

TANGENTIAL DIMENSIONS II. MEASURES

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ABSTRACT. Notions of (pointwise) tangential dimension are considered, for measures of \mathbb{R}^N . Under regularity conditions (volume doubling), the upper resp. lower dimension at a point x of a measure μ can be defined as the supremum, resp. infimum, of local dimensions of the measures tangent to μ at x . Our main purpose is that of introducing a tool which is very sensitive to the "multifractal behaviour at a point" of a measure, namely which is able to detect the "oscillations" of the dimension at a given point, even when the local dimension exists, namely local upper and lower dimensions coincide. These definitions are tested on a class of fractals, which we call translation fractals, where they can be explicitly calculated for the canonical limit measure. In these cases the tangential dimensions of the limit measure coincide with the metric tangential dimensions of the fractal defined in [7], and they are constant, i.e. do not depend on the point. However, upper and lower dimensions may differ. Moreover, on these fractals, these quantities coincide with their noncommutative analogues, defined in previous papers [5, 6], in the framework of Alain Connes' noncommutative geometry.

1. INTRODUCTION.

In this paper we continue the analysis concerning notions of tangential dimensions.

Our aim is that of finding dimensions describing the non-regularity, or fractality, of a given measure. The kind of non-regularity we study here is related to the fact that a dimension may have an oscillating behavior at a point. Indeed dimensions are often defined as limits, and an oscillating behavior means that the upper and lower versions of the considered dimension are different. Our main goal here is to associate to a measure a local dimension that is able to maximally detect such an oscillating behavior, namely for which the upper and lower determinations form a maximal dimensional interval.

Let us recall that we introduced first tangential dimensions in the framework of Alain Connes' noncommutative geometry [1], as extremal points of the singular traceability interval [5]. Their explicit formulas suggested the definition of tangential dimensions at a point for a metric space, given in [7], where we showed that, under regularity conditions (cf. Theorem 2.13 (i) and (ii)), upper, resp. lower, tangential dimension of a metric space at a given point can be equivalently defined as the supremum, resp. infimum, of local dimensions of the tangent sets *a la Gromov* at the point.

Here we define tangential dimensions at a point for measures and show that, under the volume doubling condition, upper, resp. lower, tangential dimension of

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a measure at a given point can be equivalently defined as the supremum, resp. infimum, of local dimensions of the tangent measures at the point.

Finally we compute the tangential dimensions for some classes of fractals.

Indeed we give a condition (Condition 2.16) on a measure μ on a metric space X under which the tangential dimensions for μ coincide with the tangential dimensions of X and are locally constant. Furthermore, under the same condition, tangential dimensions for metric spaces and measures are extrema of local dimensions of the corresponding tangent objects.

First we consider the class of self-similar fractals with open set condition, showing that in this case tangential dimensions do not give new information, indeed they coincide with the Hausdorff dimension.

Then we consider the class of translation fractals, show that the mentioned condition is satisfied by translation fractals with open set condition for the canonical limit measure and compute the tangential dimensions for such measure. Besides their coincidence with metric tangential dimensions, which follows by the results described above, direct inspection shows that they also coincide with the noncommutative tangential dimensions for the spectral triples associated to translation fractals computed in [5] and [6].

2. TANGENTIAL DIMENSIONS OF MEASURES

In this section we shall define upper and lower tangential dimensions of a measure on a metric space X and study some of their properties. The name tangential is motivated by the results in subsection 2.2, where we show that for Radon measures on \mathbb{R}^N , under volume doubling condition, the upper, resp. lower, tangential dimension, is simply the supremum, resp. infimum, of the (upper, resp. lower) local dimensions of the tangent measures.

2.1. Basic properties. Let (X, d) be a metric space, μ a locally finite Borel measure, namely μ is finite on bounded sets, and set $B(x, r) := \{y \in X : d(x, y) < r\}$.

Let us recall that the local dimensions of a measure at x are defined as

$$\underline{d}_\mu(x) = \liminf_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r},$$

$$\bar{d}_\mu(x) = \limsup_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r}.$$

Remark 2.1. If μ is zero on a neighborhood of x , we set $\underline{d}_\mu(x) = \bar{d}_\mu(x) = +\infty$. Indeed, let us introduce the following partial order relation on measures: $\mu <_x \nu$ if there exists a neighbourhood Ω of x such that for any positive Borel function φ supported in Ω we have $\langle \mu, \varphi \rangle \leq \langle \nu, \varphi \rangle$.

By definition the maps $\mu \mapsto \underline{d}_\mu(x)$, $\mu \mapsto \bar{d}_\mu(x)$ are decreasing, namely reverse the ordering. In particular, if x is not in the support of μ , namely μ is zero on a neighbourhood of x , the local dimensions of μ should be set to $+\infty$.

Now we introduce tangential dimensions for μ .

Definition 2.2. The lower and upper tangential dimensions of μ are defined as

$$\underline{\delta}_\mu(x) := \liminf_{\lambda \rightarrow 0} \liminf_{r \rightarrow 0} \frac{\log \left(\frac{\mu(B(x,r))}{\mu(B(x,\lambda r))} \right)}{\log 1/\lambda} \in [0, \infty],$$

$$\bar{\delta}_\mu(x) := \limsup_{\lambda \rightarrow 0} \limsup_{r \rightarrow 0} \frac{\log \left(\frac{\mu(B(x,r))}{\mu(B(x,\lambda r))} \right)}{\log 1/\lambda} \in [0, \infty].$$

In the following we shall set $f(t) = f_{x,\mu}(t) = -\log(\mu(B(x, e^{-t})))$, and $g(t, h) = f(t+h) - f(t)$. With this notation, the definitions above become

$$(2.1) \quad \underline{d}_\mu(x) = \liminf_{t \rightarrow +\infty} \frac{f(t)}{t} = \lim_{t \rightarrow \infty} \liminf_{h \rightarrow \infty} \frac{g(t, h)}{h},$$

$$(2.2) \quad \bar{d}_\mu(x) = \limsup_{t \rightarrow +\infty} \frac{f(t)}{t} = \lim_{t \rightarrow \infty} \limsup_{h \rightarrow \infty} \frac{g(t, h)}{h},$$

$$(2.3) \quad \underline{\delta}_\mu(x) = \liminf_{h \rightarrow \infty} \liminf_{t \rightarrow \infty} \frac{g(t, h)}{h},$$

$$(2.4) \quad \bar{\delta}_\mu(x) = \limsup_{h \rightarrow \infty} \limsup_{t \rightarrow \infty} \frac{g(t, h)}{h}.$$

Theorem 2.3. Let μ be a locally finite Borel measure on X . Then the following holds.

(i)

$$\underline{\delta}_\mu(x) \leq \underline{d}_\mu(x) \leq \bar{d}_\mu(x) \leq \bar{\delta}_\mu(x).$$

(ii) There exist the limits for $h \rightarrow \infty$ in equations (2.3), (2.4). Moreover,

$$\underline{\delta}_\mu(x) = \sup_{h > 0} \liminf_{t \rightarrow \infty} \frac{g(t, h)}{h},$$

$$\bar{\delta}_\mu(x) = \inf_{h > 0} \limsup_{t \rightarrow \infty} \frac{g(t, h)}{h}.$$

(iii)

$$\underline{\delta}_\mu(x) = \liminf_{(h,t) \rightarrow (\infty, \infty)} \frac{g(t, h)}{h},$$

$$\bar{\delta}_\mu(x) = \limsup_{(h,t) \rightarrow (\infty, \infty)} \frac{g(t, h)}{h}.$$

Proof. Properties (i) and (ii) follow from Proposition 1.1 in [5], now we prove (iii). Setting $p(t, h) := \frac{f(t+h) - f(t)}{h}$, we have to show that

$$\limsup_{t, h \rightarrow \infty} p(t, h) = \lim_{h \rightarrow \infty} \limsup_{t \rightarrow \infty} p(t, h).$$

