On the domain of singular traces.

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1 Introduction.

After the introduction of spectral triples in Alain Connes' Noncommutative Geometry, singular traces on $B(H)$ became a quite popular tool in operator algebras. With the aim of classifying them some papers have been written ([3],[8],[1], see also [4] for nonpositive traces), addressing in particular the question of singular traceability, namely when, for a given operator, there exists a singular trace taking a finite non-zero value on it. Such a problem has been completely solved in [1], in terms of a condition on the asymptotics of the eigenvalue sequence of the given operator, thus selecting a class of operators whose eigenvalue asymptotics are in some sense close to the sequence $1/n$.

The problem we address in this note is slightly different, requesting now that the given operator just belongs to the domain of a singular trace, the value zero being allowed. Also, a dual question is addressed, namely for which operators there exists a singular trace assuming the value ∞ on it.

A naïve answer would say that too slow asymptotics cannot be resummed by any (singular) trace, hence no singular trace may vanish on them, and conversely that too fast asymptotics are in the kernel of any (singular) trace, hence no singular trace can be infinite on them. Even though this statement will be true for singular traces generated by 'regular' asymptotics (see below), it is false in general. Indeed we show that every compact operator is in the kernel (hence in the domain) of some singular trace, and any infinite-rank operator is not in the domain (hence not in the kernel) of some singular trace.

The mentioned results will be proved for general semifinite factors, in the spirit of [6]. The proofs are based on the new characterization of singular traceability given in [7], which we extend here to the continuous case, and also on the fact that ideals can be described mainly in terms of order relations.

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2 Ideals and singular traces in a semifinite factor

Here $(\mathcal{M}, tr)$ is any σ-finite, semifinite factor with a semifinite normal faithful trace, although only in the infinite case the following discussion is non trivial.

Let us recall that a function μ is associated with any operator in $\mathcal{M}$, via nonincreasing rearrangement (cf. [5]): set $\mu_A(t) := \inf\{s \geq 0 : \lambda_A(s) \leq t\}$, $t \geq 0$, where $\lambda_A(t) := tr(e_{|A|}(t,\infty))$, and $|A| = \int_0^\infty t\; de_{|A|}(t)$ is the spectral decomposition of $|A|$. Recall that μ_A is non-increasing and right continuous, A is (Breuer-)compact if μ_A is infinitesimal, and finite-rank if μ_A is eventually zero. We also set $g_A(t) = -\log \mu_A(e^t)$.

2.1 Singular traceability

The singular traceability for a compact operator in $B(H)$, namely the existence of a singular trace on $B(H)$ which is non-trivial on (the ideal generated by) T, has been completely characterized in [1], and this result has been extended to semifinite factors in [6]. Now we will use the Matuszewska indices to give an alternate description of singular traceability. Indeed Theorems 2.2 and 2.4 have been proved in [7] for the case of $B(H)$, but all the arguments extend to the general factor case. We give the proofs for the sake of completeness.

Given an operator $A \in \mathcal{M}$, we define its integral eigenvalue function S_A as

$$S_A(x) = \begin{cases} S_A^{\uparrow}(x) := \int_0^x \mu_A(y)dy & \mu_A \notin L^1[0,\infty) \\ S_A^{\downarrow}(x) := \int_x^\infty \mu_A(y)dy & \mu_A \in L^1[0,\infty). \end{cases}$$

Recall that a singular trace on $\mathcal{M}$ is a tracial weight vanishing on finite rank operators.

Theorem 2.1. [6] *An operator $T \in \mathcal{M}$ is singularly traceable if and only if 1 is a limit point, when $x \to \infty$, of the function $\frac{S_A(\lambda x)}{S_A(x)}$, for some $\lambda > 1$. If it is true for one λ, it is indeed true for any $\lambda > 1$.*

The singular traceability condition can be reformulated as follows.

Proposition 2.2. *A is singularly traceable if and only if* $\liminf \frac{x\mu_A(x)}{S_A(x)} = 0$.

