y)$ and $(x+iy)^*(x+iy)$, we may conclude that products of Lip-elements can be arbitrarily well approximated (in norm) by Lip-elements. Now take two norm-one elements x and y in the norm closure of the space of Lip-elements. Choose two Lip-elements x_ε, y_ε, still with norm one, such that $\|x - x_\varepsilon\| < \varepsilon$, $\|y - y_\varepsilon\| < \varepsilon$, and then a Lip-element z such that $\|x_\varepsilon y_\varepsilon - z\| < \varepsilon$. We get

$$\|xy - z\| \le \|xy - x_\varepsilon y_\varepsilon\| + \|x_\varepsilon y_\varepsilon - z\| \le 3\varepsilon.$$

This means that the norm closure of the space of Lip-elements is an algebra, hence a C*-algebra. By definition of quasi Lip-normed unital C*-algebra, such closure coincides with $\mathcal{A}$. $\square$

Let us now compare condition (4.2) with the f-Leibniz condition. Let us recall that $(\mathcal{A}, L)$ satisfies the f-Leibniz condition w.r.t. a given continuous 4-variable function f if

$$L(ab) \le f(L(a), L(b), \|a\|, \|b\|), \qquad a, b \in \mathcal{A}.$$

Proposition 4.6 *Let* $(\mathcal{A}, L)$ *be a quasi Lip-normed unital C*-algebra. The following are equivalent:*

(i) $(\mathcal{A}, L)$ *satisfies the f-Leibniz condition w.r.t. some function f*

(ii) $(\mathcal{A}, L)$ *satisfies the condition*

$$\|ab\|_L \leq C\|a\|_L\|b\|_L, \qquad a, b \in \mathcal{A}$$

for some constant C

(iii) the function $\varepsilon(r)$ *defined above is zero for r large enough.*

Proof. Clearly $(ii) \Rightarrow (i)$, since $\|a\|_L = \max\{R^{-1}\|a\|, L(a)\}$, with R the radius of the state space. Conversely, if we set

$$K = \sup_{\|a\|_L \leq 1, \|b\|_L \leq 1} f(L(a), L(b), \|a\|, \|b\|),$$

and observe that K is finite by compactness, we get

$$\|ab\|_L = \max\{R^{-1}\|ab\|, L(ab)\} \leq \max\{R, K\}\|a\|_L\|b\|_L.$$

Now let us observe that (iii) means that $\varepsilon(r_0) = 0$ for some r_0, namely $\sup_{\|x\|_L \leq 1} \|x^*x\|_L \leq r_0$ or, equivalently, $\|x^*x\|_L \leq r_0\|x\|_L^2$ for any x. The latter is clearly equivalent to property (ii). $\square$

Now we characterise the existence of an inherited C*-structure. Indeed, giving a uniform sequence $\mathcal{A}_n$ of C*-algebras with Lip-norms and a free ultrafilter $\mathcal{U}$, we can construct the inclusions $\ell_L^\infty(\mathcal{A}_n, \mathcal{U}) \subset \mathcal{B}_\mathcal{U} \subset \ell^\infty(\mathcal{A}_n, \mathcal{U})$, where $\mathcal{B}_\mathcal{U}$ denotes the C*-algebra generated by $\ell_L^\infty(\mathcal{A}_n, \mathcal{U})$. By the properties proved above, $\mathcal{B}_\mathcal{U}$ is a quasi Lip-normed unital C*-algebra.

Proposition 4.7 *Let* $\{(\mathcal{A}_n, L_n)\}_{n\in\mathbb{N}}$ *be a Cauchy sequence of Lip-normed unital C*-algebras, with functions* ε_n*, and let* $\mathcal{B}_\mathcal{U}$ *the quasi Lip-normed unital C*-algebra defined above, with function* $\varepsilon_\mathcal{U}$*. Then*

