Houston Journal of Mathematics
© University of Houston
Volume , No. ,

TANGENTIAL DIMENSIONS I. METRIC SPACES

DANIELE GUIDO, TOMMASO ISOLA

Abstract. Pointwise tangential dimensions are introduced for metric spaces. Under regularity conditions, the upper, resp. lower, tangential dimensions of X at x can be defined as the supremum, resp. infimum, of box dimensions of the tangent sets, *a la Gromov*, of X at x. Our main purpose is that of introducing a tool which is very sensitive to the "multifractal behaviour at a point" of a set, namely which is able to detect the "oscillations" of the dimension at a given point. In particular we exhibit examples where upper and lower tangential dimensions differ, even when the local upper and lower box dimensions coincide. Tangential dimensions can be considered as the classical analogue of the tangential dimensions for spectral triples introduced in [7], in the framework of Alain Connes' noncommutative geometry [4].

1. Introduction.

Dimensions can be seen as a tool for measuring the non-regularity, or fractality, of a given object. Non-integrality of the dimension is a first sign of non-regularity. A second kind of non-regularity is related to the fact that the dimension is not a global constant. This may happen in two ways: either the dimension varies from point to point, or it has an oscillating behavior at a point. Indeed dimensions are often defined as limits, and an oscillating behavior means that the upper and lower versions of the considered dimension are different. Our main goal here is to introduce a local dimension that is able to maximally detect such an oscillating behavior, namely for which the upper and lower determinations form a maximal dimensional interval. With this aim, we shall define the upper and lower tangential dimension for a metric space. We mention at this point that

Received by the editors April 16, 2004.
1991 *Mathematics Subject Classification.* 28A80,28A78.
Key words and phrases. Metric dimension, tangent cone, Gromov-Hausdorff convergence, translation fractals.

such dimensions, which are presented here in a completely "classic" way, have been introduced first for noncommutative spaces [7], where their definition is purely noncommutative, depending on the oscillating behavior of the eigenvalues of the Dirac operator, which may imply that the (singular) traceability exponents form an interval, rather than a singleton.

The name tangential is motivated here by the fact that, under suitable hypotheses, such dimensions are the supremum, resp. infimum, of the local dimensions of the tangent sets for the given space. The notion of tangent set (or rather tangent cone, cf. Remark 2) for a metric space is due to Gromov [5]. A tangent set of a metric space X at a point x is any limit point of the family of its dilations, for the dilation parameter going to infinity, taken in the pointed Gromov-Hausdorff topology.

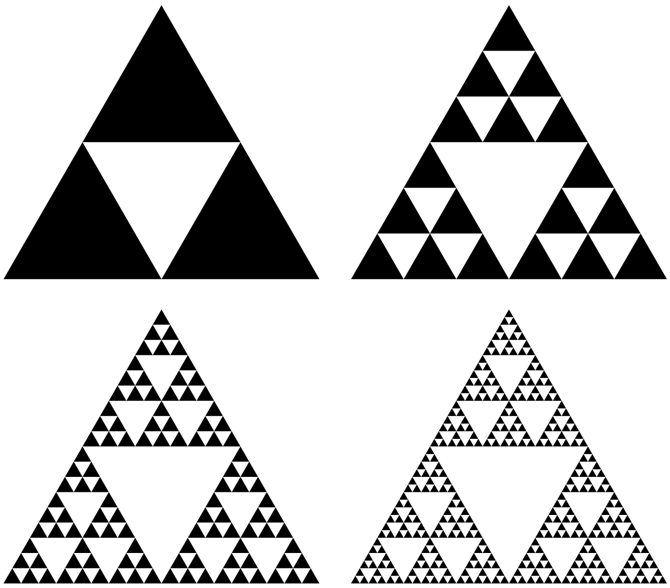

FIGURE 1. Modified Sierpinski

As an example we mention some fractals considered in [10]. They are constructed as follows. At each step the sides of an equilateral triangle are divided in $q \in \mathbb{N}$ equal parts, so as to obtain q^2 equal equilateral triangles, and then all downward pointing triangles are removed, so that $\frac{q(q+1)}{2}$ triangles are left. Setting

$q_j = 2$ if $(k-1)(2k-1) < j \leq (2k-1)k$ and $q_j = 3$ if $k(2k-1) < j \leq k(2k+1)$, $k = 1, 2, \ldots$, we get a translation fractal with dimensions given by [9]

$$\underline{\delta} = \frac{\log 3}{\log 2} < \underline{d} = \overline{d} = \frac{\log 18}{\log 6} < \overline{\delta} = \frac{\log 6}{\log 3},$$

where $\underline{\delta}, \overline{\delta}, \underline{d}, \overline{d}$ denote the lower tangential, the upper tangential, the lower local and the upper local dimensions. The first four steps ($q = 2, 3, 3, 2$) of the procedure above are shown in Figure 1.

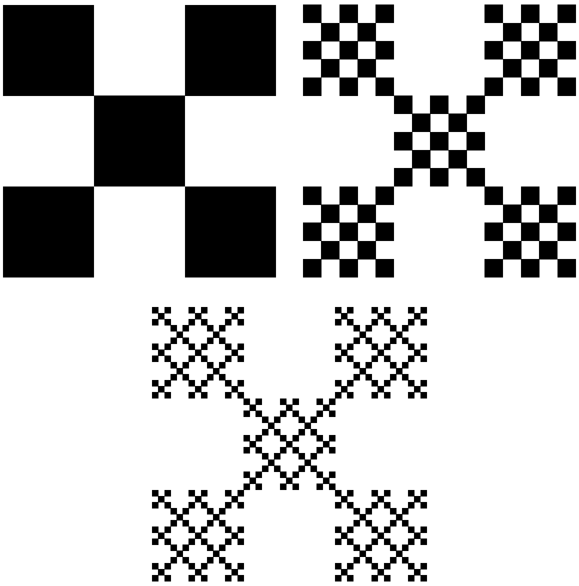

Figure 2. Modified Vicsek

The procedure considered above can, of course, be applied also to other shapes. For example, at each step the sides of a square are divided in $2q+1$, $q \in \mathbb{N}$, equal parts, so as to obtain $(2q+1)^2$ equal squares, and then $2q(q+1)$ squares are removed, so that to remain with a chessboard. In particular, we may set $q_j = 2$ if $k(2k+1) < j \leq (2k+1)(k+1)$ and $q_j = 1$ if $k(2k-1) < j \leq k(2k+1)$, $k = 0, 1, 2, \ldots$, getting a translation fractal with dimensions given by [9]

$$\underline{\delta} = \frac{\log 5}{\log 3} < \underline{d} = \overline{d} = \frac{\log 65}{\log 15} < \overline{\delta} = \frac{\log 13}{\log 5}.$$

The first three steps ($q = 1, 2, 1$) of this procedure are shown in Figure 2.

The fractals considered above belong to a general class of fractals, called translation fractals. In a forthcoming paper [9], we shall show that for such fractals the metric tangential dimensions coincide with the tangential dimensions of an invariant measure, in this way obtaining an explicit formula for the dimensions.

Translation fractals can also be studied from a noncommutative point of view, and commutative and noncommutative tangential dimensions coincide. This follows for translation fractals in $\mathbb{R}$ simply comparing the formulas given in [7] and those given in [9]. The analysis of translation fractals in $\mathbb{R}^n$ and their tangential dimensions from a noncommutative point of view is contained in [8].

2. Tangent sets of a metric space

Tangent sets of metric spaces at a point have been defined by Gromov, cf. [5, 3].

If (X,d) is a metric space, we shall denote by $B(x,r)$ the open ball $\{y \in X : d(x,y) < r\}$, by $\overline{B}(x,r)$ the closed ball $\{y \in X : d(x,y) \leq r\}$ and by $\overline{B(x,r)}$ the closure of $B(x,r)$; moreover $B_\varepsilon(E) := \{x \in X : \inf_{y\in E} d(x,y) < \varepsilon\}$, for $E \subset X$.

Let us recall that the Gromov-Hausdorff distance $d_{GH}(X,Y)$ between two metric spaces X and Y is defined as the infimum of the $\varepsilon > 0$ such that there are isometric embeddings φ_X, φ_Y of X and Y into a metric space Z for which $\varphi_X(X) \subset B_\varepsilon(\varphi_Y(Y))$ and $\varphi_Y(Y) \subset B_\varepsilon(\varphi_X(X))$. This is indeed a distance between isometry classes of compact metric spaces.