Proof. Assume first A is not trace class, i.e. $S_A(x) = S_A^{\uparrow}(x)$. Then the thesis follows by Theorem 2.1 and the following inequalities:

$$0 < \frac{S_A^{\uparrow}(2x)}{S_A^{\uparrow}(x)} - 1 \leq \frac{x\mu_A(x)}{S_A^{\uparrow}(x)} = 2\left(1 - \frac{S_A^{\uparrow}(x/2)}{S_A^{\uparrow}(x)}\right).$$

When A is trace class, i.e. $S_A(x) = S_A^{\downarrow}(x)$, we have, analogously,

$$0 < 1 - \frac{S_A^{\downarrow}(2x)}{S_A^{\downarrow}(x)} \leq \frac{x\mu_A(x)}{S_A^{\downarrow}(x)} \leq 2\left(\frac{S_A^{\downarrow}(x/2)}{S_A^{\downarrow}(x)} - 1\right)$$

and the thesis follows. □

We now define the Matuszewska indices $\underline{\delta}(A)$, $\overline{\delta}(A)$ for a compact operator A as the indices for the corresponding eigenvalue function, cf. [2]. As a consequence,

$$\frac{1}{\underline{\delta}(A)} = \lim_{h\to\infty} \limsup_{t\to\infty} \frac{g_A(t+h)-g_A(t)}{h} = \inf_{h\to\infty} \limsup_{t\to\infty} \frac{g_A(t+h)-g_A(t)}{h} \tag{2.1}$$

$$\frac{1}{\overline{\delta}(A)} = \lim_{h\to\infty} \liminf_{t\to\infty} \frac{g_A(t+h)-g_A(t)}{h} = \sup_{h\to\infty} \liminf_{t\to\infty} \frac{g_A(t+h)-g_A(t)}{h}. \tag{2.2}$$

For the existence of the limits and the equalities in the definition above, see e.g. [2]. The following Lemma holds.

Lemma 2.3.
(i) If $\underline{\delta}(A) > 1$ then $g_A(t) < c + (1-\varepsilon)t$, for suitable $c, \varepsilon > 0$. In particular A is not trace class.

(ii) If $\overline{\delta}(A) < 1$ then $g_A(t) > -c + (1+\varepsilon)t$, for suitable $c, \varepsilon > 0$. In particular A is trace class.

(iii) If $\underline{\delta}(A) = \overline{\delta}(A) = 1$ then $-c_1 + (1-\varepsilon)t \le g_A(t) \le c_2 + (1+\varepsilon)t$, for suitable $c_1, c_2, \varepsilon > 0$.

Proof. Equalities (2.1), (2.2) imply

$$\underline{\delta}(A)^{-1} \ge \limsup_{t\to\infty} \frac{g_A(t)}{t} \ge \liminf_{t\to\infty} \frac{g_A(t)}{t} \ge \overline{\delta}(A)^{-1},$$

from which the thesis follows. □

Theorem 2.4. *Let $A \in \mathcal{M}$ be compact. Then A is singularly traceable if and only if $\underline{\delta}(A) \le 1 \le \overline{\delta}(A)$.*

Proof. Let $\underline{\delta}(A) > 1$. By (2.1), this is equivalent to the existence of $h > 0$ such that $\limsup_{t\to\infty} \frac{g_A(t+h)-g_A(t)}{h} < 1$, or, equivalently, to the existence of $\lambda > 1$ for which $\lambda \liminf_{t\to\infty} \frac{\mu_A(\lambda t)}{\mu_A(t)} > 1$. Now observe that, by Lemma 2.3, A is not trace class. Therefore

$$\frac{S_A(\lambda x)}{S_A(x)} = \frac{\lambda \int_0^x \mu_A(\lambda t)dt}{\int_0^x \mu_A(t)dt} = \lambda \frac{\int_0^x \left(\frac{\mu_A(\lambda t)}{\mu_A(t)}\right) \mu_A(t)dt}{\int_0^x \mu_A(t)dt},$$

hence

$$\liminf_{x\to\infty} \frac{S_A(\lambda x)}{S_A(x)} \ge \lambda \liminf_{x\to\infty} \left(\frac{\mu_A(\lambda x)}{\mu_A(x)}\right) > 1,$$

which implies that A is not singularly traceable by Theorem 2.1.
The proof when $\overline{\delta}(A) < 1$ is analogous.
Assume now that A is not singularly traceable, namely, by Proposition 2.2,