Assume $\lim_{h \rightarrow \infty} \limsup_{t \rightarrow \infty} p(t, h) = L \in \mathbb{R}$. Let $\varepsilon > 0$, then there is $h_\varepsilon > 0$ such that, for any $h > h_\varepsilon$, $\limsup_{t \rightarrow \infty} p(t, h) > L - \varepsilon/2$, hence, for any $t_0 > 0$ there is $t = t(h, t_0) > t_0$, such that $p(t, h) > L - \varepsilon$. Hence, for any $h_0 > 0$, $t_0 > 0$ there exist $h > h_0$, $t > t_0$ such that $p(t, h) > L - \varepsilon$, namely $\limsup_{t, h \rightarrow \infty} p(t, h) \geq L$. Conversely, assume $\limsup_{t, h \rightarrow \infty} p(t, h) = L' \in \mathbb{R}$, and choose t_n, h_n such that

$\lim_{n \rightarrow \infty} p(t_n, h_n) = L'$. Then, for any $h > 0$, with k_n denoting $\lceil \frac{h_n}{h} \rceil$, we have

$$\begin{aligned} p(t_n, h_n) &\leq \frac{k_n h}{h_n} p(t_n, k_n h) = \frac{k_n h}{h_n} \frac{1}{k_n} \sum_{j=0}^{k_n-1} p(t_n + jh, h) \\ &\leq \frac{k_n h}{h_n} \max_{j=0 \dots k_n-1} p(t_n + jh, h) \leq \frac{k_n h}{h_n} \sup_{t \geq t_n} p(t, h). \end{aligned}$$

Hence, for $n \rightarrow \infty$, we get $L' \leq \limsup_{t \rightarrow \infty} p(t, h)$, which implies the equality. The other cases are treated analogously. \square

Theorem 2.4.

$$\begin{aligned} \underline{\delta}_\mu(x) &= \inf_{\{t_n\} \rightarrow \infty} \limsup_{h \rightarrow \infty} \limsup_n \frac{g(t_n, h)}{h} \\ &= \inf_{\{t_n\} \rightarrow \infty} \liminf_{h \rightarrow \infty} \liminf_n \frac{g(t_n, h)}{h}. \end{aligned}$$

The two infima are indeed minima, indeed there exists a sequence $\{\underline{t}_n\} \rightarrow \infty$ for which

$$\underline{\delta}_\mu(x) = \lim_{h \rightarrow \infty} \limsup_n \frac{g(\underline{t}_n, h)}{h} = \lim_{h \rightarrow \infty} \liminf_n \frac{g(\underline{t}_n, h)}{h},$$

and such that any subsequence is still minimizing. Analogously,

$$\begin{aligned} \bar{\delta}_\mu(x) &= \sup_{\{t_n\} \rightarrow \infty} \limsup_{h \rightarrow \infty} \limsup_n \frac{g(t_n, h)}{h} \\ &= \sup_{\{t_n\} \rightarrow \infty} \liminf_{h \rightarrow \infty} \liminf_n \frac{g(t_n, h)}{h}. \end{aligned}$$

The two suprema are indeed maxima, indeed there exists a sequence $\{\bar{t}_n\} \rightarrow \infty$ for which

$$\bar{\delta}_\mu(x) = \lim_{h \rightarrow \infty} \limsup_n \frac{g(\bar{t}_n, h)}{h} = \lim_{h \rightarrow \infty} \liminf_n \frac{g(\bar{t}_n, h)}{h}.$$

and such that any subsequence is still maximizing.

Proof. We prove the second part of the Theorem, the proof of the first part being analogous. Let us observe that the inequality

$$\bar{\delta}_\mu(x) \geq \sup_{\{t_n\} \rightarrow \infty} \limsup_{h \rightarrow \infty} \limsup_n \frac{g(t_n, h)}{h}$$

obviously holds for any $t_n \rightarrow \infty$, therefore it is enough to find a sequence $\bar{t}_n \rightarrow \infty$ for which

$$(2.5) \quad \bar{\delta}_\mu(x) \leq \liminf_{h \rightarrow \infty} \liminf_n \frac{g(\bar{t}_n, h)}{h}.$$

In [7], Proposition 5.5, we proved that, whenever dg is bounded by a constant S , where

$$dg(t, h, k) = g(t, h+k) - g(t+h, k) - g(t, h),$$

then, for any $\kappa > 0$, there exists a sequence $\{\bar{t}_n\} \rightarrow \infty$ for which

$$(2.6) \quad \limsup_{h \rightarrow +\infty} \limsup_{t \rightarrow +\infty} \frac{g(t, h)}{h} \leq \liminf_{h \rightarrow +\infty} \liminf_{n \in \mathbb{N}} \frac{g(\bar{t}_n, h)}{h} + \frac{2S}{\kappa}.$$

Since in our case $dg = 0$, $\{\bar{t}_n\}$ is the required sequence. Clearly inequality (2.5) is preserved when passing to a subsequence. \square

2.2. Tangential dimensions on \mathbb{R}^N . In this subsection μ is a Radon measure on \mathbb{R}^N . Let us recall that the cone $T_x(\mu)$ of measures tangent to μ at $x \in \mathbb{R}^N$ is the set of non-zero limit points in the vague topology of sequences $c_n \mu \circ D_x^{\lambda_n}$, where λ_n decreases to 0 and $c_n > 0$, and D_x^λ is the dilation with center $x \in \mathbb{R}^N$ and factor $\lambda > 0$. In this case vague topology is the weak topology determined by continuous functions with compact support.

Let us consider the following properties: we say that μ satisfies the *volume doubling* condition at x if

$$(2.7) \quad \limsup_{r \rightarrow 0} \frac{\mu(B(x, 2r))}{\mu(B(x, r))} < \infty,$$

and the *weak volume doubling* condition at x if there exists an infinitesimal sequence r_n such that, for any $\lambda > 0$

$$(2.8) \quad \limsup_{n \rightarrow \infty} \frac{\mu(B(x, \lambda r_n))}{\mu(B(x, r_n))} < \infty.$$

Then the following proposition holds.

Proposition 2.5. *Let μ be a Radon measure on \mathbb{R}^N .*

- (i) $T_x(\mu) \neq \emptyset$ iff μ satisfies the weak volume doubling condition at x .
- (ii) Assume volume doubling at x . Then any tangent measure of μ at x is of the form

$$(2.9) \quad \nu^{\{r_n\}} = \lim_n \frac{\mu \cdot D_x^{r_n}}{\mu(B(x, r_n))},$$

for a suitable infinitesimal sequence $\{r_n\}$. Moreover, for any $r_n \searrow 0$ there exists a subsequence r_{n_k} giving rise to a tangent measure $\nu^{\{r_{n_k}\}}$ as above.

- (iii) Any tangent measure of μ at x is of the form $\nu^{\{r_n\}} \cdot D_x^\lambda$ for a suitable infinitesimal sequence r_n and $\lambda > 0$.
- (iv) Volume doubling at x is equivalent to $\bar{\delta}_\mu(x) < \infty$. In particular,

$$\limsup_{r \rightarrow 0} \frac{\mu(B(x, 2r))}{\mu(B(x, r))} = A \Rightarrow \bar{\delta}_\mu(x) \leq \log_2 A.$$

Proof. The first two properties are proved in [8], where it is also shown that any tangent measure is of the form

$$\lim_n \frac{\mu \cdot D_x^{r_n}}{\mu(B(x, \lambda r_n))},$$

from which (iii) follows.