In case of noncompact (proper) metric spaces one considers the pointed Gromov-Hausdorff topology, which can be equivalently defined as

(1) a neighbourhood base consists of the sets $U^\varepsilon(X,x)$, (X,x) a pointed metric space, $\varepsilon \in (0,\frac{1}{2})$, where $U^\varepsilon(X,x) := \{(Y,y) : d_{pGH}((X,x),(Y,y)) < \varepsilon\}$, and $d_{pGH}((X,x),(Y,y))$ is the infimum of the $\varepsilon > 0$ for which there is a compatible metric d on the disjoint union of X and Y s.t. $d(x,y) < \varepsilon$, $\overline{B}_X(x,\frac{1}{\varepsilon}) \subset B_\varepsilon(Y)$, $\overline{B}_Y(y,\frac{1}{\varepsilon}) \subset B_\varepsilon(X)$.

(2) a neighbourhood base consists of the sets $V^{R,\varepsilon}(X,x)$, (X,x) a pointed metric space, $R > 0$, $\varepsilon \in (0,1)$, where $V^{R,\varepsilon}(X,x) := \{(Y,y) : d^R((X,x),(Y,y)) < \varepsilon\}$, and $d^R((X,x),(Y,y))$ is defined as the infimum of the $\varepsilon > 0$ such that there are isometric embeddings φ_X, φ_Y of X and Y into a metric space (Z,d) for which $d(\varphi_X(x),\varphi_Y(y)) < \varepsilon$, $\varphi_X(\overline{B}_X(x,R)) \subset B_\varepsilon(\varphi_Y(Y))$ and $\varphi_Y(\overline{B}_Y(y,R)) \subset B_\varepsilon(\varphi_X(X))$.

On the isometry classes of proper metric spaces it is a Hausdorff topology. Since this topology is separable, it is determined by its converging sequences; indeed it is equivalently defined as follows.

Proposition 2.1. [5] *(X_n, x_n) converges to (X, x) in the pointed Gromov-Hausdorff topology if and only if, for any $R > 0$ there exists a positive infinitesimal sequence ε_n such that, for any $\eta > 0$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$ there are isometric embeddings φ_n, φ of $\overline{B}_{X_n}(x_n, R+\varepsilon_n)$ and $\overline{B}_X(x, R)$ into a metric space (Z_n, d_n) for which $d_n(\varphi_n(x_n), \varphi(x)) < \eta$, $\varphi_n(\overline{B}_{X_n}(x_n, R+\varepsilon_n)) \subset B_\eta(\varphi(\overline{B}_X(x, R))$ and $\varphi(\overline{B}_X(x, R))) \subset B_\eta(\varphi_n(\overline{B}_{X_n}(x, R + \varepsilon_n)))$.*

From the previous characterization one easily gets

Proposition 2.2.
(i) If (X_n, x_n) converge to (X, x) in the pointed Gromov-Hausdorff topology, then, possibly passing to a subsequence, $\overline{B_{X_n}(x_n, R)} \xrightarrow[GH]{} B$, with $\overline{B_X(x, R)} \subseteq B \subseteq \overline{B}_X(x, R)$.
(ii) If X_n is an increasing sequence of proper spaces such that the completion X of $\cup_n X_n$ is proper, then $(X_n, x) \xrightarrow[pGH]{} (X, x)$ for any $x \in X_1$.

Definition 1. Let (X, d) be a metric space, $x \in X$. A *tangent set* of X at x is any limit point, for $t \to \infty$, of (X, x, td) in the pointed Gromov-Hausdorff topology, where td denotes the rescaled distance by the parameter t. We write also tX for (X, x, td) when the metric and x are clear from the context. We shall denote by $\mathcal{T}_x X$, and call the *tangent cone* of X at x, the family of tangent sets of X at x. A *tangent ball* of X at x is any ball centered in x of some tangent set $T \in \mathcal{T}_x X$.

Proposition 2.3. *Let (X, x) be such that*

$$\limsup_{r \to 0} n_{\lambda r}(\overline{B}_X(x, r)) < \infty \quad \forall \lambda > 0. \tag{2.1}$$

Then $\mathcal{T}_x X$ is not empty. Indeed, given any sequence $t_n \to +\infty$, there exists a subsequence t_{n_k} for which $(X, x, t_{n_k} d)$ converges to a unique proper space in the pointed Gromov-Hausdorff topology.

Proof. Follows from the Gromov compactness criterion [5]. □

Remark.
(i) A tangent set cannot be empty, since it necessarily contains x. It may happen that $\mathcal{T}_x X$ is empty, namely that (X, x, td) has no limit points.
(ii) If X is a manifold, the tangent set at x is unique and coincides with the ordinary tangent space (cf. [5]).
(iii) $\mathcal{T}_x X$ is indeed a cone, namely it is dilation invariant. In fact, if (T, d_T) is a tangent set of X at x given by the converging sequence $(X, x, t_n d)$, and $\alpha > 0$, then $(X, x, \alpha t_n d)$, converges to $(T, \alpha d_T)$. As a consequence, if $\mathcal{T}_x X$ consists of a unique set, such set is a cone. Since this case has been usually considered, one usually refers to such metric space with the name of Gromov tangent cone.

(iv) If all the metric spaces X_n are subsets of the same proper metric space Z, pointed Gromov-Hausdorff convergence may be replaced by the Attouch-Wets convergence, [2]. Let us note that in this case we do not need to specify a point in Z.
(v) If the ambient space Z is dilation invariant, e.g. $Z = \mathbb{R}^n$, then the dilations of a given subset are still subsets of Z, hence the tangent sets can be defined as Attouch-Wets limits, and are subsets of Z, as in [1]. Even if the two topologies do not coincide, the families of tangent sets at a given point do.

We conclude this section by computing explicitly the tangent cone of a self-similar fractal at the points which are invariant for some of the dilations generating the fractal.

Theorem 2.4. *Let F be a self-similar fractal, w one of the generating similarities, $x = wx$. The tangent cone $\mathcal{T}_x F$ consists exactly of all dilations of $Z = \overline{\bigcup_{n\in\mathbb{N}} w^{-n} F}$.*

Proof. Let us observe that, given $r_n \to +\infty$, $\lambda > 1$, we may find $n_k, m_k \in \mathbb{N}$, $c > 0$ such that $c\lambda^{m_k}/r_{n_k} \to 1$. Hence, if T is a tangent set of F at x, with $r_n F \underset{\text{pGH}}{\longrightarrow} T$, we have $\lambda^{m_k} F \underset{\text{pGH}}{\longrightarrow} (1/c)T$. Therefore it is enough to show that $\lambda^m F \underset{\text{pGH}}{\longrightarrow} Z$. Indeed $(\lambda^m F, x)$ is isometric to $(w^{-n}F, x)$ as pointed metric spaces, and $w^{-n}F$ is an increasing sequence of proper metric spaces, therefore the thesis follows from Proposition 2.2 (ii). ☐

Let $\vec{q} = \{q_j\}$ be a sequence of natural numbers, and $S(\vec{q})$ be the corresponding fractal constructed as in Fig. 1. Let us observe that if $q_j \equiv q$, then we get the q-Sierpinski triangle $S(q)$.

Theorem 2.5. *If for any $p \in \mathbb{N}$ there is a $\overline{j}$ such that $q_j = q$ for $\overline{j} \le j \le \overline{j} + p$, then the tangent cone of $S(\vec{q})$ at one of its extreme points contains the tangent space of $S(q)$ at one of its extreme points.*

Proof. Clearly for any p we may indeed find an increasing sequence j_n such that $q_j = q$ for $j_n - p \le j \le j_n + p$. If we set $Q_j = \prod_{i=1}^{j} q_i$, then, for any $r < q^p$, the Hausdorff distance between the ball of radius r of $Q_{j_n} S(\vec{q})$ and the ball of radius r of $Q_{j_n} S(q)$ is less than q^{-p}, which implies that the pointed Gromov-Hausdorff limit of $Q_{j_n} S(\vec{q})$ coincides with the pointed Gromov-Hausdorff limit of $Q_{j_n} S(q)$. ☐

Remark. While in the first example, the tangent sets are described by one (dilation) parameter, in the second example a second parameter q appears. In a sense this shows that the higher is the regularity of the set (around the point x), the smaller is its tangent cone. In the case of the Sierpinski triangle S, the explicit description of the tangent set to a point x can be extended easily to all points which are obtained by applying a product of similarities to one of the three extremal points of S.