$$\varepsilon_\mathcal{U}(r) = \lim_\mathcal{U} \varepsilon_n(r).$$

Proof. Given $r > 0$, $n \in \mathbb{N}$, let $x_n, y_n \in \mathcal{A}_n$ realise the worst element with Lip-norm ≤ 1 and the best approximation of $x_n^*x_n$ with Lip-norm $\leq r$ respectively, hence $\|x_n^*x_n - y_n\| = \varepsilon_n(r)$, and then set $x = \lim_\mathcal{U} x_n, y = \lim_\mathcal{U} y_n, \varepsilon(r) = \lim_\mathcal{U} \varepsilon_n(r)$. This implies that $\|x^*x - y\| = \varepsilon(r)$. An element $\tilde{y} \in \ell_L^\infty(\mathcal{A}_n, \mathcal{U})$, $\|\tilde{y}\|_L \leq r$, giving the best approximation of x^*x, could be obtained as $\tilde{y} = \lim_\mathcal{U} \tilde{y}_n$, with $\|\tilde{y}_n\|_L \leq r$, as shown in the proof of Lemma 2.9. Since $\varepsilon_n(r) \leq \|x_n^*x_n - \tilde{y}_n\| \to_\mathcal{U} \|x^*x - \tilde{y}\| \leq \varepsilon_\mathcal{U}(r)$, we get $\varepsilon(r) \leq \varepsilon_\mathcal{U}(r)$.

Conversely, let $x \in \ell_L^\infty(\mathcal{A}_n, \mathcal{U})$ realise the worst element with Lip-norm ≤ 1, and, as above, obtain it as $x = \lim_\mathcal{U} x_n$, $\|x_n\|_L \leq 1$. Then let y_n be the best approximation of $x_n^*x_n$ with Lip-norm $\leq r$, hence $\|x_n^*x_n - y_n\| \leq \varepsilon_n(r)$. Setting $y = \lim_{\mathcal{U}} y_n$, we get $\|y\|_L \leq r$ and $\varepsilon_{\mathcal{U}}(r) \leq \|x^*x - y\| \leq \varepsilon(r)$. $\square$

Corollary 4.8 *Let $\{(\mathcal{A}_n, L_n)\}_{n\in\mathbb{N}}$ be a Cauchy sequence of Lip-normed unital C*-algebras, with functions ε_n. The following are equivalent:*

(i) the limit inherits a C-structure*

(ii) $\lim\limits_{r\to\infty} \lim\limits_{\mathcal{U}} \varepsilon_n(r) = 0$ for some free ultrafilter $\mathcal{U}$

(iii) there exists a subsequence n_k such that

$$\lim_{r\to\infty} \limsup_{k} \ \varepsilon_{n_k}(r) = 0.$$

Proof. By the results above, (ii) amounts to saying that the quasi Lip-normed unital C*-algebra $\mathcal{B}_{\mathcal{U}}$ is indeed Lip-normed, hence coincides with $\ell^\infty_L(\mathcal{A}_n, \mathcal{U})$, which is therefore a C*-algebra.

$(iii) \implies (ii)$ For any free ultrafilter $\mathcal{U}$ such that $\{n_k : k \in \mathbb{N}\} \in \mathcal{U}$, we have $\lim\limits_{r\to\infty} \lim\limits_{\mathcal{U}} \varepsilon_n(r) = 0$.

$(ii) \implies (iii)$ Choose a sequence $\{n_k^1\}_{k\in\mathbb{N}} \in \mathcal{U}$ such that $\exists \lim_k \varepsilon_{n_k^1}(1) = \varepsilon_{\mathcal{U}}(1)$, and then, inductively, $\{n_k^j\}_{k\in\mathbb{N}} \in \mathcal{U}$ as a subsequence of n_k^{j-1} such that $\exists \lim_k \varepsilon_{n_k^j}(j) = \varepsilon_{\mathcal{U}}(j)$. For the diagonal subsequence $n_k := n_k^k$, we get $\lim_k \varepsilon_{n_k}(j) = \varepsilon_{\mathcal{U}}(j)$ for any j. Then

$$\limsup_k \varepsilon_{n_k}(r) \leq \limsup_k \varepsilon_{n_k}([r]) = \varepsilon_{\mathcal{U}}([r]) \to 0, \quad r \to \infty.$$

$\square$

We observe here that, by making use of the function ε considered above, it is possible to construct complete metrics on the family of equivalence classes of Lip-normed unital C*-algebras.