Let us prove (iv). By definition, $\limsup_{r \rightarrow 0} \frac{\mu(B(x, 2r))}{\mu(B(x, r))} = A$ can be rewritten as $\limsup_{t \rightarrow \infty} f(t + \log 2) - f(t) = \log A$, from which $\bar{\delta}_\mu(x) \leq \log_2 A$ easily follows. Conversely, if $\limsup_{t \rightarrow \infty} f(t + \log 2) - f(t) = \infty$, then $\limsup_{t \rightarrow \infty} f(t + h) - f(t) = \infty$ for any $h \geq \log 2$, hence $\bar{\delta}_\mu(x) = \infty$. \square

Proposition 2.6. *Let us consider a tangent measure of μ at x of the form $\nu = \nu^{\{r_n\}} \cdot D_x^\lambda$ as in Proposition 2.5. Then, with $f(t) = -\log \mu(B(x, e^{-t}))$, and $t_n :=$*

$-\log r_n$, we have

$$\begin{aligned}\underline{d}_\nu(x) &= \liminf_{h \rightarrow \infty} \lim_n \frac{f(t_n + h) - f(t_n)}{h}, \\ \bar{d}_\nu(x) &= \limsup_{h \rightarrow \infty} \lim_n \frac{f(t_n + h) - f(t_n)}{h}.\end{aligned}$$

Proof. First let us notice that

$$\underline{d}_{\nu^{\{r_n\}} \cdot D_x^\lambda}(x) = \underline{d}_{\nu^{\{r_n\}}}(x).$$

Then, let us recall that tangent measures are defined in terms of vague convergence, namely weak convergence on continuous functions with compact support. Then, let φ be a continuous function verifying $\chi_{B(x,1)} \leq \varphi \leq \chi_{B(x,2)}$. Setting $h = -\log \lambda$, we get

$$\begin{aligned}\nu^{\{r_n\}}(B(x, \lambda/2)) &= \nu^{\{r_n\}} \cdot D_x^{\lambda/2}(B(x, 1)) \\ &\leq \langle \nu^{\{r_n\}} \cdot D_x^{\lambda/2}, \varphi \rangle \\ &= \lim_n \frac{\langle \mu \cdot D_x^{\lambda r_n/2}, \varphi \rangle}{\mu(B(x, r_n))} \\ &\leq \lim_n \frac{\mu(B(x, \lambda r_n))}{\mu(B(x, r_n))} \\ &= \lim_n \exp[-(f(t_n + h) - f(t_n))] \\ &\leq \langle \nu^{\{r_n\}} \cdot D_x^\lambda, \varphi \rangle \leq \nu^{\{r_n\}}(B(x, 2\lambda)).\end{aligned}$$

Then,

$$(2.10) \quad \frac{\log \nu^{\{r_n\}}(B(x, \lambda/2))}{\log \lambda} \leq \lim_n \frac{f(t_n + h) - f(t)}{h} \leq \frac{\log \nu^{\{r_n\}}(B(x, 2\lambda))}{\log \lambda},$$

from which the thesis immediately follows. \square

Theorem 2.7. *Let μ be a Radon measure on \mathbb{R}^N , satisfying the volume doubling condition at x . Then*

$$\begin{aligned}\underline{\delta}_\mu(x) &= \inf_{\nu \in T_x(\mu)} \underline{d}_\nu(x), \\ \bar{\delta}_\mu(x) &= \sup_{\nu \in T_x(\mu)} \bar{d}_\nu(x).\end{aligned}$$

Let us remark that volume doubling implies weak volume doubling, namely the set of tangent measures at x is non-empty.

Proof. Let us give the proof for $\underline{\delta}_\mu(x)$, the other case being proved analogously. Let $T(\mu, x)$ be the set of sequences $t_n \rightarrow \infty$ such that $r_n = e^{-t_n}$ generates a tangent measure as in Lemma 2.5 (ii). Then, from Proposition 2.6, we get

$$(2.11) \quad \inf_{\nu \in T_x(\mu)} \underline{d}_\nu(x) = \inf_{\{t_n\} \in T(\mu, x)} \liminf_{h \rightarrow \infty} \liminf_n \frac{f(t_n + h) - f(t_n)}{h}.$$

So the equality is proved if we show that $T(\mu, x)$ contains one of the minimizing sequences of theorem 2.4. This holds true, since any minimizing sequence of theorem 2.4 has a subsequence giving rise to a tangent measure by Proposition 2.5 (ii), and such subsequence inherits the minimizing property. \square

2.3. Further properties. Tangential dimensions are invariant under bi-Lipschitz maps.

Proposition 2.8. *Let X, Y be metric spaces, $f : X \rightarrow Y$ be a bi-Lipschitz map i.e. there is $L > 0$ such that $L^{-1}d_X(x, x') \leq d_Y(f(x), f(x')) \leq Ld_X(x, x')$, for $x, x' \in X$. Let μ be a finite Borel measure on X and set $\nu := \mu \circ f^{-1}$, which is a finite Borel measure on Y . Then $\underline{\delta}_\mu(x) = \underline{\delta}_\nu(f(x))$ and $\overline{\delta}_\mu(x) = \overline{\delta}_\nu(f(x))$, for all $x \in X$.*

Proof. Observe that, for any $x \in X, y \in Y, r > 0$, we have

$$\begin{aligned} B(f(x), r/L) &\subset f(B(x, r)) \subset B(f(x), rL) \\ B(f^{-1}(y), r/L) &\subset f^{-1}(B(y, r)) \subset B(f^{-1}(y), rL) \end{aligned}$$

which implies

$$\mu(B(x, R/L)) \leq \nu(B(f(x), R)) = \mu(f^{-1}(B(f(x), R))) \leq \mu(B(x, RL)).$$

Therefore, taking $\limsup_{R \rightarrow 0}$, then $\lim_{\lambda \rightarrow 0}$, and doing some algebra, we get

$$\lim_{\lambda \rightarrow 0} \limsup_{R \rightarrow 0} \frac{\log \frac{\mu(B(x, R))}{\mu(B(x, \lambda R))}}{\log L^2/\lambda} \leq \lim_{\lambda \rightarrow 0} \limsup_{R \rightarrow 0} \frac{\log \frac{\nu(B(x, R))}{\nu(B(x, \lambda R))}}{\log 1/\lambda} \leq \lim_{\lambda \rightarrow 0} \limsup_{R \rightarrow 0} \frac{\log \frac{\mu(B(x, R))}{\mu(B(x, \lambda R))}}{\log 1/(L^2\lambda)}$$

which means $\overline{\delta}_\mu(x) = \overline{\delta}_\nu(f(x))$. The other equality is proved in the same manner. \square

The following propositions show some properties of tangential dimensions, i.e. their behaviour under the operations of sum or tensor product of measures.

Proposition 2.9. *Let μ_1, μ_2 be finite Borel measures on X . Then*

$$\begin{aligned} \underline{\delta}_{\mu_1 + \mu_2}(x) &\geq \min\{\underline{\delta}_{\mu_1}(x), \underline{\delta}_{\mu_2}(x)\} \\ \overline{\delta}_{\mu_1 + \mu_2}(x) &\leq \max\{\overline{\delta}_{\mu_1}(x), \overline{\delta}_{\mu_2}(x)\}. \end{aligned}$$

Proof. As

$$\frac{1}{2} \min\left\{\frac{a}{c}, \frac{b}{d}\right\} \leq \frac{a+b}{c+d} \leq 2 \max\left\{\frac{a}{c}, \frac{b}{d}\right\}$$

we get

$$\begin{aligned} &-\frac{\log 2}{\log 1/\lambda} + \min\left\{\frac{\log \frac{\mu_1(B(x, r))}{\mu_1(B(x, \lambda r))}}{\log 1/\lambda}, \frac{\log \frac{\mu_2(B(x, r))}{\mu_2(B(x, \lambda r))}}{\log 1/\lambda}\right\} \\ &\leq \frac{\log \frac{\mu_1(B(x, r)) + \mu_2(B(x, r))}{\mu_1(B(x, \lambda r)) + \mu_2(B(x, \lambda r))}}{\log 1/\lambda} \\ &\leq \frac{\log 2}{\log 1/\lambda} + \max\left\{\frac{\log \frac{\mu_1(B(x, r))}{\mu_1(B(x, \lambda r))}}{\log 1/\lambda}, \frac{\log \frac{\mu_2(B(x, r))}{\mu_2(B(x, \lambda r))}}{\log 1/\lambda}\right\}. \end{aligned}$$