3. Tangential dimensions

3.1. **Definition of tangential dimensions and connection with tangent sets.** Let (X, d) be a metric space, $E \subset X$. Let us denote by $n(r, E) \equiv n_r(E)$, resp. $\overline{n}(r, E) \equiv \overline{n}_r(E)$, the minimum number of open, resp. closed, balls of radius r necessary to cover E, and by $\nu(r, E) \equiv \nu_r(E)$ the maximum number of disjoint open balls of E of radius r contained in E.

Definition 2. Let (X, d) be a metric space, $E \subset X$, $x \in E$. We call *upper, resp. lower tangential dimension* of E at x the (possibly infinite) numbers

$$\underline{\delta}_E(x) := \liminf_{\lambda\to 0} \liminf_{r\to 0} \frac{\log n(\lambda r, E \cap \overline{B}(x, r))}{\log 1/\lambda},$$

$$\overline{\delta}_E(x) := \limsup_{\lambda\to 0} \limsup_{r\to 0} \frac{\log n(\lambda r, E \cap \overline{B}(x, r))}{\log 1/\lambda}.$$

Proposition 3.1. *Nothing changes in the previous definition if one replaces n with ν or with $\overline{n}$, or $E \cap \overline{B}(x, r)$ with $E \cap B(x, r)$. Moreover, if E is closed in X, one can replace $E \cap \overline{B}(x, r)$ also with $\overline{E \cap B(x, r)}$.*

Proof. The statements about ν and $\overline{n}$ follow from (see e.g. [6])

$$n_{2r}(E) \leq \nu_r(E) \leq n_r(E) \tag{3.1}$$

$$n_{2r}(E) \leq \overline{n}_r(E) \leq n_r(E). \tag{3.2}$$

From $B(x, r) \subset \overline{B}(x, r) \subset B(x, 2r)$, and $E \cap B(x, r) \subset \overline{E \cap B(x, r)} \subset E \cap \overline{B}(x, r)$, if E is closed, follow the other statements. ☐

We want to give a geometric interpretation of the (lower and upper) tangential dimensions. We need some auxiliary results.

Proposition 3.2.
(i) For any $r > 0$, the function $X \mapsto n_r(X)$ is upper semicontinuous on compact sets in the Gromov-Hausdorff topology.
(ii) For any $r > 0$, $R > 0$, the function $(X, x) \mapsto n_r(\overline{B}_X(x, R))$ is upper semicontinuous on proper spaces in the pointed Gromov-Hausdorff topology.

(iii) For any $r > 0$, the function $X \mapsto \overline{n}_r(X)$ is lower semicontinuous on compact sets in the Gromov-Hausdorff topology.
(iv) For any $r > 0$, $R > 0$, the function $(X, x) \mapsto \overline{n}_r(\overline{B_X(x,R)})$ is lower semicontinuous on proper spaces in the pointed Gromov-Hausdorff topology.

Proof. (i). Since n_r is integer valued, the statement is equivalent to: $\forall K$ compact $\exists \delta > 0$ s.t. $d_{GH}(J,K) < \delta$, imply $n_r(J) \leq n_r(K)$. Then, if $\cup_{j=1}^{n_r(K)} B(x_j, r)$ is a minimal covering for K with open balls of radius r, and we set

$$R = \max_{x \in K} \min_{j=1,\dots n_r(K)} d_K(x, x_j),$$

then $\delta = r - R > 0$ and $n_\rho(K) = n_r(K)$ for any $r \geq \rho > R$. Therefore $d_H(J,K) < \delta/2$ implies J and K may be embedded in a metric space Z where $J \subset B_Z(K, \delta/2)$, $K \subset B_Z(J, \delta/2)$, hence we may find points $y_1, \dots, y_n \in J$ with $d_Z(x_i, y_i) < \delta/2$. Finally

$$\cup_{j=1}^{n_r(K)} B(y_j, r) \supset \cup_{j=1}^{n_r(K)} B(x_j, r - \delta/2) \supset B_Z(K, \delta/2),$$

namely $n_r(J) \leq n_r(K)$.

(ii). Assume $(X_n, x_n) \underset{\text{pGH}}{\longrightarrow} (X, x)$. Then, by Proposition 2.1 for any given R, there exists $\varepsilon_n \to 0$ such that $\overline{B}_{X_n}(x_n, R + \varepsilon_n) \underset{\text{GH}}{\longrightarrow} \overline{B}_X(x, R)$. Eventually, by (i),

$$n_r(\overline{B}_{X_n}(x_n, R)) \leq n_r(\overline{B}_{X_n}(x_n, R + \varepsilon_n)) \leq n_r(\overline{B}_X(x, R)).$$

(iii). We have to show that, for any $p \in \mathbb{N}$, $\mathcal{B} := \{X$ proper metric space : $\overline{n}_r(X) \leq p\}$ is closed. Let $\{X_n\} \subset \mathcal{B}$, $X_n \underset{\text{GH}}{\longrightarrow} X$, and possibly passing to a subsequence, we may assume that $\overline{n}_r(X_n) = q \leq p$, all $n \in \mathbb{N}$. According to [3] we may describe X as an ultralimit, namely, given any free ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we may set $d_{\mathcal{U}}(\{x_n\}, \{y_n\}) = \lim_{\mathcal{U}} d_{X_n}(x_n, y_n)$ where $x_n, y_n \in X_n$, and X is isometric to the space of equivalence classes $x_{\mathcal{U}}$ of $\{x_n\}$ obtained by identifying points with zero distance. Now let x_n^j, $j = 1, \dots, q$ be the centers of balls of radius r covering X_n, and set $x_{\mathcal{U}}^j := [x_n^j]$, $j = 1, \dots, q$. Given any $x_{\mathcal{U}} = [x_n] \in X$, setting $N_j := \{n \in \mathbb{N} : d_{X_n}(x_n, x_n^j) \leq r\}$, there is $j_0 \in \{1, \dots, q\}$ such that $N_{j_0} \in \mathcal{U}$, so that

$$d_{\mathcal{U}}(x_{\mathcal{U}}^{j_0}, x_{\mathcal{U}}) = \lim_{\mathcal{U}} d_{X_n}(x_n^{j_0}, x_n) \leq r,$$

hence $\overline{n}_r(X) \leq q$, that is $X \in \mathcal{B}$.

(iv). Assume $(X_n, x_n) \underset{\text{pGH}}{\longrightarrow} (X, x)$. Then, for any given R, and possibly passing to a subsequence, $\overline{B_{X_n}(x_n, R)} \underset{\text{GH}}{\longrightarrow} B$ with $\overline{B_X(x,R)} \subset B \subset \overline{B}_X(x, R)$ (cf. Proposition 2.2). Therefore, by (iii),

$$\overline{n}_r\left(\overline{B_X(x,R)}\right) \leq \overline{n}_r(B) \leq \liminf \overline{n}_r\left(\overline{B_{X_n}(x_n,R)}\right).$$

□

Theorem 3.3. *Let (X, d) be a metric space, and let $x \in X$ be such that the sufficient condition (2.1) is satisfied. The following formulas hold:*

$$\overline{\delta}_X(x) = \limsup_{r\to 0} \sup_{T\in\mathcal{T}_x X} \frac{\log n_r(\overline{B}_T(x,1))}{\log 1/r},$$

$$\underline{\delta}_X(x) = \liminf_{r\to 0} \inf_{T\in\mathcal{T}_x X} \frac{\log n_r(\overline{B}_T(x,1))}{\log 1/r}.$$

Proof. Let us denote by tX the metric space (X, td). Fix $\lambda > 0$ and choose $r_n \to 0$ such that $\limsup_{r\to 0} n_{\lambda r}(\overline{B}_X(r)) = \limsup_n n_{\lambda r_n}(\overline{B}_X(r_n))$ and $r_n^{-1}X$ is converging in the pointed Gromov-Hausdorff topology, say to a tangent set T. By the proposition above,

$$\begin{aligned} n_\lambda(B_T(1)) &\geq \limsup_n n_\lambda(\overline{B}_{r_n^{-1}X}(1)) = \limsup_n n_{\lambda r_n}(\overline{B}_X(r_n)) \\ &= \limsup_{r\to 0} n_{\lambda r}(\overline{B}_X(r)). \end{aligned}$$

Taking the $\limsup_{\lambda\to 0} \sup_{T\in\mathcal{T}_x(X)}$ we get

$$\overline{\delta}_X(x) \leq \limsup_{r\to 0} \sup_{T\in\mathcal{T}_x X} \frac{\log n_r(\overline{B}_T(x,1))}{\log 1/r}.$$

Conversely, for any $T \in \mathcal{T}_x(X)$, with $r_n^{-1}X \xrightarrow[\text{pGH}]{} T$, we get $\overline{n}_\lambda(\overline{B_T(1)}) \leq \liminf_n \overline{n}_{\lambda r_n}(\overline{B_X(r_n)}) \leq \limsup_r \overline{n}_{\lambda r}(\overline{B_X(r)})$.