$\liminf_{t\to\infty} \frac{t\mu_A(t)}{S_A(t)} = \inf_t \frac{t\mu_A(t)}{S_A(t)} = k > 0$. If A is not trace-class, i.e. $S_A = S_A^\uparrow$, since μ_A is the derivative of $S_A^\uparrow$, the hypothesis means that

$$\frac{d}{dt} \log S_A^\uparrow(t) \geq \frac{k}{t}, \forall t.$$

Integrating on the interval $[x, \lambda x]$ one gets $S_A^\uparrow(\lambda x) \geq \lambda^k S_A^\uparrow(x)$. Since $\frac{x\mu_A(x)}{S_A^\uparrow(x)} \leq 1$, one obtains

$$\frac{\mu_A(x)}{\mu_A(\lambda x)} \leq \frac{\frac{S_A^\uparrow(x)}{x}}{\frac{kS_A^\uparrow(\lambda x)}{\lambda x}} \leq \frac{\lambda^{1-k}}{k}.$$

As a consequence,

$$\begin{aligned}\underline{\delta}(A)^{-1} &= \lim_{\lambda\to\infty} \frac{1}{\log\lambda} \limsup_{x\to\infty} \log \frac{\mu_A(x)}{\mu_A(\lambda x)} \\ &\leq \lim_{\lambda\to\infty} \frac{-\log k + (1-k)\log\lambda}{\log\lambda} = 1-k < 1\end{aligned}$$

If A is trace-class, i.e. $S_A = S_A^\downarrow$, since $-\mu_A$ is the derivative of $S_A^\downarrow$, we may prove, in analogy with the previous argument, that $S_A^\downarrow(\lambda x) \leq \lambda^{-k} S_A^\downarrow(x)$. Since $S_A^\downarrow(t) \geq \int_t^{2t} \mu_A(s)ds \geq t\mu_A(2t)$, which implies $\frac{x\mu_A(2x)}{S_A^\downarrow(x)} \leq 1$, one obtains

$$\frac{\mu_A(2\lambda x)}{\mu_A(x)} \leq \frac{\frac{S_A^\downarrow(\lambda x)}{\lambda x}}{\frac{kS_A^\downarrow(x)}{x}} \leq \frac{\lambda^{-1-k}}{k},$$

namely

$$\frac{\mu_A(x)}{\mu_A(\lambda x)} \geq k \left(\frac{\lambda}{2}\right)^{1+k},$$

As a consequence,

$$\begin{aligned}\overline{\delta}(A)^{-1} &= \lim_{\lambda\to\infty} \frac{1}{\log\lambda} \liminf_{x\to\infty} \log \frac{\mu_A(x)}{\mu_A(\lambda x)} \\ &\geq \lim_{\lambda\to\infty} \frac{\log k + (1+k)\log 2 + (1+k)\log\lambda}{\log\lambda} = 1+k > 1.\end{aligned}$$

□

Remark 2.5. We say that A is *regular* if $\underline{\delta}(A) = \overline{\delta}(A) =: \delta(A)$. As a consequence, for a regular A, singular traceability is equivalent to $\delta(A) = 1$.

2.2 Ideals

Let us introduce the set M of non-increasing infinitesimal right continuous functions defined on the interval $[0, \infty)$, and the set G of $(-\infty, +\infty]$-valued functions on $\mathbb{R}$ which are non-decreasing, right continuous, bounded from below and unbounded from above. Clearly the map

$$\mu \to g(t) = -\log \mu(e^t)$$

gives an order-reversing one-to-one correspondence from M to G.

Consider the action $\lambda \to D_\lambda f$ of the multiplicative group $\mathbb{R}_+$ on M given by:

$$D_\lambda f(t) = \lambda f(\lambda t) \ , \qquad \lambda, t \in \mathbb{R}_+$$

We say that a face F in M, namely a hereditary subcone of M is *dilation invariant* if $f \in F \Rightarrow D_\lambda f \in F$, $\lambda \in \mathbb{R}_+$.