Therefore, taking $\limsup_{r \rightarrow 0}$, then using the equality $\limsup_{r \rightarrow 0} \max\{f(r), g(r)\} = \max\{\limsup_{r \rightarrow 0} f(r), \limsup_{r \rightarrow 0} g(r)\}$, and finally taking $\lim_{\lambda \rightarrow 0}$, we obtain

$$\overline{\delta}_{\mu_1 + \mu_2}(x) \leq \max\{\overline{\delta}_{\mu_1}(x), \overline{\delta}_{\mu_2}(x)\}.$$

Besides, taking $\liminf_{r \rightarrow 0}$, then using the equality $\liminf_{r \rightarrow 0} \min\{f(r), g(r)\} = \min\{\liminf_{r \rightarrow 0} f(r), \liminf_{r \rightarrow 0} g(r)\}$, and finally taking $\lim_{\lambda \rightarrow 0}$, we obtain

$$\underline{\delta}_{\mu_1 + \mu_2}(x) \geq \min\{\underline{\delta}_{\mu_1}(x), \underline{\delta}_{\mu_2}(x)\}.$$

□

Proposition 2.10. *Let X, Y be metric spaces, μ, ν finite Borel measures on X and Y respectively. Then*

$$\begin{aligned}\underline{\delta}_{\mu \otimes \nu}((x, y)) &\geq \underline{\delta}_{\mu}(x) + \underline{\delta}_{\nu}(y) \\ \overline{\delta}_{\mu \otimes \nu}((x, y)) &\leq \overline{\delta}_{\mu}(x) + \overline{\delta}_{\nu}(y).\end{aligned}$$

Proof. Endow $X \times Y$ with the metric

$$(2.12) \quad d((x_1, y_1), (x_2, y_2)) := \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$$

which is by-Lipschitz equivalent to the product metric. Then

$$(2.13) \quad B_{X \times Y}((x, y), R) = B_X(x, R) \times B_Y(y, R),$$

which implies

$$\mu \otimes \nu(B_{X \times Y}((x, y), R)) = \mu(B_X(x, R))\nu(B_Y(y, R)),$$

and

$$\frac{\log \frac{\mu \otimes \nu(B_{X \times Y}((x, y), r))}{\mu \otimes \nu(B_{X \times Y}((x, y), \lambda r))}}{\log 1/\lambda} = \frac{\log \frac{\mu(B_X(x, r))}{\mu(B_X(x, \lambda r))}}{\log 1/\lambda} + \frac{\log \frac{\nu(B_Y(y, r))}{\nu(B_Y(y, \lambda r))}}{\log 1/\lambda},$$

from which the thesis follows. □

The following theorem examines the dependence of tangential dimensions on the point $x \in X$.

Theorem 2.11. *The function*

$$\underline{\delta}_{\mu} : x \in X \rightarrow \lim_{\lambda \rightarrow 0} \liminf_{r \rightarrow 0} \frac{\log \left(\frac{\mu(B(x, r))}{\mu(B(x, \lambda r))} \right)}{\log 1/\lambda} \in [0, \infty)$$

is Borel-measurable. The same is true of $\overline{\delta}_{\mu}$ with \liminf replaced by \limsup .

Proof. Set, for $r > 0, \lambda \in (0, 1)$, $f_{r, \lambda}(x) := \frac{\mu(B(x, r))}{\mu(B(x, \lambda r))}$, which is Borel-measurable by [2], proof of 1.5.9. Then we must prove that

$$f(x) := \lim_{\lambda \rightarrow 0} \frac{1}{\log 1/\lambda} \log \left(\liminf_{r \rightarrow 0} f_{r, \lambda}(x) \right),$$

is Borel-measurable. First

$$f_{\lambda}(x) := \liminf_{r \rightarrow 0} f_{r, \lambda}(x) = \lim_{r \rightarrow 0} \inf_{0 < r' < r} f_{r', \lambda}(x)$$

is Borel-measurable, because, from $\{r_n\} \subset \mathbb{Q}, r_n \nearrow r$, it follows $f_{r_n, \lambda}(x) \rightarrow f_{r, \lambda}(x)$, and

$$\lim_{r \rightarrow 0} \inf_{0 < r' < r} f_{r', \lambda}(x) = \lim_{r \rightarrow 0} \inf_{0 < r' < r, r' \in \mathbb{Q}} f_{r', \lambda}(x) = \lim_{n \rightarrow \infty} \inf_{0 < r' < \frac{1}{n}} f_{r', \lambda}(x).$$

Then

$$f(x) = \lim_{\lambda \rightarrow 0} \frac{\log f_{\lambda}(x)}{\log 1/\lambda} = \lim_{n \rightarrow \infty} \frac{\log f_{\frac{1}{n}}(x)}{\log n}$$

is Borel-measurable. □

2.4. Relations between tangential dimensions of metric spaces and measures.

Definition 2.12. Let (X, d) be a metric space, $E \subset X$, $x \in E$. Let us denote by $n(r, E) \equiv n_r(E)$ the minimum number of open balls of radius r necessary to cover E , and by $\nu(r, E) \equiv \nu_r(E)$ the maximum number of disjoint open balls of E of radius r contained in E . We call *upper, resp. lower tangential dimension* of E at x the (possibly infinite) numbers

$$\underline{\delta}_E(x) := \liminf_{\lambda \rightarrow 0} \liminf_{r \rightarrow 0} \frac{\log n(\lambda r, E \cap \overline{B}(x, r))}{\log 1/\lambda},$$

$$\overline{\delta}_E(x) := \limsup_{\lambda \rightarrow 0} \limsup_{r \rightarrow 0} \frac{\log n(\lambda r, E \cap \overline{B}(x, r))}{\log 1/\lambda}.$$

Observe that we obtain the same numbers if we use ν in place of n .

Theorem 2.13 ([7]). *Let us assume the following conditions*

- (i) $\limsup_{r \rightarrow 0} n_{\lambda r}(\overline{B}_X(x, r)) < \infty \quad \forall \lambda > 0$,
- (ii) *there exist constants $c \geq 1$, $a \in (0, 1]$ such that, for any $r \leq a$, $\lambda, \mu \leq 1$, $y, z \in B_X(x, r)$, $n(\lambda \mu r, B_X(y, \lambda r)) \leq cn(\lambda \mu r, B_X(z, \lambda r))$.*

Then

$$(2.14) \quad \underline{\delta}_X(x) = \inf_{T \in \mathcal{T}_x X} \underline{d}(T) = \inf_{T \in \mathcal{T}_x X} \overline{d}(T),$$

$$(2.15) \quad \overline{\delta}_X(x) = \sup_{T \in \mathcal{T}_x X} \underline{d}(T) = \sup_{T \in \mathcal{T}_x X} \overline{d}(T).$$

Lemma 2.14. *Let μ be a finite Borel measure on the metric space X , $x \in X$. The following inequalities hold:*

$$(2.16) \quad \mu(B(x, r)) \leq n(\lambda r, B(x, r)) \sup_{y \in B(x, r)} \mu(B(y, \lambda r)).$$

$$(2.17) \quad \mu(B(x, r)) \geq \nu(\lambda r, B(x, r)) \inf_{y \in B(x, r)} \mu(B(y, \lambda r)).$$

Proof. The first inequality follows from

$$B(x, r) \subset \bigcup_{i=1}^{n(\lambda r, B(x, r))} B(y_i, \lambda r),$$

the second follows from

$$B(x, r) \supset \bigcup_{i=1}^{\nu(\lambda r, B(x, r))} B(y_i, \lambda r).$$

□

Proposition 2.15. *Let μ be a finite Borel measure on X and define*

$$m_x(r, R) := \inf\{\mu(B(y, r)) : B(y, r) \subset B(x, R)\}$$

$$M_x(r, R) := \sup\{\mu(B(y, r)) : B(y, r) \subset B(x, R)\}.$$

If

$$\limsup_{\lambda \rightarrow 0} \limsup_{r \rightarrow 0} \frac{\log \frac{M_x(\lambda r, r)}{m_x(\lambda r, r)}}{\log 1/\lambda} = 0$$

then $\underline{\delta}_X(x) = \underline{\delta}_\mu(x)$ and $\overline{\delta}_X(x) = \overline{\delta}_\mu(x)$.