Taking the $\limsup_{\lambda\to 0} \sup_{T\in\mathcal{T}_x(X)}$ we get

$$\overline{\delta}_X(x) \geq \limsup_{r\to 0} \sup_{T\in\mathcal{T}_x X} \frac{\log \overline{n}_r(\overline{B_T(x,1)})}{\log 1/r}.$$

The thesis easily follows. □

3.2. **Further properties of tangential dimensions.** Tangential dimensions are invariant under bi-Lipschitz maps.

Proposition 3.4. *Let X, Y be metric spaces, $f : X \to Y$ be a bi-Lipschitz map i.e. there is $L > 0$ such that $L^{-1}d_X(x, x') \leq d_Y(f(x), f(x')) \leq Ld_X(x, x')$, for $x, x' \in X$. Then $\underline{\delta}_X(x) = \underline{\delta}_Y(f(x))$ and $\overline{\delta}_X(x) = \overline{\delta}_Y(f(x))$, for all $x \in X$.*

Proof. Observe that, for any $x \in X$, $y \in Y$, $r > 0$, we have

$$B(f(x), r/L) \subset f(B(x,r)) \subset B(f(x), rL) \tag{3.3}$$

$$B(f^{-1}(y), r/L) \subset f^{-1}(B(y,r)) \subset B(f^{-1}(y), rL) \tag{3.4}$$

so that

$$f(B(x,R)) \subset B(f(x),RL) \subset \bigcup_{i=1}^{n(r/L,B(f(x),RL))} B(y_i,r/L)$$

and

$$B(x,R) \subset \bigcup_{i=1}^{n(r/L,B(f(x),RL))} f^{-1}(B(y_i,r/L)) \subset \bigcup_{i=1}^{n(r/L,B(f(x),RL))} B(f^{-1}(y_i),r)$$

from which it follows $n(r,B(x,R)) \leq n(r/L,B(f(x),RL))$. Exchanging the roles of f and f^{-1}, we obtain $n(r,B(f(x),R)) \leq n(r/L,B(x,RL))$, so that

$$n(rL,B(f(x),\frac{R}{L})) \leq n(r,B(x,R)) \leq n(r/L,B(f(x),RL)).$$

Therefore, taking $\limsup_{R\to 0}$, then $\limsup_{\lambda\to 0}$, and doing some algebra, we get

$$\begin{aligned}\limsup_{\lambda\to 0}\limsup_{R\to 0}\frac{\log n(\lambda R,B(f(x),R))}{\log L^2/\lambda} &\leq \limsup_{\lambda\to 0}\limsup_{R\to 0}\frac{\log n(\lambda R,B(x,R))}{\log 1/\lambda}\\ &\leq \limsup_{\lambda\to 0}\limsup_{R\to 0}\frac{\log n(\lambda R,B(f(x),R))}{\log 1/(L^2\lambda)}\end{aligned}$$

which means $\overline{\delta}_X(x) = \overline{\delta}_Y(f(x))$. The other equality is proved in the same manner. □

The following proposition shows that the functions $\underline{\delta}_X$ and $\overline{\delta}_X$ satisfy properties which are characteristic of a dimension function. Denote by $B_Y(x,r) := Y \cap B_X(x,r)$, if $Y \subset X$.

Proposition 3.5.
(i) Let $Y \subset X$ and $x \in Y$. Then $\underline{\delta}_Y(x) \leq \underline{\delta}_X(x)$, and $\overline{\delta}_Y(x) \leq \overline{\delta}_X(x)$. Equality holds if there is $R_0 > 0$ such that $B_X(x,R_0) \subset Y$.
(ii) Let $X_1,\ X_2 \subset X$ and $x \in X_1 \cap X_2$. Then

$$\begin{aligned}&\underline{\delta}_{X_1\cup X_2}(x) \geq \max\{\underline{\delta}_{X_1}(x),\underline{\delta}_{X_2}(x)\}\\ &\overline{\delta}_{X_1\cup X_2}(x) = \max\{\overline{\delta}_{X_1}(x),\overline{\delta}_{X_2}(x)\}.\end{aligned}$$

(iii) Let $X,\ Y$ be metric spaces, $x \in X$, $y \in Y$. Then $\underline{\delta}_{X\times Y}((x,y)) \geq \underline{\delta}_X(x) + \underline{\delta}_Y(y)$, and $\overline{\delta}_{X\times Y}((x,y)) \leq \overline{\delta}_X(x) + \overline{\delta}_Y(y)$.

Proof. (i) As $B_Y(x,R) \subset B_X(x,R)$, we get $n_r(B_Y(x,R)) \leq n_r(B_X(x,R))$, and analogously for ν_x, and the claim follows. The second statement is obvious.

(ii) The inequalities $\geq$ follow from (i). It remains to prove $\overline{\delta}_{X_1\cup X_2}(x) \leq \max\{\overline{\delta}_{X_1}(x),\overline{\delta}_{X_2}(x)\}$, and we can assume $a := \overline{\delta}_{X_1}(x) < \infty$ and $b := \overline{\delta}_{X_2}(x) < \infty$, otherwise there is nothing to prove. Assume for definiteness that $a \leq b$. Then, for any $\varepsilon > 0$, there is $\lambda_0 > 0$ such that, for any $\lambda \in (0, \lambda_0)$, there exists $r_0 = r_0(\varepsilon, \lambda)$ such that, for any $r \in (0, r_0)$ we get

$$\begin{aligned} n_{\lambda r}(B_{X_1}(x, r)) &\leq \frac{1}{\lambda^{a+\varepsilon}} \\ n_{\lambda r}(B_{X_2}(x, r)) &\leq \frac{1}{\lambda^{b+\varepsilon}}. \end{aligned}$$

As $B_{X_1 \cup X_2}(x, R) \subset B_{X_1}(x, R) \cup B_{X_2}(x, R)$, we get

$$\begin{aligned} n_{\lambda r}(B_{X_1 \cup X_2}(x, r)) &\leq n_{\lambda r}(B_{X_1}(x, r)) + n_{\lambda r}(B_{X_2}(x, r)) \\ &\leq \frac{1}{\lambda^{a+\varepsilon}} + \frac{1}{\lambda^{b+\varepsilon}} \\ &= \frac{1}{\lambda^{b+\varepsilon}}(1 + \lambda^{b-a}). \end{aligned}$$

Therefore

$$\frac{\log n_{\lambda r}(B_{X_1 \cup X_2}(x, r))}{\log 1/\lambda} \leq b + \varepsilon + \frac{\log(1 + \lambda^{b-a})}{\log 1/\lambda},$$

so that $\overline{\delta}_{X_1 \cup X_2}(x) \leq b + \varepsilon$, and the thesis follows by the arbitrariness of ε.