Definition 4.9 *Let $\mathcal{A}$, $\mathcal{B}$ be Lip-normed unital C*-algebras, with ε-functions $\varepsilon_{\mathcal{A}}$, $\varepsilon_{\mathcal{B}}$, and set*

$$\mathrm{dist}_p^\varepsilon(\mathcal{A}, \mathcal{B}) := \max\{\mathrm{dist}_p(\mathcal{A}, \mathcal{B}), \|\varepsilon_{\mathcal{A}} - \varepsilon_{\mathcal{B}}\|\},$$

where the norm is the sup norm.

Corollary 4.10 $\mathrm{dist}_p^\varepsilon$*, $p \geq 2$, is a complete metric on the family of equivalence classes of Lip-normed unital C*-algebras.*

Proof. The properties of a metric are obviously satisfied. Given a sequence $\mathcal{A}_n$ of Lip-normed unital C*-algebras, Cauchy w.r.t. $\mathrm{dist}_p^\varepsilon$, the corresponding sequence ε_n is uniformly convergent, hence condition (iii) of Corollary 4.8 is satisfied, implying that $\ell_L^\infty(\mathcal{A}_n, \mathcal{U})$ is a C^*-algebra. By Proposition 4.2 we get the thesis. $\square$

4.2 Counterexamples

This section is mainly devoted to the proof of Theorem 4.3 via suitable counterexamples. Also, examples showing the non-equivalence of the f-Leibniz condition with the $\varepsilon(r) \to 0$ condition are given.

4.2.1 Example 1

We give here an example of a Cauchy sequence of Lip-normed unital C*-algebras w.r.t. the complete quantum Gromov-Hausdorff distance dist_∞ which does not converge to a C*-algebra.

Let us denote by $\mathcal{C}$ the algebra of 2×2 matrices, and by $\mathcal{C}_0$ the subspace of $\mathcal{C}$ consisting of all matrices whose diagonal part is a multiple of the identity. Then we let $\mathcal{B}$ be a C*-algebra acting faithfully on a Hilbert space $\mathcal{K}$, and denote by $\mathcal{A}$ the C*-algebra $\mathcal{C} \otimes \mathcal{B}$, acting on $\mathcal{H} := \mathbb{C}^2 \otimes \mathcal{K}$, and by $\mathcal{A}_0$ the subspace of $\mathcal{A}$ given by $\mathcal{C}_0 \otimes \mathcal{B}$.

Let us now assume that $\mathcal{B}$ is Lip-normed, with Lip-seminorm L, and define on $\mathcal{A}$ the functionals

$$\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\|_n := \max \left\{ \left\| \frac{a+d}{2} \right\|_L, n \left\| \frac{a-d}{2} \right\|_L, \|b\|_L, \|c\|_L \right\}, \quad a, b, c, d \in \mathcal{B}$$

$$L^n \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) := \inf_{\lambda \in \mathbb{R}} \left\| \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} \right\|_n, \quad a, b, c, d \in \mathcal{B}$$

Let us remark that in the following, besides the trivial case $\mathcal{B} = \mathbb{C}I$, we shall consider the case in which $\mathcal{B}$ is UHF (cf. Remark 1). The existence of a Lip-seminorm on such algebras has been proved in [3].

Lemma 4.11 *L^n is a Lip-seminorm on $\mathcal{A}$. All these seminorms coincide on $\mathcal{A}_0$.*

Proof. Obvious. $\square$

Theorem 4.12 *The sequence $(\mathcal{A}, L^n)$ converges in the complete quantum Gromov-Hausdorff distance* dist_∞ *to* $(\mathcal{A}_0, L^1)$.

Proof. Let us consider the seminorms $\tilde{L}^n$ on $\mathcal{A}_0 \oplus \mathcal{A}$:

$$\tilde{L}^n(A_0 \oplus A) = \max\{L^1(A_0), L^n(A), n\|A - A_0\|_1\}, \quad A_0 \in \mathcal{A}_0, A \in \mathcal{A}.$$

Clearly

$$\min_{A_0 \in \mathcal{A}_0} \tilde{L}^n(A_0 \oplus A) = L^n(A), \quad \min_{A \in \mathcal{A}} \tilde{L}^n(A_0 \oplus A) = L^1(A_0),$$

where the first minimum is attained for $A_0 = \begin{pmatrix} (a_{11}+a_{22})/2 & a_{12} \\ a_{21} & (a_{11}+a_{22})/2 \end{pmatrix}$,
and the second minimum is attained for $A = A_0$. This means that $\tilde{L}^n$ induces L^1 on $\mathcal{A}_0$ and L^n on $\mathcal{A}$.