Proof. From the definition of M_x we get

$$\mu(B(x, R)) \leq \sum_{i=1}^{n_r(B(x, R))} \mu(B(y_i, r)) \leq n_r(B(x, R))M_x(r, R+r).$$

From the definition of m_x we get

$$\mu(B(x, R)) \geq \sum_{i=1}^{\nu_r(B(x, R))} \mu(B(y_i, r)) \geq \nu_r(B(x, R))m_x(r, R).$$

Therefore

$$\nu_{\lambda R}(B(x, R)) \frac{m_x(\lambda R, R)}{M_x(\lambda R, R)} \leq \frac{\mu(B(x, R))}{\mu(B(x, \lambda R))} \leq n_{\lambda R}(B(x, R)) \frac{M_x(\lambda R, (\lambda+1)R)}{m_x(\lambda R, (\lambda+1)R)}$$

and

$$\begin{aligned} \frac{\log \nu_{\lambda R}(B(x, R))}{\log 1/\lambda} - \frac{\log \frac{M_x(\lambda R, R)}{m_x(\lambda R, R)}}{\log 1/\lambda} &\leq \frac{\log \frac{\mu(B(x, R))}{\mu(B(x, \lambda R))}}{\log 1/\lambda} \\ &\leq \frac{\log n_{\lambda R}(B(x, R))}{\log 1/\lambda} + \frac{\log \frac{M_x(\lambda R, (\lambda+1)R)}{m_x(\lambda R, (\lambda+1)R)}}{\log 1/\lambda}. \end{aligned}$$

Taking $\limsup_{R \rightarrow 0}$ and then $\limsup_{\lambda \rightarrow 0}$ and doing some algebra we get

$$\begin{aligned} &\limsup_{\lambda \rightarrow 0} \limsup_{R \rightarrow 0} \frac{\log \nu_{\lambda R}(B(x, R))}{\log 1/\lambda} - \liminf_{\lambda \rightarrow 0} \liminf_{R \rightarrow 0} \frac{\log \frac{M_x(\lambda R, R)}{m_x(\lambda R, R)}}{\log 1/\lambda} \\ &\leq \lim_{\lambda \rightarrow 0} \limsup_{R \rightarrow 0} \frac{\log \frac{\mu(B(x, R))}{\mu(B(x, \lambda R))}}{\log 1/\lambda} \\ &\leq \limsup_{\lambda \rightarrow 0} \limsup_{R \rightarrow 0} \frac{\log n_{\lambda R}(B(x, R))}{\log 1/\lambda} + \limsup_{\lambda \rightarrow 0} \limsup_{R \rightarrow 0} \frac{\log \frac{M_x(\lambda R, R)}{m_x(\lambda R, R)}}{\log 1/\lambda} \end{aligned}$$

and the thesis $\bar{\delta}_X(x) = \bar{\delta}_\mu(x)$ follows. The proof of the other equality is analogous. \square

Condition 2.16. Let μ be a finite Borel measure on X . For any $x \in X$ there are constants $R, C > 0$, depending on x , such that, for any $y \in B(x, R)$, $r \in (0, R)$, it holds

$$C^{-1}\mu(B(x, r)) \leq \mu(B(y, r)) \leq C\mu(B(x, r)).$$

Corollary 2.17. Let μ be a finite Borel measure on X satisfying Condition 2.16. Then $\underline{\delta}_X(x) = \underline{\delta}_\mu(x)$ and $\bar{\delta}_X(x) = \bar{\delta}_\mu(x)$, and these functions are locally constant.

Proof. As $m_x(\lambda r, r) = \inf\{\mu(B(y, \lambda r)) : B(y, \lambda r) \subset B(x, r)\} \geq C^{-1}\mu(B(x, \lambda r))$, and $M_x(\lambda r, r) \leq C\mu(B(x, \lambda r))$, for any $\lambda \in (0, 1)$, $r \in (0, R)$, we get

$$\limsup_{\lambda \rightarrow 0} \limsup_{r \rightarrow 0} \frac{\log \frac{M_x(\lambda r, r)}{m_x(\lambda r, r)}}{\log 1/\lambda} \leq \limsup_{\lambda \rightarrow 0} \frac{C^2}{\log 1/\lambda} = 0,$$

and the thesis follows from the previous Proposition. Moreover, $\forall y \in B(x, R)$, $\underline{\delta}_\mu(y) = \underline{\delta}_\mu(x)$, and $\bar{\delta}_\mu(y) = \bar{\delta}_\mu(x)$. \square

We now show that Condition 2.16 also implies property (ii) of Theorem 2.13 and volume doubling, hence tangential dimensions are indeed suprema, resp. infima, of dimensions of tangent objects. We first need some Lemmas.

Lemma 2.18. *The following inequality holds, for $0 \leq \lambda, \mu \leq 1$:*

$$n(\lambda\mu r, B_X(x, r)) \leq n(\lambda r, B_X(x, r)) \sup_{y \in B_X(x, r)} n(\lambda\mu r, B_X(y, \lambda r)).$$

Proof. Let us note that we may realize a covering of $B_X(x, r)$ with balls of radius $\lambda\mu r$ as follows: first choose an optimal covering of $B_X(x, r)$ with balls of radius λr , and then cover any covering ball optimally with balls of radius $\lambda\mu r$. The thesis follows. \square

Lemma 2.19. *Let X be a subset of \mathbb{R}^N . Then, for any $\lambda \leq 1$, there exists a constant K_λ such that*

$$n(\lambda r, B_X(x, r)) \leq K_\lambda, \quad \forall r > 0, x \in X.$$

Proof. Since (cf. e.g. [3]) the inequality

$$(2.18) \quad n(r, B_X(x, R)) \geq \nu(r, B_X(x, R)) \geq n(2r, B_X(x, R))$$

holds, we get

$$n(\lambda r, B_X(x, r)) \leq \nu\left(\frac{\lambda}{2}r, B_X(x, r)\right) \leq \nu\left(\frac{\lambda}{2}r, B_{\mathbb{R}^N}(x, r)\right) = \nu\left(\frac{\lambda}{2}, B_{\mathbb{R}^N}(1)\right),$$

where we used the dilation invariance of \mathbb{R}^N in the last equation, and omitted the irrelevant reference to the point x in the last term. \square

Lemma 2.20. *Let X be a closed subset of \mathbb{R}^N , $x \in X$. Then property (ii) of Theorem 2.13 is equivalent to the following:*

For any $\vartheta > 0$, or equivalently for some $\vartheta > 0$, there exist constants $c_\vartheta \geq 1$, $a_\vartheta \in (0, 1]$ such that, for any $r \leq a_\vartheta$, $\lambda, \mu \leq 1$, $y, z \in B_X(x, r)$,

$$(2.19) \quad n(\lambda\mu r, B_X(y, \lambda r)) \leq c_\vartheta n(\lambda\mu r\vartheta, B_X(z, \lambda r)).$$

Proof. First we show that property (ii) of Theorem 2.13 implies (2.19) for $\vartheta > 1$, hence for all $\vartheta > 0$. Indeed, for $a_\vartheta = \frac{a}{\vartheta}$, we get

$$\begin{aligned} \frac{n(\lambda\mu r, B_X(y, \lambda r))}{n(\lambda\mu r\vartheta, B_X(z, \lambda r))} &\leq c^2 \frac{n(\lambda\mu r, B_X(x, \lambda r))}{n(\lambda\mu r\vartheta, B_X(x, \lambda r))} \\ &\leq c^3 n(\lambda\mu r, B_X(x, \lambda\mu r\vartheta)) \leq c^3 K_{1/\vartheta}, \end{aligned}$$

where we used Lemmas 2.18, 2.19. We get (2.19) with $c_\vartheta = c^3 K_{1/\vartheta}$.