(iii) Endow $X \times Y$ with the metric

$$d((x_1, y_1), (x_2, y_2)) := \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\} \tag{3.5}$$

which is by-Lipschitz equivalent to the product metric. Then

$$B_{X\times Y}((x, y), R) = B_X(x, R) \times B_Y(y, R). \tag{3.6}$$

Therefore $\nu_r(B_{X\times Y}((x, y), R)) \geq \nu_r(B_X(x, R))\nu_r(B_Y(y, R))$, and

$$\begin{aligned} \liminf_{\lambda\to 0} \liminf_{r\to 0} \frac{\log \nu_{\lambda r}(B_{X\times Y}((x, y), r))}{\log 1/\lambda} &\geq \liminf_{\lambda\to 0} \liminf_{r\to 0} \frac{\log \nu_{\lambda r}(B_X(x, r))}{\log 1/\lambda} \\ &\quad + \liminf_{\lambda\to 0} \liminf_{r\to 0} \frac{\log \nu_{\lambda r}(B_Y(y, r))}{\log 1/\lambda}. \end{aligned}$$

Moreover $n_r(B_{X\times Y}((x, y), R)) \leq n_r(B_X(x, R))n_r(B_Y(y, R))$, and

$$\begin{aligned} \limsup_{\lambda\to 0} \limsup_{r\to 0} \frac{\log n_{\lambda r}(B_{X\times Y}((x, y), r))}{\log 1/\lambda} &\leq \limsup_{\lambda\to 0} \limsup_{r\to 0} \frac{\log n_{\lambda r}(B_X(x, r))}{\log 1/\lambda} \\ &\quad + \limsup_{\lambda\to 0} \limsup_{r\to 0} \frac{\log n_{\lambda r}(B_Y(y, r))}{\log 1/\lambda}. \end{aligned}$$

□

4. Local dimensions of tangent sets

4.1. **A different formula for tangential dimensions.** There is another notion of dimension naturally associated with the tangent cone. One may indeed take the infimum, resp. supremum, of the lower, resp. upper, box dimension of the tangent balls at a given point. We give below a sufficient condition for them to coincide with the tangential dimensions defined above. However, this equality does not hold in general, as shown in subsection 4.2.

Let X be a metric space, $x \in X$. We shall consider the following.

Assumption 4.1. *There exist constants $c \geq 1$, $a \in (0,1]$ such that, for any $r \leq a$, $\lambda, \mu \leq 1$, $y, z \in B_X(x,r)$,*

$$n(\lambda\mu r, B_X(y,\lambda r)) \leq cn(\lambda\mu r, B_X(z,\lambda r)). \tag{4.1}$$

Let us observe that the previous inequality is trivially satisfied when $\mu \geq 1$.

Let us recall some notions of dimension.

The lower and upper box dimensions of X are

$$\underline{d}(X) = \lim_{R\to\infty} \liminf_{r\to 0} \frac{\log n(r, B_X(x,R))}{\log 1/r},$$
$$\overline{d}(X) = \lim_{R\to\infty} \limsup_{r\to 0} \frac{\log n(r, B_X(x,R))}{\log 1/r},$$

while the (lower and upper) local (box) dimensions of X at a point x are defined as

$$\underline{d}_X(x) = \lim_{R\to 0} \underline{d}(B_X(x,R)) = \lim_{R\to 0} \liminf_{r\to 0} \frac{\log n(r, B_X(x,R))}{\log 1/r},$$
$$\overline{d}_X(x) = \lim_{R\to 0} \overline{d}(B_X(x,R)) = \lim_{R\to 0} \limsup_{r\to 0} \frac{\log n(r, B_X(x,R))}{\log 1/r}.$$

Remark. (i) For the box dimensions to be non-trivial, (the completion of) X has to be proper; for the local box dimensions at x to be non-trivial, x needs to have a compact (totally bounded) neighborhood.

(ii) We obtain the same definition if we replace n with ν or with $\overline{n}$, and/or $B_X(x,r)$ with $\overline{B}_X(x,r)$ or with $\overline{B_X(x,r)}$. The proof is the same as that of Proposition 3.1.

Now we set

$$g(t,h) = \log n(e^{-(t+h)}, B_X(x,e^{-t})). \tag{4.2}$$

Clearly $h \mapsto g(t,h)$ is non-decreasing for any t. The tangential dimensions can be rewritten as

$$\underline{\delta}_X(x) = \liminf_{h\to+\infty} \liminf_{t\to+\infty} \frac{g(t,h)}{h},$$
$$\overline{\delta}_X(x) = \limsup_{h\to+\infty} \limsup_{t\to+\infty} \frac{g(t,h)}{h}.$$

The local dimensions can be rewritten as

$$\underline{d}_X(x) = \lim_{t\to+\infty} \liminf_{h\to+\infty} \frac{g(t,h)}{h},$$
$$\overline{d}_X(x) = \lim_{t\to+\infty} \limsup_{h\to+\infty} \frac{g(t,h)}{h}.$$

We define the *coboundary* of g as the three-variable function

$$dg(t,h,k) = g(t,h+k) - g(t+h,k) - g(t,h),$$

and note that g is a *cocycle*, namely $dg = 0$, if and only if $g(t,h) = g(0,t+h) - g(0,t)$, namely if it is a coboundary where, given $t \to f(t)$, we set $df(t,h) = f(t+h) - f(t)$.

We shall show that our assumption implies a bound on dg.

Lemma 4.2. *The following inequality holds:*

$$n(\lambda\mu r, B_X(x,r)) \leq n(\lambda r, B_X(x,r)) \sup_{y\in B_X(x,r)} n(\lambda\mu r, B_X(y,\lambda r)).$$

Proof. Let us note that we may realize a covering of $B_X(x,r)$ with balls of radius $\lambda\mu r$ as follows: first choose an optimal covering of $B_X(x,r)$ with balls of radius λr, and then cover any covering ball optimally with balls of radius $\lambda\mu r$. The thesis follows. ☐

Let us recall that the function $\nu(r, B_X(x,R))$, denotes the maximum number of disjoint open balls of X of radius r centered in the open ball of center x and radius R of X.

Lemma 4.3. *The following inequality holds:*

$$\nu(\lambda\mu r, B_X(x,r)) \geq \nu(\lambda r, B_X(x,r)) \inf_{y\in B_X(x,r)} \nu(\lambda\mu r, B_X(y,\lambda r)).$$

Proof. Indeed we may find disjoint open balls of X of radius $\lambda\mu r$ centered in $B_X(x,r)$ as follows: first find a maximal set of disjoint open balls of X of radius λr centered in $B_X(x,r)$, and then, for any such ball, find a maximal set of disjoint open balls of X of radius $\lambda\mu r$ centered in it. This implies the thesis. ☐

Proposition 4.4. *Assumption 4.1 and condition* (2.1) *for* (X, x) *imply that* dg *is bounded, for* $t > t_0$*,* $h, k > 0$*.*

Proof. Let us observe that it is enough to find a bound for h, k sufficiently large. By the assumption and Lemma 4.2, for $r \le a$,

$$\begin{aligned} n(\lambda\mu r, B_X(x,r)) &\le n(\lambda r, B_X(x,r)) \sup_{y\in B_X(x,r)} n(\lambda\mu r, B_X(y,\lambda r)) \\ &\le cn(\lambda r, B_X(x,r))n(\lambda\mu r, B_X(x,\lambda r)). \end{aligned}$$

Therefore, if we set $r = e^{-t}$, $\lambda = e^{-h}$, $\mu = e^{-k}$, we get, for $t \ge \log 1/a$,

$$dg(t,h,k) = g(t,h+k) - g(t+h,k) - g(t,h) \le \log c. \tag{4.3}$$

Let us now find a bound from below. By Lemma 4.3, the inequalities (3.1), and assumption 4.1, we get, for $r \le a$,

$$\begin{aligned} n(\lambda\mu r, B_X(x,r)) &\ge \nu(\lambda\mu r, B_X(x,r)) \\ &\ge \nu(\lambda r, B_X(x,r)) \inf_{y\in B_X(x,r)} \nu(\lambda\mu r, B_X(y,\lambda r)) \\ &\ge n(2\lambda r, B_X(x,r)) \inf_{y\in B_X(x,r)} n(2\lambda\mu r, B_X(y,\lambda r)) \\ &\ge \frac{1}{c} n(2\lambda r, B_X(x,r))n(2\lambda\mu r, B_X(x,\lambda r)). \end{aligned}$$

As a consequence, for $t \ge \log 1/a$,

$$g(t,h+k) - g(t,h-\log 2) - g(t+h,k-\log 2) \ge -\log c,$$

which implies

$$dg(t,h,k) \ge -\log c - (g(t,h) - g(t,h-\log 2)) - (g(t+h,k) - g(t+h,k-\log 2)).$$

The result follows if we show that $g(t,h+\log 2) - g(t,h)$ is bounded from above. Indeed, by the upper bound (4.3),

$$\begin{aligned} g(t,h+\log 2) - g(t,h) &= dg(t,h,\log 2) + g(t+h,\log 2) \\ &\le \log c + g(t+h,\log 2). \end{aligned}$$