We proved in [6] that proper two-sided ideals in $\mathcal{M}$ are in one-to-one correspondence with dilation invariant faces in M, namely for any ideal $\mathcal{I}$ there exists a face F such that

$$A \in \mathcal{I} \Leftrightarrow \mu_A \in F,$$

and conversely if F is a dilation invariant face then $\{A \in \mathcal{M} : \mu_A \in F\}$ is a two sided ideal in $\mathcal{M}$.

We note here that the additivity property of F does not really matter, indeed since $\mu \vee \nu \leq \mu + \nu \leq 2(\mu \vee \nu)$, homogeneity and the closure under $\vee$ are equivalent to linearity. As a consequence ideals of $\mathcal{M}$ can be described by subsets of G, namely the following corollary holds.

Corollary 2.6. *There is a one to one correspondence between ideals in $\mathcal{M}$ and subsets H of G such that*

$f, g \in H \Rightarrow f \wedge g \in H$

$f \in H$ *and* $g \in G$ *with* $g \geq f \Rightarrow g \in H$

$f \in H \Rightarrow f + a \in H$, $\forall a \in \mathbb{R}$

$f \in H \Rightarrow f^a \in H$, $\forall a \in \mathbb{R}$, *where* $f^a(t) = f(t-a)$.

In particular, given such an H, the set $\{A \in \mathcal{M} : g_A \in H\}$ is a two sided ideal in $\mathcal{M}$.

According to the previous correspondence, the ideal $\mathcal{K}(\mathcal{M})$ of compact operators corresponds to the whole set G, while the ideal $\mathcal{F}(\mathcal{M})$ of finite rank operators corresponds to eventually infinite elements of G. The principal ideal generated by an operator B corresponds to the set $H(B)$ defined as follows:

$$f \in H(B) \Leftrightarrow \exists a, b \in \mathbb{R} : f \geq b + g_B^a. \tag{2.3}$$

We now introduce the notion of kernel of an ideal. If $\mathcal{I}$ is an ideal in $\mathcal{M}$, we define its kernel $\mathcal{I}_0$ as the set of $A \in \mathcal{M}$ for which there exists $T \in \mathcal{I}_+$ such that

$$\forall \varepsilon > 0 \ \exists F_\varepsilon \in \mathcal{F}(\mathcal{M}) : |A| < \varepsilon T + F_\varepsilon.$$

The following statement is an immediate consequence of the definition of $\mathcal{I}_0$.

Proposition 2.7. *Let $\mathcal{I}$ be an ideal in $\mathcal{M}$. Then $\mathcal{I}_0$ is an ideal contained in $\mathcal{I}$, and any singular trace defined on $\mathcal{I}$ vanishes on $\mathcal{I}_0$.*

Clearly, if the face F corresponds to $\mathcal{I}$, the face F_0 corresponding to $\mathcal{I}_0$ is defined as the set of $\mu \in M$ for which there exists $\nu \in F$ such that

$$\forall \varepsilon > 0 \ \exists x_0 > 0 : \mu(x) < \varepsilon \nu(x), \ x > x_0,$$

hence the subset H_0 of G, corresponding to F_0, consists of the elements $g \in G$ for which there exists $h \in H$ such that

$$\forall c > 0 \ \exists t_0 \in \mathbb{R} : g(t) > c + h(t), \ t > t_0.$$

If $H = H(B)$ corresponds to the ideal generated by B, then $g \in H_0(B)$ if

$$\exists a \in \mathbb{R} : \forall c > 0 \exists t_0 \in \mathbb{R} : g(t) > c + g_B^a, \ t > t_0. \tag{2.4}$$

Let us prove the following.