Since $\mathcal{A}_0 \subset \mathcal{A}$, $UCP_p(\mathcal{A})$ projects onto $UCP_p(\mathcal{A}_0)$, the projection being simply the restriction to $\mathcal{A}_0$: $\varphi_0 := \varphi|_{\mathcal{A}_0}$, $\varphi \in UCP_p(\mathcal{A})$. Therefore, the distance between $UCP_p(\mathcal{A}_0)$ and $UCP_p(\mathcal{A})$ induced by $\tilde{L}^n$ is majorised by the supremum, on $\varphi \in UCP_p(\mathcal{A})$, of the distance between φ and $\varphi_0 = \varphi|_{\mathcal{A}_0}$. Now

$$\begin{aligned}
\rho_{\tilde{L}^n}(\varphi_0 \oplus 0, 0 \oplus \varphi) &= \sup_{\|A_0 \oplus A\|_{\tilde{L}^n} \le 1} \|\langle \varphi, A_0 - A\rangle\| \\
&\le \sup_{\|A_0 \oplus A\|_{\tilde{L}^n} \le 1} \|A_0 - A\| \\
&\le \sup_{\|A_0 \oplus A\|_{\tilde{L}^n} \le 1} c\|A_0 - A\|_1 \le \frac{c}{n},
\end{aligned}$$

where we may take c equal to the diameter of $S(\mathcal{B})$ w.r.t. L. This implies that

$$\mathrm{dist}_\infty((\mathcal{A}, L^n), (\mathcal{A}_0, L^1)) \le \sup_{p \in \mathbb{N}} \rho^H_{\tilde{L}^n}(UCP_p(\mathcal{A}_0), UCP_p(\mathcal{A})) \le \frac{c}{n},$$

i.e. the thesis. $\square$

We prove now that $\mathcal{A}_0$ is not a C*-algebra up to complete order isomorphism. To do so, we need the notion of injective envelope for operator systems, due to Hamana [6]

Theorem 4.13 $\mathcal{A}_0$ *is not completely order isomorphic to a C*-algebra.*

Lemma 4.14 *The injective envelope of* $\mathcal{A}_0$ *contains* $\mathcal{A}$.

Proof. Let $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{I}(\mathcal{A}_0)$ be a completely positive projection, existing by injectivity of $\mathcal{I}(\mathcal{A}_0)$. We will show that π is the identity on $\mathcal{A}$. Choose $b \in \mathcal{B}_+$ and a unit vector ξ in the Hilbert space $\mathcal{K}$. If u denotes the injection of $\mathbb{C} \to \mathcal{K}$ such that $\lambda \mapsto \lambda\xi$, we may construct the map $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{C}$ given by

$$\varphi(a) = \begin{pmatrix} u^* & 0 \\ 0 & u^* \end{pmatrix} a \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}.$$

Let us observe that φ is completely positive and that when a is written as a $\mathcal{B}$-valued 2×2 matrix we have

$$\varphi \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} (\xi, a_{11}\xi) & (\xi, a_{12}\xi) \\ (\xi, a_{21}\xi) & (\xi, a_{22}\xi) \end{pmatrix}.$$