Since $L = \limsup_{t\to+\infty} g(t,\log 2) < +\infty$, there exists $t_0 > \log 1/a$ such that, for $t > t_0$, $g(t,\log 2) \le 2L$. ∎

Proposition 4.5. *Let us assume property 4.1.*

(i) If condition (2.1) holds, the $\liminf_{\lambda\to 0}$*, resp.* $\limsup_{\lambda\to 0}$*, in the definition of* $\underline{\delta}$*, resp.* $\overline{\delta}$*, are indeed limits:*

$$\underline{\delta}_X(x) = \lim_{\lambda\to 0} \liminf_{r\to 0} \frac{\log n(\lambda r, B(x,r))}{\log 1/\lambda},$$
$$\overline{\delta}_X(x) = \lim_{\lambda\to 0} \limsup_{r\to 0} \frac{\log n(\lambda r, B(x,r))}{\log 1/\lambda}.$$

(ii) If condition (2.1) holds, the following inequalities hold:

$$\underline{\delta}_X(x) \le \underline{d}_X(x) \le \overline{d}_X(x) \le \overline{\delta}_X(x).$$

(iii) Condition (2.1) is equivalent to the finiteness of $\overline{\delta}_X(x)$*.*

Proof. All the statements follow directly by Proposition 5.2. □

Lemma 4.6. *Let* $\lambda_n \to 0$ *be a sequence such that* $(\frac{1}{\lambda_n}X, x) \underset{pGH}{\longrightarrow} (T, x)$*. Then,*

$$\underline{d}(\overline{B}_T(x,1)) = \liminf_{h\to\infty} \liminf_n \frac{g(t_n,h)}{h} = \liminf_{h\to\infty} \limsup_n \frac{g(t_n,h)}{h}, \tag{4.4}$$

$$\overline{d}(\overline{B}_T(x,1)) = \limsup_{h\to\infty} \liminf_n \frac{g(t_n,h)}{h} = \limsup_{h\to\infty} \limsup_n \frac{g(t_n,h)}{h}, \tag{4.5}$$

where we posed $t_n = -\log\lambda_n$*.*

Proof. In the following we shall omit the reference to the point x. By definition, setting $h = \log 1/r$,

$$\begin{aligned}\limsup_n \frac{g(t_n,h)}{h} &= \limsup_n \frac{\log n(\lambda_n r, \overline{B}_X(\lambda_n))}{\log 1/r}\\ &= \limsup_n \frac{\log n(r, \overline{B}_{1/\lambda_n X}(1))}{\log 1/r}\\ &\le \frac{\log n(r, \overline{B}_T(1))}{\log 1/r},\end{aligned}$$

where we used the upper semicontinuity in Proposition 3.2 (ii). Analogously,

$$\begin{aligned}\liminf_n \frac{\log \overline{n}(\lambda_n r, \overline{B_X(\lambda_n)})}{\log 1/r} &= \liminf_n \frac{\log \overline{n}(r, \overline{B_{1/\lambda_n X}(1)})}{\log 1/r}\\ &\ge \frac{\log \overline{n}(r, \overline{B_T(1)})}{\log 1/r},\end{aligned}$$

namely

$$\begin{aligned}\frac{\log \overline{n}(r, B_T(1))}{\log 1/r} &\leq \frac{\log \overline{n}(r, \overline{B_T(1)})}{\log 1/r} \\ &\leq \liminf_n \frac{\log \overline{n}(\lambda_n r, \overline{B_X(\lambda_n)})}{\log 1/r} \\ &\leq \limsup_n \frac{\log n(\lambda_n r, \overline{B}_X(\lambda_n))}{\log 1/r} \\ &\leq \frac{\log n(r, \overline{B}_T(1))}{\log 1/r}.\end{aligned}$$

Recalling Proposition 3.1 and Remark 4.1 (ii), and taking the $\liminf$ for $r \to 0$ we get the equalities (4.4), taking the $\limsup$ for $r \to 0$ we get the equalities (4.5). □

Theorem 4.7. *Under the Assumption 4.1 and condition (2.1)*

$$\underline{\delta}_X(x) = \inf_{T \in \mathcal{T}_x X} \underline{d}(T) = \inf_{T \in \mathcal{T}_x X} \overline{d}(T), \tag{4.6}$$

$$\overline{\delta}_X(x) = \sup_{T \in \mathcal{T}_x X} \underline{d}(T) = \sup_{T \in \mathcal{T}_x X} \overline{d}(T). \tag{4.7}$$

Proof. We only prove (4.7), the proof of (4.6) being analogous. Let us observe that the property satisfied by the sequence $\overline{t}_n \to \infty$ described in Proposition 5.5 remains valid for any subsequence. We may therefore assume that $\overline{t}_n$ produces a tangent set, namely $e^{\overline{t}_n} X$ converges to a tangent set T in the pointed Gromov-Hausdorff topology. Then, by Lemma 4.6 and Proposition 5.5, for any κ there exists a tangent set T such that $\overline{\delta}_X(x) - \frac{2S}{\kappa} \leq \overline{d}(\overline{B}_T(1)) \leq \overline{\delta}_X(x)$, hence

$$\overline{\delta}_X(x) = \sup_{T \in \mathcal{T}_x X} \overline{d}(\overline{B}_T(1)).$$

Since $\mathcal{T}_x X$ is globally dilation invariant, and the box dimensions are dilation invariant, for any tangent set T and any $r > 0$ there exists a tangent set S for which

$$\overline{d}(\overline{B}_T(r)) = \overline{d}(r\overline{B}_S(1))) = \overline{d}(\overline{B}_S(1)),$$

namely

$$\sup_{T \in \mathcal{T}_x X} \overline{d}(\overline{B}_T(1)) = \sup_{\substack{T \in \mathcal{T}_x X \\ r > 0}} \overline{d}(\overline{B}_T(r)) = \sup_{T \in \mathcal{T}_x X} \overline{d}(T).$$

□

4.2. **A counterexample.** Now we show that the equality shown above under hypothesis 4.1 does not hold in general. In the example below we construct a subset of $\mathbb{R}^3$ for which any tangent set at a given point is zero-dimensional, but $\overline{\delta}$ is positive.

First set $a_n^k := \mathrm{e}^{-\left(\frac{(n+k)(n+k+1)}{2}+k\right)^2}$, for k, $n \in \mathbb{N}$.

Let now $S^2 := \{x \in \mathbb{R}^3 : \|x\| = 1\}$, and choose, for any $k \in \mathbb{N}$, $S_k \subset S^2$ such that

- The diameter of S_k is $1/k^2$,
- $\#S_k = k^2$,
- $d(v, w) \geq \frac{1}{k^3}$, $v, w \in S_k$, $v \neq w$,
- $\min\{d(v, w) : v \in S_k, w \in S_h\} \geq \frac{1}{k^3}$, $h \geq k$,
- $\lim_{k\to\infty} S_k =: S_\infty = \{v_\infty\} \subset S^2$ in the Hausdorff topology.

Set, for any $k \in \mathbb{N}$, $A_k := \{a_n^k v : v \in S_k, n \in \mathbb{N}\}$, and $F := \overline{\cup_{k=1}^\infty A_k} \subset \mathbb{R}^3$.

Lemma 4.8. *The tangent cone of F at 0 consists, up to dilations, of the set $\{0\}$ and of the sets $S_k \cup \{0\}$, with $k \in \mathbb{N} \cup \{\infty\}$.*

Proof. If $(\lambda_n F, 0) \underset{\mathrm{pGH}}{\longrightarrow} (T, 0)$, then the tangent set T does not consist of the sole $\{0\}$ if and only if, for suitable sequences $n(p)$, $k(p) \in \mathbb{N}$, $v(p) \in S_{k(p)}$, $\lambda_p a_{n(p)}^{k(p)} v(p)$ converges, when $p \to \infty$, and $\lambda_p a_{n(p)}^{k(p)} \to c \in (0, \infty)$.

Assume $\{k(p)\}$ is bounded. Since $\lambda_p a_{n(p)}^{k(p)} v(p)$ converges, then $k(p)$ has to be eventually equal to some k_0, namely we may replace λ_p with a subsequence of $c\left(a_n^{k_0}\right)^{-1}$. This implies that $T \supseteq cS_k$.