Lemma 2.8. *Let B be regular. Then a compact operator A is in the ideal $\mathcal{I}(B)$ generated by B if and only if there exists $T \in \mathcal{I}(B)$ regular such that $g_A \geq g_T$.*

Proof. By equation (2.3), $A \in \mathcal{I}(B)$ if there exist $a, b \in \mathbb{R}$ such that $g_A \geq b + g_B^a$. Clearly $\underline{\delta}(g_B) = \underline{\delta}(b + g_B^a)$, and $\overline{\delta}(g_B) = \overline{\delta}(b + g_B^a)$, therefore, choosing T such that $g_T = b + g_B^a$, we obtain that T is regular. □

3 Domains and kernels of singular traces

Theorem 3.1.

(i) Let A be a compact operator. Then there is a singular trace vanishing on A.

(ii) Let A be an infinite rank operator. Then there is a singular trace which is infinite on A.

Proof. (i) The statement is proved if we show that there exists a singularly traceable operator B such that $A \in \mathcal{I}_0(B)$. According to the previous discussion this amounts to find an element B such that g_A satisfies condition (2.4).
Choose inductively an increasing sequence t_n such that $t_{n+1} - t_n > n$ and $g_A^{1/2}(t_{n+1}) - g_A^{1/2}(t_n) > n$, and set $g(t) = g_A^{1/2}(t_n)$ when $t \in [t_n, t_{n+1})$. Obviously $g \in G$. Then choose B such that $g_B = g$. Since $g \leq g_A^{1/2}$ and $\forall c > 0 \ \exists t_0 \in \mathbb{R}$: $g_A(t) > c + g_A^{1/2}(t)$, $t > t_0$, then a fortiori g_A satisfies (2.4). Moreover one easily gets $\underline{\delta}(g) = 0$ and $\overline{\delta}(g) = \infty$, which implies that B is singularly traceable.
(ii) The statement is proved if we show that there exists a singularly traceable operator B such that $A \notin \mathcal{I}(B)$. In analogy with the previous proof, this amounts to find a function $g \in G$ such that

$$\forall c > 0 \ \exists t_0 \in \mathbb{R} : g_A(t) < c + g(t), \ t > t_0, \tag{3.1}$$

and then choosing B such that $g_B = g$.
Choose inductively an increasing sequence t_n such that $t_{n+1} - t_n > n$ and $g_A^2(t_{n+1}) - g_A^2(t_n) > n$, and set $g(t) = g_A^2(t_{n+1})$ when $t \in [t_n, t_{n+1})$. Obviously $g \in G$. Since g_A^2 clearly satisfies (3.1) and $g \geq g_A^2$, a fortiori g satisfies (3.1). Moreover one easily gets $\underline{\delta}(g) = 0$ and $\overline{\delta}(g) = \infty$, which implies that B is singularly traceable. □

Theorem 3.2. *Let A be not singularly traceable, and B satisfy $\delta(B) = 1$.*

(i) If A is not trace class, then A does not belong to the ideal $\mathcal{I}(B)$ generated by B, namely any singular trace on $\mathcal{I}(B)$ is infinite on A.

(ii) If A is trace class, then A belongs to the kernel $\mathcal{I}_0(B)$ of $\mathcal{I}(B)$, namely any singular trace on $\mathcal{I}(B)$ is zero on A.

Proof. (i) By Theorem 2.4, $\underline{\delta}(A) > 1$, hence, by Lemma 2.3 there exist $\varepsilon > 0$ and $t_0 > 0$ such that $g_A(t) \leq (1-\varepsilon)t$ when $t > t_0$. Now, for any regular $T \in \mathcal{I}(B)$, we have, by Lemma 2.3, $g_T(t) \geq c(1-\varepsilon/2))t$. This implies that eventually $g_T(t) \geq g_A(t)$. The thesis then follows from Lemma 2.8.
(ii) By Theorem 2.4, $\overline{\delta}(A) < 1$, hence, by Lemma 2.3, there exist $\varepsilon > 0$ and $t_0 > 0$ such that $g_A(t) \geq (1+\varepsilon)t$ when $t > t_0$. Now, we have, by Lemma 2.3, $g_B(t) \leq c(1+\varepsilon/2))t$. This clearly implies g_A satisfies property (2.4) with $a = 0$, hence the thesis. □

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