We then consider the map $\psi : A \in \mathcal{C} \to \psi(A) \in \mathcal{C}$ given by $\psi(A) = \varphi(\pi(A \otimes b))$, and notice that ψ is completely positive and, when $A \in \mathcal{C}_0$, we have $\pi(A \otimes b) = A \otimes b$, hence

$$\psi(A) = (\xi, b\xi)A. \tag{4.3}$$

Let us show that this relation holds for any $A \in \mathcal{C}$. Indeed this is clearly true when $(\xi, b\xi) = 0$, since a positive map vanishing on the identity is zero. When $(\xi, b\xi) \neq 0$, the map $\frac{1}{(\xi,b\xi)}\psi$ is a completely positive map from $\mathcal{C}$ to $\mathcal{C}$ which is the identity on $\mathcal{C}_0$ and, since the injective envelope of $\mathcal{C}_0$ is $\mathcal{C}$, it has to be the identity anywhere. A simple calculation shows that relation (4.3) may be rewritten as $(\xi, (\pi(A \otimes b)_{ij} - a_{ij}b)\xi) = 0$, $i, j = 1, 2$. By the arbitrariness of ξ we get $\pi(A \otimes b) = A \otimes b$, and by the arbitrariness of $b \in \mathcal{B}$ we get the thesis. $\square$

Proof. [Proof of Theorem 4.13] Let us recall Proposition 15.10 in [12]: given an inclusion $\mathcal{B} \subseteq S \subset \mathcal{B}(\mathcal{H})$, where $\mathcal{B}$ is a unital C*-algebra and S is an operator system, then $\mathcal{B}$ is a subalgebra of $\mathcal{I}(S)$. This implies that if S is an operator system that can be represented as a unital C*-algebra $\mathcal{B}$ acting on $\mathcal{H}$, the immersion of $\mathcal{B}$ in $\mathcal{I}(\mathcal{B})$ is a *-monomorphism, namely the product structure of S making it a C*-algebra is the one given by the immersion in its injective envelope.

Then, posing $S = \mathcal{I}(\mathcal{A}_0)$ and $\mathcal{B} = \mathcal{A}$ in the same Proposition, one gets that the product on $\mathcal{A}_0$ given by the immersion in $\mathcal{I}(\mathcal{A}_0)$ is the same as that given by the immersion in $\mathcal{A}$, namely $\mathcal{A}_0$ is not a subalgebra of its injective envelope. By the argument above, it is not an algebra. $\square$

Corollary 4.15 *The space of equivalence classes of C*-algebras endowed with the metric* dist_∞ *is not complete.*

Remark 1 *The preceding example works well also in the case $\mathcal{B} = \mathbb{C}$. However, in the finite-dimensional case, the replacement of the distance between state spaces with the distance between (the closure of) pure states, like the distance* dist_q^e *considered by Rieffel in [15] after Proposition 4.9, would destroy* *the example, since the sequence is not Cauchy w.r.t. such distance. One could therefore think that, endowing C*-algebras with the appropriate distance, completeness may follow. But this is not true, since, choosing $\mathcal{B}$ as a UHF algebra, the pure states are dense, namely the mentioned replacement would have no effect.*

Let us also mention that when $\mathcal{B} = \mathbb{C}$, namely $\mathcal{A}_0 = \mathcal{C}_0$, such operator system is not even order isomorphic to a C-algebra. Indeed its state space is two dimensional and has the convex structure of a disc, while the only C*-algebra with two-dimensional state-space is $\mathbb{C}^3$, whose state space has the convex structure of a triangle. This means that even replacing* dist_∞ *with* dist_p *the set of Lip-normed unital C*-algebras is still non-complete.*

4.2.2 Example 2

We give here an example of a Cauchy sequence of Lip-normed unital C*-algebras w.r.t. the complete quantum Gromov-Hausdorff distance dist_∞ which converges to a C*-algebra, but the C*-structure is not inherited.

The sequence $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ is made of the constant algebra $\mathbb{C}^3$ endowed with the following seminorms:

$$L_n(a,b,c) = \|\frac{a-b}{2}, n(\frac{a+b}{2} - c)\|_2,$$

where $\|\cdot\|_2$ is the Euclidean norm. It is not difficult to show that the sequence converges, in any dist_p, to the Lip-normed unital C*-algebra $\mathcal{A}_\infty$ consisting of $\mathbb{C}^2$ with the seminorm $L_\infty(\alpha,\beta) = |\frac{\alpha-\beta}{2}|$. Indeed, let us consider on $\mathbb{C}^3 \oplus \mathbb{C}^2$ the seminorm

$$\tilde{L}_n(a,b,c,\alpha,\beta) = \max\{L_n(a,b,c), L_\infty(\alpha,\beta), n|a-\alpha|, n|b-\beta|, n|c - \frac{\alpha+\beta}{2}|\}.$$

Clearly $\tilde{L}_n$ induces L_n on $\mathcal{A}_n$ and L_∞ on $\mathcal{A}_\infty$ and, reasoning as in the previous example, we get $\mathrm{dist}_\infty(\mathcal{A}_n, \mathcal{A}_\infty) \le \frac{1}{n}$.