Now we prove (2.19), for some $\vartheta < 1$, implies property (ii) of Theorem 2.13. We set $a = \vartheta a_\vartheta$. Then, reasoning as in the previous case, we get

$$\begin{aligned} \frac{n(\lambda\mu r, B_X(y, \lambda r))}{n(\lambda\mu r, B_X(z, \lambda r))} &\leq c_\vartheta^3 n(\lambda\mu r\vartheta^2, B_X(x, \lambda\mu r/\vartheta)) \\ &\leq c_\vartheta^3 K_{\vartheta^3}. \end{aligned}$$

Finally we observe that (2.19), for some $\vartheta > 0$, implies (2.19), for some $\vartheta < 1$, hence, because of what has already been proved, it implies (2.19), for all $\vartheta > 0$. The thesis follows. \square

Proposition 2.21. *Let μ be a finite Borel measure on X satisfying Condition 2.16. Then property (ii) of Theorem 2.13 and Volume Doubling hold.*

Proof. Let us show volume doubling. Indeed, if $r \leq R/2$,

$$\mu(B(x, 2r)) \leq n(r, B(x, 2r)) \sup_{y \in B(x, 2r)} \mu(B(y, r)) \leq Cn(r, B(x, 2r))\mu(B(x, r)),$$

hence, by Lemma 2.19,

$$\frac{\mu(B(x, 2r))}{\mu(B(x, r))} \leq Cn(r, B(x, 2r)) \leq CK_{1/2}.$$

Now we prove property (ii) of Theorem 2.13 in the equivalent form (2.19), for $\vartheta = 1/2$, with $c_{1/2} = C^4$ and $a_{1/2} = R/2$. Indeed, let $\lambda, \mu \leq 1$, $r \leq R/2$, $y, z \in B(x, r)$. By (2.16), we get

$$\begin{aligned} n(\lambda\mu r/2, B(z, \lambda r)) &\geq \frac{\mu(B(z, \lambda r))}{\sup_{w \in B(z, \lambda r)} \mu(B(w, \lambda\mu r/2))} \\ &\geq \frac{\mu(B(z, \lambda r))}{\sup_{w \in B(z, 2r)} \mu(B(w, \lambda\mu r/2))} \geq \frac{1}{C^2} \frac{\mu(B(x, \lambda r))}{\mu(B(x, \lambda\mu r/2))}. \end{aligned}$$

Analogously, by (2.17), we get

$$\begin{aligned} \nu(\lambda\mu r/2, B(y, \lambda r)) &\leq \frac{\mu(B(y, \lambda r))}{\inf_{w \in B(y, \lambda r)} \mu(B(w, \lambda\mu r/2))} \\ &\leq \frac{\mu(B(y, \lambda r))}{\inf_{w \in B(y, 2r)} \mu(B(w, \lambda\mu r/2))} \leq C^2 \frac{\mu(B(x, \lambda r))}{\mu(B(x, \lambda\mu r/2))}. \end{aligned}$$

Finally, making use of (2.18), we get

$$\frac{n(\lambda\mu r, B(y, \lambda r))}{n(\lambda\mu r/2, B(z, \lambda r))} \leq C^4.$$

□

Corollary 2.22. *If Condition 2.16 holds for a measure μ on F , then tangential dimensions for μ are extrema of local dimensions of tangent measures, and tangential dimensions for F are extrema of local dimensions of tangent sets.*

3. COMPUTATION OF TANGENTIAL DIMENSIONS

3.1. Self-similar fractals. We compute here the tangential dimensions for self-similar fractals with open set condition, showing that Condition 2.16 is satisfied, and that upper and lower tangential dimensions for the Hausdorff measure are equal, hence coincide with the Hausdorff dimension. This means that self-similar fractals are too regular to give rise to a dimension interval, and a different class has to be considered, see the next subsection.

Let us recall that a self-similar fractal F is the fixed point of a map

$$W : K \in \text{Comp}(\mathbb{R}^N) \rightarrow \cup_{j=1}^p w_j(K) \in \text{Comp}(\mathbb{R}^N),$$

where w_j are similarities with similarity parameter λ_j , and that it satisfies the *open set condition* if there exists an open set V s.t. $w_j V \subset V$. It is well known that the Hausdorff dimension d of F satisfies $\sum_{j=1}^p \lambda_j^d = 1$, that the corresponding Hausdorff measure \mathcal{H}_d is nontrivial on F and the normalized restriction of \mathcal{H}_d to F is the unique self-similar probability measure on F . In particular $\mathcal{H}_d(w_j(F)) = \lambda_j^d$.

If Σ_n is the set of multi-indices $\sigma = (\sigma_1, \dots, \sigma_n)$ of length n , we denote by w_σ the product $w_{\sigma_1} \cdots w_{\sigma_n}$, and by λ_σ the product $\lambda_{\sigma_1} \cdots \lambda_{\sigma_n}$. Also we use the notations $F_\sigma = w_\sigma F$, $V_\sigma = w_\sigma V$. We note that $\mathcal{H}_d(F_\sigma) = \mathcal{H}_d(V_\sigma) = \lambda_\sigma^d$.

Lemma 3.1. *Let F be a self-similar fractal with open set condition. There exists a constant $C > 0$ s.t., for any $x \in F$, $r > 0$,*

$$C^{-1}r^d \leq \mathcal{H}_d(B_F(x, r)) \leq Cr^d.$$

Proof. It is not restrictive to assume that the diameter of V is 1, hence V_σ has diameter λ_σ . If σ is a multi-index of length n , in the following we shall denote by $\bar{\sigma}$ the multi-index $(\sigma_1, \dots, \sigma_{n-1})$. Let us consider the set $\Sigma(r)$ of multi-indices σ s.t. $\lambda_\sigma < r \leq \lambda_{\bar{\sigma}}$. Clearly the F_σ 's, $\sigma \in \Sigma(r)$ give a covering of F , and the V_σ , $\sigma \in \Sigma(r)$, are pairwise disjoint. Then, if $x \in F_\sigma$, $\sigma \in \Sigma(r)$, $B_F(x, r) \supset F_\sigma$, whence

$$\mathcal{H}_d(B_F(x, r)) \geq \lambda_\sigma^d \geq r^d \underline{\Delta}^d,$$

where $\underline{\Delta} = \min(\lambda_1, \dots, \lambda_p)$. Conversely, set $\Sigma(r, x) = \{\sigma \in \Sigma(r) : F_\sigma \cap B_F(x, r) \neq \emptyset\}$. Then

$$\mathcal{H}_d(B_F(x, r)) \leq \sum_{\sigma \in \Sigma(r, x)} \lambda_\sigma^d \leq \#\Sigma(r, x) r^d.$$

Observe now that $\cup_{\sigma \in \Sigma(r, x)} V_\sigma \subset B(x, 2r)$, therefore

$$\omega_N(2r)^N := \text{vol}(B(x, 2r)) \geq \sum_{\sigma \in \Sigma(r, x)} \text{vol}(V) \lambda_\sigma^N \geq \text{vol}(V) \#\Sigma(r, x) r^N \underline{\Delta}^N,$$

therefore

$$\#\Sigma(r, x) \leq \frac{\omega_N 2^N}{\text{vol}(V) \underline{\Delta}^N}.$$

The thesis follows. \square

Corollary 3.2. *Let F be a self-similar fractal satisfying open set condition, d its Hausdorff dimension. Then, for any $x \in F$,*

$$d = \underline{\delta}_F(x) = \bar{\delta}_F(x) = \underline{\delta}_{\mathcal{H}_d}(x) = \bar{\delta}_{\mathcal{H}_d}(x).$$

Proof. Since Condition 2.16 is satisfied for \mathcal{H}_d , it is enough to compute the tangential dimensions relative to the Hausdorff measure. Indeed Lemma 3.1 may be rephrased as $dt - \log C \leq f(t) \leq dt + \log C$, where $f(t) = -\log \mathcal{H}_d(B(x, e^{-t}))$, from which it follows

$$\lim_{h \rightarrow \infty} \lim_{t \rightarrow \infty} \frac{f(t+h) - f(t)}{h} = d.$$

\square

3.2. Translation fractals. Now we compute the tangential dimensions for translation fractals defined in [4] (cf. also [5, 6]).