Let us observe that two infinitesimal subsequences c_n, c'_n contained in $\{a_n^k : k, n \in \mathbb{N}\}$ such that $\frac{c_n}{c'_n} \to \chi \neq 0$ eventually coincide. From this it is not difficult to derive that all limit points in $(\lambda_n F, 0)$ belong to cS_k.

If $\{k(p)\}$ is not bounded, it has to diverge, namely $T \supseteq cS_\infty$. Reasoning as before, one gets $T = cS_\infty$. □

Proposition 4.9. *Let F be as above. Then*

$$\overline{\delta}_F(0) > \sup_{T \in \mathcal{T}_0 F} \overline{d}(T) = 0.$$

Proof. By Lemma 4.8 we get $\sup_{T\in\mathcal{T}_0 F} \overline{d}(T) = 0$. Now let $k \in \mathbb{N}$, and let $\{\lambda_n\} \subset (0, \infty)$ be an increasing diverging sequence s.t. $X := \lim_{n\to\infty} \lambda_n F$ exists and $\overline{B}_X(0, 1)$ consists of $k + 1$ points, all belonging to $\{tv : v \in S_k, t \geq 0\}$. As $n_{1/k^2}(\overline{B}_X(0, 1)) = k + 1$, we obtain

$$\frac{\log n_{1/k^2}(\overline{B}_X(0, 1))}{\log k^2} \geq \frac{1}{2},$$

so that

$$\overline{\delta}_F(0) = \limsup_{r\to 0} \sup_{T\in\mathcal{T}_0 F} \frac{\log n_r(\overline{B}_T(0,1))}{\log 1/r} \geq \frac{1}{2}.$$

□

5. Appendix

Here we collect some results on the two-variable functions $g(t,h)$. Throughout this section we assume that g is non-decreasing in the h variable and that, for a suitable constant t_0,

$$S = \sup_{\substack{t>t_0 \\ h,g>0}} |dg(t,h,k)| < \infty,$$

where $dg(t,h,k) = g(t,h+k) - g(t+h,k) - g(t,h)$.

Lemma 5.1. *Given $t > t_0$, $h_1, \dots h_n > 0$, we have*

$$\left| g(t, \sum_{i=1}^{n} h_i) - \sum_{k=1}^{n} g(t + \sum_{i=1}^{k-1} h_i, h_k) \right| \leq (n-1)S. \tag{5.1}$$

Proof. A straightforward computation gives

$$g(t, \sum_{i=1}^{n} h_i) = \sum_{k=1}^{n} g(t + \sum_{i=1}^{k-1} h_i, h_k) + \sum_{k=1}^{n-1} dg(t + \sum_{i=1}^{k-1} h_i, h_k, \sum_{i=k+1}^{n} h_i). \tag{5.2}$$

The thesis follows. □

Proposition 5.2.
(i) The quantities

$$\limsup_{t\to\infty} \frac{g(t,h)}{h}, \quad \liminf_{t\to\infty} \frac{g(t,h)}{h},$$

have a limit when $h \to \infty$.
(ii) The following inequalities hold:

$$\lim_{t\to\infty} \limsup_{h\to\infty} \frac{g(t,h)}{h} \leq \lim_{h\to\infty} \limsup_{t\to\infty} \frac{g(t,h)}{h},$$
$$\lim_{t\to\infty} \liminf_{h\to\infty} \frac{g(t,h)}{h} \geq \lim_{h\to\infty} \liminf_{t\to\infty} \frac{g(t,h)}{h}.$$

iii The quantity $\lim_{h\to\infty} \limsup_{t\to\infty} \frac{g(t,h)}{h}$ is infinite if and only if the quantity $\limsup_{t\to\infty} g(t,h)$ is infinite for one (and in fact for any) $h > 0$.

PROOF. (i). Let us set $\underline{g}(h) = \liminf_{t\to\infty} g(t,h)$. Then, by eq. (5.1), we get

$$\frac{\underline{g}(nh)}{nh} \geq \frac{\underline{g}(h)}{h} - \frac{S}{h}. \tag{5.3}$$

Therefore,

$$\frac{\underline{g}(s)}{s} \geq \frac{\underline{g}\left(\lfloor \frac{s}{r} \rfloor r\right)}{s} \geq \lfloor \frac{s}{r} \rfloor \frac{r}{s} \left(\frac{\underline{g}(r)}{r} - \frac{S}{r} \right).$$

Taking the $\liminf_{s\to\infty}$, we get

$$\liminf_{s\to\infty} \frac{\underline{g}(s)}{s} \geq \frac{\underline{g}(r)}{r} - \frac{S}{r}. \tag{5.4}$$

Then we take the $\limsup_{r\to\infty}$, and obtain

$$\liminf_{s\to\infty} \frac{\underline{g}(s)}{s} \geq \limsup_{r\to\infty} \frac{\underline{g}(r)}{r},$$

which proves the existence of $\lim_{h\to\infty} \liminf_{t\to\infty} \frac{g(t,h)}{h}$. The existence of the other limit is proved analogously.

(ii). Since g is non-decreasing in h, for any $\kappa > 0$ we have

$$\liminf_{h\to\infty} \frac{g(t,h)}{h} = \liminf_{n\in\mathbb{N}} \frac{g(t,n\kappa)}{n\kappa}.$$

Then, again by eq. (5.1), we get, for $t > t_0$,

$$\frac{g(t,n\kappa)}{n\kappa} \geq \frac{1}{\kappa} \left(\frac{1}{n} \sum_{k=1}^{n} g(t+(k-1)\kappa, \kappa) - S. \right) \tag{5.5}$$

Taking the $\liminf$ on $n \in \mathbb{N}$ we get

$$\liminf_{n\in\mathbb{N}} \frac{g(t,n\kappa)}{n\kappa} \geq \liminf_{t\to\infty} \frac{g(t,\kappa)}{\kappa} - \frac{S}{\kappa},$$

from which

$$\lim_{t\to\infty} \liminf_{h\to\infty} \frac{g(t,h)}{h} \geq \lim_{h\to\infty} \liminf_{t\to\infty} \frac{g(t,h)}{h}$$

follows. The other inequality is proved in the same way.

(iii). Sufficiency is obvious. Conversely, set $\overline{g}(h) = \limsup_{t\to\infty} g(t,h)$. Then, by eq. (5.1), and in analogy with (5.3), we get

$$\frac{\overline{g}(nh)}{nh} \leq \frac{\overline{g}(h)}{h} + \frac{S}{h}, \tag{5.6}$$

hence, taking the $\lim_{n\to\infty}$,

$$\overline{g}(h) \geq h \lim_{h'\to\infty} \limsup_{t\to\infty} \frac{g(t,h')}{h'} - S,$$

from which the thesis follows. □

In the following κ is a given positive number, and we set $p(t,h) = g(t,h)/h$.

Lemma 5.3. *Let us define*

$$\begin{aligned} V_h^d &= \{t > 0 : p(t,h) > d\}, \\ V^d &= \{h \in \kappa\mathbb{N} : \sup V_h^d = +\infty\}, \\ V &= \{d \in \mathbb{R} : \sup V^d = +\infty\}. \end{aligned}$$

Then,

$$\sup V = \limsup_{h\in\kappa\mathbb{N}} \limsup_{t\to+\infty} p(t,h).$$

Proof. Let us observe that if $L = \limsup_{x\to\infty} f(x)$, we have

$$L = \sup\{T \in \mathbb{R} : \{x \in \mathbb{R} : f(x) > T\} \text{ is unbounded}\}.$$

Then, setting

$$U^d = \{h \in \kappa\mathbb{N} : \limsup_{t\to\infty} p(t,h) > d\}, \qquad U = \{d : \sup U^d = +\infty\},$$

we have

$$\limsup_{h\in\kappa\mathbb{N}} \limsup_{t\to+\infty} p(t,h) = \sup U,$$

and

$$\limsup_{t\to+\infty} p(t,h) = \sup\{d : \sup V_h^d = +\infty\},$$

hence

$$\limsup_{t\to+\infty} p(t,h) > d \Rightarrow \sup V_h^d = +\infty \Rightarrow \limsup_{t\to+\infty} p(t,h) \geq d,$$

which implies

$$U^d \subseteq \{h \in \kappa\mathbb{N} : \sup V_h^d = +\infty\} \subseteq \bigcap_{\varepsilon>0} U^{d-\varepsilon}.$$

Finally,

$$\sup U^d = +\infty \Rightarrow \sup V^d = +\infty \Rightarrow \sup U^{d-\varepsilon} = +\infty,\ \forall\varepsilon > 0,$$

from which the thesis follows. □

Lemma 5.4. *Let us define*

$$\begin{aligned} \tilde V_h^d &= \{t > 0 : p(t,j) > d, j \in \kappa\mathbb{N}, j \leq h\}, \\ \tilde V^d &= \{h \in \kappa\mathbb{N} : \sup \tilde V_h^d = +\infty\}, \\ \tilde V &= \{d \in \mathbb{R} : \sup \tilde V^d = +\infty\}. \end{aligned}$$

Then $0 \leq \sup V - \sup \tilde V \leq \frac{2S}{\kappa}$.