Now we compute the ultraproducts. Since we have a sequence of finite-dimensional constant spaces, for any free ultrafilter $\mathcal{U}$, the ultraproduct coincides with $\mathbb{C}^3$, where we can represent any element with the constant sequence [1]. Then the Lip-ultraproduct consists of those sequences constantly equal to (a,b,c) for which $L_n(a,b,c)$ is bounded, i.e. $c = \frac{a+b}{2}$. Therefore, setting

$$\mathcal{A}_0 = \{(\alpha, \beta, \frac{\alpha+\beta}{2}) \in \mathbb{C}^3 : \alpha, \beta \in \mathbb{C}\},$$

the inclusion of the Lip-ultraproduct in the ultraproduct is given by $\mathcal{A}_0 \subset \mathbb{C}^3$, for any free ultrafilter $\mathcal{U}$. Since $\mathcal{A}_0$ is not a subalgebra of $\mathbb{C}^3$, the limit does not inherit the C*-structure.

Let us remark that the previous results are not in contradiction, since the map $(a,b) \in \mathbb{C}^2 \mapsto (a,b,(a+b)/2)$ is clearly a complete order isomorphism, namely $\mathcal{A}_0$ and $\mathcal{A}_\infty$ are completely order isomorphic.

Remark 2 *The previous example consists of abelian C*-algebras converging to an abelian C*-algebra, therefore one could expect it corresponds to the Gromov-Hausdorff convergence of the spectra. But if this were true the Lip-ultraproduct would correspond to the ultralimit, hence would be a C*-algebra in a natural way. This apparent contradiction is due to the fact that the approximating state spaces (triangles) converge to the limit state space (segment) like a triangle flattening on its base, namely the upper vertex converges to the middle point of the basis. Therefore the spectra do not converge Gromov-Hausdorff.*

4.2.3 Example 3

We conclude with an example of a converging sequence of C*-algebras where the limit inherits the C*-structure, however no f-Leibniz condition is satisfied, namely there is no function f such that all algebras satisfy the same f-Leibniz condition. According to Proposition 4.6, it is sufficient to exhibit a converging sequence for which the functions ε_n are eventually zero, but converge pointwise to a nowhere zero function infinitesimal at $+\infty$.

As in the previous examples, the sequence will consist of a constant algebra with varying Lip-seminorms.

The C*-algebra $\mathcal{A}$ is made of sequences $A = \{A_k\}_{k\in\mathbb{N}}$ of 2×2 matrices converging to a multiple of the identity.

On the C*-algebra $\mathcal{A}$ let us consider the (possibly infinite) functionals

$$\begin{aligned} \|A\| &= \sup_k \|A_k\|, \\ |||A||| &= \sup_k k\|A_k\|, \\ L(A) &= \min_{\lambda\in\mathbb{C}} |||A - \lambda I|||. \end{aligned}$$

and the dense subspace $\mathcal{A}_0$ of the elements for which $L(A) < \infty$.

Let us observe that if $|||A - \alpha I||| < \infty$ then $A_k \to \alpha I$, hence

$$|\alpha| = \lim_k \|A_k\| \le \sup_k \|A_k\| = \|A\|. \tag{4.4}$$

Lemma 4.16 *L is a Lip-seminorm, and satisfies the inequality*

$$L(AB) \leq L(A)\|B\| + \|A\|L(B). \tag{4.5}$$

Proof. Clearly $|||\cdot|||$ is a lower semicontinuous norm on $\mathcal{A}_0$, hence L is a lower semicontinuous seminorm vanishing only on the multiples of the identity. Let us observe that $\mathcal{B} := \{B : |||B||| \leq 1\}$ is the image of the unit ball under the compact operator sending $\{A_k\} \mapsto \{\frac{1}{k}A_k\}$, hence it is totally bounded. Consider $\{A : L(A) \leq 1, \|A\| \leq 1\}$. Then $|||A - \alpha I||| \leq 1$ for a suitable α. Making use of inequality (4.4), we get $A \in \cup_{|\alpha|\leq 1}(\alpha I + \mathcal{B})$, showing that such set is totally bounded, i.e. L is a Lip-seminorm.