Let $\{w_{n_j}\}$, $n \in \mathbb{N}$, $j = 1, \dots, p_n$, be contracting similarities of \mathbb{R}^N , with contraction parameter $\lambda_n \in (0, 1)$ only depending on n , and assume they verify the *regular open set condition*, namely there exists a nonempty bounded open set V in \mathbb{R}^N for which $w_{n_j}(V) \subset V$, the Lebesgue measure of V is equal to the Lebesgue measure of its closure C and V is equal to the interior of C . Setting $W_n : K \in \text{Comp}(\mathbb{R}^N) \rightarrow \cup_{j=1}^{p_n} w_{n_j}(K) \in \text{Comp}(\mathbb{R}^N)$, we get a sequence of compact sets $\{W_1 \circ W_2 \circ \dots \circ W_n(\bar{V})\}$ contained in \bar{V} , we call the Hausdorff limit F a *translation fractal*. Since the sequence is indeed decreasing, F can be equivalently defined

as the intersection. To avoid triviality we assume $p_j \geq 2$, which implies $2\lambda_j^N \leq 1$, i.e. $\lambda_j \leq 2^{-1/N}$.

As an example of fractals in our class consider the following construction. It is obtained by applying a sequence of either a Carpet step or a Vicsek step.

The Carpet step ($q = 1$) is obtained by dividing the sides of a square in 3 equal parts, so as to obtain 9 equal squares, and then the central square is removed.

The Vicsek step ($q = 2$) is obtained by dividing the sides of a square in 3 equal parts, so as to obtain 9 equal squares, and then 4 squares are removed, so that to remain with a chessboard.

In particular, we may set $q_j = 1$, if $(k-1)(2k-1) < j \leq (2k-1)k$ and $q_j = 2$, if $k(2k-1) < j \leq k(2k+1)$, $k = 1, 2, \dots$, getting a translation fractal with dimensions given by (cf. Theorem 3.4 below)

$$\underline{\delta} = \frac{\log 5}{\log 3} < \underline{d} = \bar{d} = \frac{\log 40}{\log 9} < \bar{\delta} = \frac{\log 8}{\log 3}$$

The first four steps ($q = 1, 2, 2, 1$) of the procedure above are shown in Figure 1.

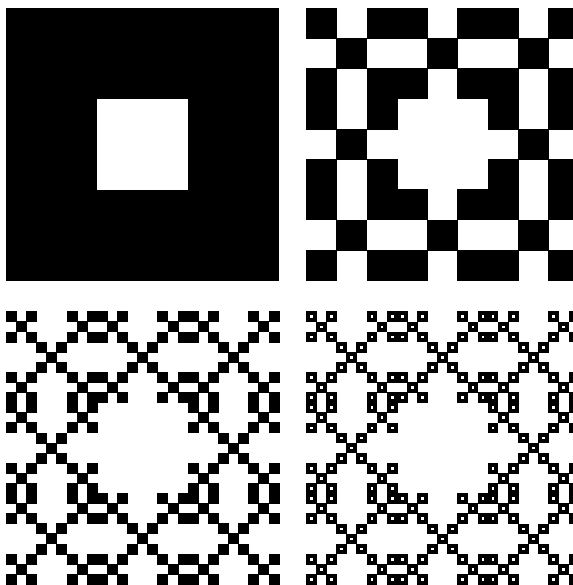


FIGURE 1. Carpet-Vicsek

More examples are contained in [7].

We set $\Lambda_n = \prod_{i=1}^n \lambda_i$, $P_n = \prod_{i=1}^n p_i$, $\Sigma_n := \{\sigma : \{1, \dots, n\} \rightarrow \mathbb{N} : \sigma(k) \in \{1, \dots, p_k\}, k = 1, \dots, n\}$, $\Sigma := \cup_{n \in \mathbb{N}} \Sigma_n$, and write $w_\sigma := w_{1\sigma(1)} \circ w_{2\sigma(2)} \circ \dots \circ w_{n\sigma(n)}$, for any $\sigma \in \Sigma_n$, and $V_\sigma := w_\sigma V$, $C_\sigma := w_\sigma C$.

On F there is a canonical limit measure μ , that can be defined as the weak*-limit of the sequence

$$(3.1) \quad \mu_n(A) = \sum_{|\sigma|=n} P_n^{-1} \mu_0(w_\sigma^{-1}(A)),$$

the limit being independent of the starting probability Borel measure μ_0 . In [6], Theorem 1.7, we proved that, when V is regular, μ can be characterized via the following property: for any subset \mathcal{J} of Σ_n

$$(3.2) \quad \mu(V_{\mathcal{J}}) \leq \frac{\#\mathcal{J}}{P_n} \leq \mu(C_{\mathcal{J}}),$$

where we set $C_{\mathcal{J}} = \cup_{\sigma \in \mathcal{J}} C_{\sigma}$, $V_{\mathcal{J}}$ equal to the interior of $C_{\mathcal{J}}$ relative to C .

Theorem 3.3. *Let F be a translation fractal with regular open set condition, with the notation above, and assume $p := \sup_n p_n < \infty$. Then Condition 2.16 holds for the limit measure μ , therefore tangential dimensions for F and for μ coincide, and they are extremal dimensions for the corresponding tangent objects.*

Proof. Denote by $B_F(x, r) := F \cap B(x, r)$, $x \in F$, $r > 0$. We may assume without restriction that the diameter of V is equal to one. Then set $a := \frac{\text{vol}(V)}{\text{vol}(B(0,2))}$, where vol denotes the Lebesgue measure. Then the number of disjoint copies of V intersecting a ball of radius 1 is not greater than the number of disjoint copies of V contained in a ball of radius 2 which is in turn lower equal than a^{-1} .

As a consequence, for any $x \in F$, if $\mathcal{J}(x, n) := \{\sigma \in \Sigma_n : V_{\sigma} \cap B_F(x, \Lambda_n) \neq \emptyset\}$, then

$$(3.3) \quad \#\mathcal{J}(x, n) \leq a^{-1}.$$

Clearly, by the regularity of V , $B_F(x, \Lambda_n) \subseteq V_{\mathcal{J}(x, n)}$, hence

$$(3.4) \quad \mu(B_F(x, \Lambda_n)) \leq \mu(V_{\mathcal{J}(x, n)}) \leq \frac{1}{aP_n}.$$

On the other hand, if $x \in F$, there is $\sigma(x) \in \Sigma_n$ such that $x \in \overline{V_{\sigma(x)}}$, therefore, for any $r > \Lambda_n$, $B_F(x, r) \supseteq C_{\sigma(x)}$, hence

$$(3.5) \quad \mu(B_F(x, r)) \geq \mu(C_{\sigma(x)}) \geq \frac{1}{P_n}.$$

Then, for any $r > 0$, if $n = n_r \in \mathbb{N}$ is such that $\Lambda_n < r \leq \Lambda_{n-1}$, we get, for $x, y \in F$,

$$\begin{aligned} \frac{a}{p} &\leq \frac{a}{p_n} = \frac{1/P_n}{1/(aP_{n-1})} \\ &\leq \frac{\mu(B_F(x, r))}{\mu(B_F(y, \Lambda_{n-1}))} \leq \frac{\mu(B_F(x, r))}{\mu(B_F(y, r))} \leq \frac{\mu(B_F(x, \Lambda_{n-1}))}{\mu(B_F(y, r))} \\ &\leq \frac{1/(aP_{n-1})}{1/P_n} = \frac{p_n}{a} \leq \frac{p}{a}. \end{aligned}$$

This proves Condition 2.16, therefore the other statements follow from Corollaries 2.17, 2.22. \square

Theorem 3.4. *Let F be a translation fractal with regular open set condition, with the notation above, and assume $p := \sup_n p_n < \infty$. Then*

(i)

$$\begin{aligned} \underline{\delta}_{\mu}(x) &= \liminf_{n, k \rightarrow \infty} \frac{\log P_{n+k} - \log P_n}{\log 1/\Lambda_{n+k} - \log 1/\Lambda_n}, \\ \overline{\delta}_{\mu}(x) &= \limsup_{n, k \rightarrow \infty} \frac{\log P_{n+k} - \log P_n}{\log 1/\Lambda_{n+k} - \log 1/\Lambda_n}. \end{aligned}$$

(ii)

$$\underline{d}_\mu(x) = d_H(F) = \liminf_{n \rightarrow \infty} \frac{\log P_n}{\log 1/\Lambda_n},$$

$$\overline{d}_\mu(x) = \limsup_{n \rightarrow \infty} \frac{\log P_n}{\log 1/\Lambda_n}.$$

Moreover the Hausdorff measure corresponding to $d := d_H(F)$ is non trivial if and only if $\liminf(\log P_n - d \log 1/\Lambda_n)$ is finite.