PROOF. Since $\tilde{V}_h^d \subset V_h^d$, we have $\sup \tilde{V} \leq \sup V$. Let us assume that $\sup \tilde{V} < d_1 < d_2 < \sup V$, for suitable constants d_1, d_2. Now $d_1 \notin \tilde{V}$, hence there exists $\overline{h} \in \kappa\mathbb{N}$ such that $\sup \tilde{V}_{\overline{h}}^{d_1} < +\infty$, namely

$$\exists \overline{t} : \forall t > \overline{t}, \exists j_t \in \kappa\mathbb{N}, j_t \leq \overline{h} : p(t, j_t) \leq d_1. \tag{5.7}$$

Also, $d_2 \in V$, hence we may find $\tilde{h} \in \kappa\mathbb{N}$ such that $\sup V_{\tilde{h}}^{d_2} = +\infty$ and so large that

$$\tilde{h} > \frac{2d_1}{d_2 - d_1}\overline{h}.$$

Therefore we may find $t_0 > \overline{t}$ such that $p(t_0, \tilde{h}) > d_2$.
By equation (5.7), we can now construct inductively a sequence $j_i \in \kappa\mathbb{N}$, $j_i \leq \overline{h}$, such that, setting

$$t_k = t_0 + \sum_{i=1}^{k} j_i,$$

we get $p(t_k, j_{k+1}) \leq d_1$. Since $t_n \geq t_0 + n\kappa$, there exists $\overline{n} \in \mathbb{N}$ such that

$$t_{\overline{n}} - \overline{h} \leq t_{\overline{n}-1} \leq t_0 + \tilde{h} < t_{\overline{n}}.$$

Now, by equation (5.1), one gets

$$\begin{aligned} d_2 < p(t_0, \tilde{h}) &\leq \frac{g(t_0, \sum_{i=1}^{\overline{n}} j_i)}{\tilde{h}} \\ &\leq \frac{1}{\tilde{h}} \sum_{k=1}^{\overline{n}} j_k p(t_{k-1}, j_k) + \frac{\overline{n}-1}{\tilde{h}} S \\ &\leq \frac{\sum_{i=1}^{\overline{n}} j_i}{\tilde{h}} d_1 + \frac{S}{\kappa} \\ &\leq \left(1 + \frac{\overline{h}}{\tilde{h}}\right) d_1 + \frac{S}{\kappa} \leq \frac{d_2 + d_1}{2} + \frac{S}{\kappa}. \end{aligned}$$

The thesis follows. □

Proposition 5.5. *For any sequence $t_n \to \infty$,*

$$\limsup_{h \to +\infty} \limsup_{n \in \mathbb{N}} \frac{g(t_n, h)}{h} \leq \limsup_{h \to +\infty} \limsup_{t \to +\infty} \frac{g(t, h)}{h}. \tag{5.8}$$

Moreover, for any $\kappa > 0$, there exists a sequence $\{\overline{t}_n\} \to \infty$ for which

$$\limsup_{h \to +\infty} \limsup_{t \to +\infty} \frac{g(t, h)}{h} \leq \liminf_{h \to +\infty} \liminf_{n \in \mathbb{N}} \frac{g(\overline{t}_n, h)}{h} + \frac{2S}{\kappa}. \tag{5.9}$$

PROOF. The first inequality is obvious. We shall prove the second.
For any given $\kappa > 0$, let $d < \sup \tilde{V}$. Then $\sup \tilde{V}^d = +\infty$, i.e. there is $\{h_n\} \subset \kappa\mathbb{N}$, $h_n \to \infty$, such that $\sup \tilde{V}^d_{h_n} = +\infty$. It is not restrictive to assume $h_n > n$. Correspondingly we find sequences $t_{nk} \to +\infty$ for $k \to \infty$ such that

$$p(t_{nk}, j) > d, \quad j \leq h_n,\ j \in \kappa\mathbb{N}.$$

Again, it is not restrictive to assume $t_{nk} > k$. Now we make explicit the dependence on d, setting h_{np} for the sequence h_n associated to $d = \sup \tilde{V} - 1/p$, and t_{nkp} for the sequence t_{nk} corresponding to the same d. We have

$$p(t_{nkp}, j) > \sup \tilde{V} - \frac{1}{p}, \quad j \leq h_{np},\ j \in \kappa\mathbb{N}.$$

Since we assumed $h_{np} > n$, this implies

$$p(t_{nkp}, j) > \sup \tilde{V} - \frac{1}{p}, \quad j \leq n,\ j \in \kappa\mathbb{N}.$$

Setting $\overline{t}_n = t_{nnn}$, we have $\overline{t}_n > n$ hence $\overline{t}_n \to \infty$, and

$$\liminf_{n\in\mathbb{N}} p(\overline{t}_n, h) \geq \sup \tilde{V}, \quad \forall h \in \kappa\mathbb{N}.$$

Then, by the proof of Proposition 5.2 (ii), and Lemma 5.4, we have, for any $h \in \kappa\mathbb{N}$,

$$\limsup_{h\to+\infty} \limsup_{t\to+\infty} p(t, h) = \sup V \leq \sup \tilde{V} + \frac{2S}{\kappa} \leq \liminf_{n\in\mathbb{N}} p(\overline{t}_n, h) + \frac{2S}{\kappa}.$$

Finally we observe that the function $\underline{g}(h)$ defined as $\underline{g}(h) = \liminf_n g(\overline{t}_n, h)$ is increasing, therefore, if $\lfloor\cdot\rfloor$ denotes the lower integer part, we get $\underline{g}(h) \geq \underline{g}(\lfloor \frac{h}{\kappa} \rfloor \kappa)$, from which

$$\liminf_{h\to\infty} \frac{\underline{g}(h)}{h} \geq \liminf_{h\to\infty} \frac{\underline{g}\left(\lfloor \frac{h}{\kappa} \rfloor \kappa\right)}{\lfloor \frac{h}{\kappa} \rfloor \kappa} \frac{\lfloor \frac{h}{\kappa} \rfloor \kappa}{h} = \liminf_{h\in\kappa\mathbb{N}} \frac{\underline{g}(h)}{h},$$

and the thesis follows. $\square$

References

[1] T. Bedford, A. M. Fisher. *On the magnification of Cantor sets and their limit models.* Monatsh. Math. **121**, (1996) 11-40.

[2] G. Beer. *Topologies on closed and closed convex sets.* Kluwer Acad. Publ., Dordrecht, 1993.

[3] M.R. Bridson, A. Haefliger. *Metric spaces of non-positive curvature.* Springer-Verlag, Berlin, 1999.

[4] A. Connes. *Non Commutative Geometry.* Academic Press, 1994.

[5] M. Gromov. *Metric structures for Riemannian and non-Riemannian spaces.* Birkhäuser, Boston, MA, 1999.

[6] D. Guido, T. Isola. *An asymptotic dimension for metric spaces, and the* 0*-th Novikov-Shubin invariant*. Pacific J. Math. **204**, (2002) 43-59.

[7] D. Guido, T. Isola. *Dimensions and singular traces for spectral triples, with applications to fractals*. Journ. Funct. Analysis **203**, (2003) 362-400.

[8] D. Guido, T. Isola. *Dimensions and spectral triples for fractals in* $\mathbb{R}^N$. Preprint math.OA/0404295.

[9] D. Guido, T. Isola. *Tangential dimensions II. Measures*. Preprint math.FA/0405174.

[10] B. Hambly. *Brownian motion on a homogeneous random fractal*. Prob. Th. Rel. Fields, **94**, (1992), 1-38.

Dipartimento di Matematica, Università di Roma "Tor Vergata", I–00133 Roma, Italy.

E-mail address: guido@mat.uniroma2.it, isola@mat.uniroma2.it