Concerning inequality (4.5), we have, for $A, B \in \mathcal{A}_0$ with $|||A - \alpha||| = L(A)$, $|||B - \beta||| = L(B)$,

$$\begin{aligned} L(AB) &\leq |||AB - \alpha\beta||| = |||(A-\alpha)B + \alpha(B-\beta)||| \\ &\leq L(A)\|B\| + |\alpha|L(B) \leq L(A)\|B\| + \|A\|L(B), \end{aligned}$$

where we used inequality (4.4). □

Now we consider a new sequence of Lip-seminorms on $\mathcal{A}$:

$$L_n(A) = \max\{L(A), \sup_{k<n} \ell_k(A_k)\}, \quad n \in \mathbb{N} \cup \{\infty\},$$

where

$$\ell_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} = k^3|a-d|.$$

Clearly each L_n is again a Lip-seminorm, and, for finite n, it still satisfies an f-Leibniz condition (cf. Proposition 4.6), being a finite rank perturbation of L.

In the following we shall denote by $\mathcal{A}_n$ the Lip-normed unital C*-algebra $(\mathcal{A}, L_n)$, $n \in \mathbb{N} \cup \{\infty\}$.

First we observe that, for any free ultrafilter $\mathcal{U}$, we may identify the Lip-ultraproduct $\ell^\infty_L(\mathcal{A}_n, \mathcal{U})$ with $\mathcal{A}_\infty$. Indeed, given $\{A^n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ with $\|A^n\| \leq 1$ and $L_n(A^n) \leq 1$, we have shown that it lies in a compact set, namely $\lim_{\mathcal{U}} A^n$ exists, and we call it A. We can therefore identify the class of the sequence $\{A^n\}$ in $\ell^\infty_L(\mathcal{A}_n, \mathcal{U})$ with the class of the sequence constantly equal to A. As a consequence the C*-structure is inherited.

Now we show that indeed $\{\mathcal{A}_n\}$ converges in the complete quantum Gromov-Hausdorff distance dist_∞ to $\mathcal{A}_\infty$. Take on $\mathcal{A} \oplus \mathcal{A}$ the seminorm

$$\tilde{L}_n(A, B) = \max\{L_n(A), L_\infty(B), n\|A - B\|\},$$

which is clearly a Lip-seminorm. It is easy to see that it induces L_n on the first summand, the minimum being attained for $B_k = A_k$, $k \leq n$, $B_k = \alpha I$, $k > n$, with $L(A) = |||A - \alpha I|||$. Analogously, it induces L_∞ on the second summand.

As in the first example, we get

$$\rho_{\tilde{L}^n}(\varphi \oplus 0, 0 \oplus \varphi) \leq \sup_{\|A \oplus B\|_{\tilde{L}^n} \leq 1} \|A - B\| \leq \frac{1}{n},$$

hence

$$\mathrm{dist}_\infty(\mathcal{A}_n, \mathcal{A}_\infty) \leq \sup_{p \in \mathbb{N}} \rho^H_{\tilde{L}^n}(UCP_p(\mathcal{A} \oplus 0), UCP_p(0 \oplus \mathcal{A})) \leq \frac{1}{n},$$

i.e. the thesis.

It only remains to show that $(\mathcal{A}, L_\infty)$ does not satisfy any f-Leibniz condition, i.e. by Proposition 4.6, that we can find an element A with finite Lip-seminorm such that $L_\infty(A^*A)$ is infinite. Taking $A = \{A_k\}$, $A_k = \begin{pmatrix} 0 & 1/k \\ 0 & 0 \end{pmatrix}$, we have $L_\infty(A) = L(A) = 1$, but $L_\infty(A^*A) = \infty$.

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