Proof. (i) Let us denote by $\text{Cl}(p, \lambda)$ the set of limit points, for $n, k \rightarrow +\infty$, of $\frac{\log P_{n+k} - \log P_n}{\log 1/\Lambda_{n+k} - \log 1/\Lambda_n}$, and by $\text{Cl}(f)$ the set of limit points, for $t, h \rightarrow +\infty$, of $\frac{f(t+h) - f(t)}{h}$, where we set $f(t) := -\log \mu(B(x, e^{-t}))$. Recalling Theorem 2.3, (iii), the formulas are proved if we show that $\text{Cl}(p, \lambda) \subseteq \text{Cl}(f)$, and that for any $c \in \text{Cl}(f)$ there exist $c', c'' \in \text{Cl}(p, \lambda)$ such that $c' \leq c \leq c''$. Concerning the inclusion, from (3.4) and (3.5) we have

$$\begin{aligned} -\log 1/a + f(\log 1/\Lambda_{n+k}) - f(\log 1/\Lambda_n) &\leq \log P_{n+k} - \log P_n \\ &\leq \log 1/a + f(\log 1/\Lambda_{n+k}) - f(\log 1/\Lambda_n). \end{aligned}$$

As a consequence, if n_j, k_j are subsequences giving rise to a limit point in $\text{Cl}(p, \lambda)$, and we set $t'_j = \log 1/\Lambda_{n_j+k_j}$, $t_j = \log 1/\Lambda_{n_j}$, we obtain that $\frac{f(t'_j) - f(t_j)}{t'_j - t_j}$ converges to the same limit, where we used that $h_j := \log 1/\Lambda_{n_j+k_j} - \log 1/\Lambda_{n_j} \rightarrow \infty$, since is minorized by $\frac{\log 2}{N} k_j$.

Now let $c \in \text{Cl}(f)$. Then we find two sequences, t_k and t'_k , such that $t_k \rightarrow \infty$ and $(t'_k - t_k) \rightarrow \infty$, and $c = \lim_k \frac{f(t'_k) - f(t_k)}{t'_k - t_k}$. If $\log 1/\Lambda_{n_k}$ is the best approximation from below of t_k and $\log 1/\Lambda_{n'_k}$ is the best approximation from below of t'_k , we get

$$\begin{aligned} f(t'_k) - f(t_k) &\leq f(\log 1/\Lambda_{n'_k+1}) - f(\log 1/\Lambda_{n_k}) \\ &\leq \log P_{n'_k+1} - \log P_{n_k} - \log a \\ &\leq \log P_{n'_k} - \log P_{n_k+1} + 2 \log p - \log a, \end{aligned}$$

which shows in particular that $P_{n'_k}/P_{n_k+1} \rightarrow \infty$, hence, for the bound on p_j , also $n'_k - (n_k + 1) \rightarrow \infty$. We also get

$$t'_k - t_k \geq \log 1/\Lambda_{n'_k} - \log 1/\Lambda_{n_k+1}$$

therefore

$$\begin{aligned} \lim_k \frac{f(t'_k) - f(t_k)}{t'_k - t_k} &\leq \limsup_k \frac{\log P_{n'_k} - \log P_{n_k+1}}{t'_k - t_k} \\ &\leq \limsup_k \frac{\log P_{n'_k} - \log P_{n_k+1}}{\log 1/\Lambda_{n'_k} - \log 1/\Lambda_{n_k+1}}. \end{aligned}$$

Possibly passing to a subsequence we obtain $c \leq c'' \in \text{Cl}(p, \lambda)$. The point c' is obtained analogously.

(ii) Let $\{t_k\}$ be an increasing sequence of positive real numbers s.t. $\underline{d}_\mu(x) = \lim_{k \rightarrow \infty} \frac{f(t_k)}{t_k}$, and let $\{n_k\}$ be an increasing sequence of natural numbers s.t.

$\log \frac{1}{\Lambda_{n_k}} \leq t_k < \log \frac{1}{\Lambda_{n_{k+1}}}$. Then

$$\begin{aligned} \frac{f(t_k)}{t_k} &\geq \frac{f\left(\frac{1}{\Lambda_{n_k}}\right)}{\log \frac{1}{\Lambda_{n_{k+1}}}} \geq \frac{\log P_{n_k} + \log a}{\log \frac{1}{\Lambda_{n_{k+1}}}} \\ &\geq \frac{\log P_{n_{k+1}} - \log p + \log a}{\log \frac{1}{\Lambda_{n_{k+1}}}} \end{aligned}$$

so that $\underline{d}_\mu(x) \geq \liminf_{n \rightarrow \infty} \frac{\log P_n}{\log \frac{1}{\Lambda_n}}$. Conversely, let $\{n_k\}$ be an increasing sequence of natural numbers s.t. $\lim_{k \rightarrow \infty} \frac{\log P_{n_k}}{\log \frac{1}{\Lambda_{n_k}}} = \liminf_{n \rightarrow \infty} \frac{\log P_n}{\log \frac{1}{\Lambda_n}}$ and set $t_k := \log \frac{1}{\Lambda_{n_k}}$.

Then

$$\begin{aligned} \underline{d}_\mu(x) &= \liminf_{t \rightarrow \infty} \frac{f(t)}{t} \leq \liminf_{k \rightarrow \infty} \frac{f(t_k)}{t_k} \\ &= \liminf_{k \rightarrow \infty} \frac{f\left(\frac{1}{\Lambda_{n_k}}\right)}{\log \frac{1}{\Lambda_{n_k}}} \leq \liminf_{k \rightarrow \infty} \frac{\log P_{n_k}}{\log \frac{1}{\Lambda_{n_k}}} \\ &= \liminf_{n \rightarrow \infty} \frac{\log P_n}{\log \frac{1}{\Lambda_n}}. \end{aligned}$$

The equation for $\overline{d}_\mu(x)$ is proved similarly.

The equality $\underline{d}_\mu(x) = d_H(F)$ and the last statement follow from Theorem 1.8 in [6]. \square

REFERENCES

- [1] A. Connes. *Non Commutative Geometry*. Academic Press, 1994.
- [2] G. A. Edgar. *Integral, probability, and fractal measures*. Springer, New York, 1997.
- [3] D. Guido, T. Isola. *An asymptotic dimension for metric spaces, and the 0-th Novikov-Shubin invariant*. Pacific J. Math. **204**, (2002) 43-59.
- [4] D. Guido, T. Isola. *Fractals in Noncommutative Geometry*, in the Proceedings of the Conference "Mathematical Physics in Mathematics and Physics", Siena 2000, Edited by R. Longo, Fields Institute Communications, Vol. 30 American Mathematical Society, Providence, RI, 2001
- [5] D. Guido, T. Isola. *Dimensions and singular traces for spectral triples, with applications to fractals*. Journ. Funct. Analysis **203**, (2003) 362-400.
- [6] D. Guido, T. Isola. *Dimensions and spectral triples for fractals in \mathbb{R}^N* . Preprint math.OA/0404295.
- [7] D. Guido, T. Isola. *Tangential dimensions I. Metric spaces*. Accepted for publication in Houston J. Math.
- [8] P. Mattila. *Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability*. Cambridge Studies in Advanced Mathematics, 44. Cambridge University Press, Cambridge, 1995.

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