Dimensions and singular traces for spectral triples, with applications to fractals

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November 12, 2002

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Abstract

Given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, the functionals on $\mathcal{A}$ of the form $a \mapsto \tau_\omega(a|D|^{-\alpha})$ are studied, where τ_ω is a singular trace, and ω is a generalised limit. When τ_ω is the Dixmier trace, the unique exponent d giving rise possibly to a non-trivial functional is called Hausdorff dimension, and the corresponding functional the (d-dimensional) Hausdorff functional.

It is shown that the Hausdorff dimension d coincides with the abscissa of convergence of the zeta function of $|D|^{-1}$, and that the set of α's for which there exists a singular trace τ_ω giving rise to a non-trivial functional is an interval containing d. Moreover, the endpoints of such traceability interval have a dimensional interpretation. The corresponding functionals are called Hausdorff-Besicovitch functionals.

These definitions are tested on fractals in $\mathbb{R}$, by computing the mentioned quantities and showing in many cases their correspondence with classical objects. In particular, for self-similar fractals the traceability interval consists only of the Hausdorff dimension, and the corresponding Hausdorff-Besicovitch functional gives rise to the Hausdorff measure. More generally, for any limit fractal, the described functionals do not depend on the generalized limit ω.

*Supported in part by GNAMPA and MIUR

0 Introduction.

The concept of spectral triple, introduced by Alain Connes as a framework for noncommutative geometry [6], is wide enough to describe non smooth, or even fractal spaces.

While further axioms can be, and have been, added to describe the noncommutative analogues of Riemannian (or spin) manifolds (see [16] and references therein), spectral triples have been attached to fractals in $\mathbb{R}$ and to quasi-circles, and, using the Hausdorff dimension as the exponent for the infinitesimal length element, and the Dixmier trace, Connes, resp. Connes-Sullivan proved that one obtains the Hausdorff measure for Cantor sets, resp. quasi-circles (cf. [6], IV.3).

Our aim in this paper is twofold. On the one hand we intend to show how spectral triples can provide a framework for noncommutative Hausdorff-Besicovitch theory. On the other hand we investigate how the noncommutative quantities we introduce give back classical known quantities, or even produce new ones, when applied to spectral triples associated to fractals.

Let us recall that a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ consists of a *-algebra $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$ and of a selfadjoint operator D with compact resolvent, the Dirac operator, such that $[D, a]$ is bounded for any $a \in \mathcal{A}$.

Concerning the first point, we define, as suggested by the examples of Connes, the α-dimensional Hausdorff functional as the functional $a \mapsto \mathrm{Tr}_\omega(a|D|^{-\alpha})$, where Tr_ω denotes the Dixmier trace, namely the singular trace summing logarithmic divergences.

Once Hausdorff functionals are defined, the Hausdorff dimension of a spectral triple is easily defined (cf. Definition 2.6). We show moreover that such dimension is equal to the abscissa of convergence of the zeta function of $|D|^{-1}$, $\zeta_{|D|^{-1}}(s) = \mathrm{tr}\,|D|^{-s}$. This result turns out to be a useful formula for computing the Hausdorff dimension, and also opens the way to general Tauberian formulas for singular traces [24], see also [5] for related results.

However our main goal here is to enlarge the class of geometric measures, in the same way as Besicovitch measures generalize Hausdorff measures.

Let us recall that the Hausdorff-Besicovitch measures replace the power law for the volume of the balls of the Hausdorff measures with a general infinitesimal behaviour.

Following the idea that the powerlike asymptotics for $\mu_n(|D|)$, which give rise to non trivial logarithmic Dixmier traces, are the noncommutative counterpart of the powerlike asymptotics of the volume of the balls with small radius, which corresponds to some nontrivial Hausdorff measure, it is clear that, in order to define Hausdorff-Besicovitch functionals, we have to pass from the logarithmic singular trace to a general singular trace.

The trace theorem of Connes ([6], IV.2) shows that the logarithmic Dixmier trace produces a trace functional on the C*-algebra $\overline{\mathcal{A}}$, corresponding to the Riemann volume on manifolds. The proof of this theorem given in [8] however, works for any positive trace functional whose domain contains the principal ideal generated by $|D|^{-1}$, which is singular, namely vanishes on finite rank operators, and for which the Hölder inequality holds.

We show here that Hölder inequality holds for any singular trace, possibly up to a constant (cf. Appendix), namely any such trace produces a Hausdorff-Besicovitch functional on $\overline{\mathcal{A}}$.

Then a new question arises: given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, characterize the set $\{\alpha > 0 : |D|^{-\alpha}$ is singularly traceable $\}$, namely describe when nontrivial Hausdorff-Besicovitch functionals can be produced. We give a complete answer to this question, namely prove that such set is a relatively closed interval in $(0, \infty)$, whose endpoints $\underline{\delta}$ and $\overline{\delta}$ coincide with the Matuszewska indices of the eigenvalue function of $|D|^{-1}$, and satisfy $\underline{\delta} \leq d \leq \overline{\delta}$, where d denotes the Hausdorff dimension of the spectral triple.

This means in particular that when d is finite nonzero, it gives rise to a nontrivial Hausdorff-Besicovitch functional (see Theorem 2.4). Besides, when the traceability exponent is unique, it has to coincide with the Hausdorff dimension. We remark that the singular trace associated with d is not necessarily logarithmic Dixmier, indeed $|D|^{-d}$ may also be trace class. The existence of a nontrivial Hausdorff-Besicovitch functional on $|D|^{-d}$ therefore depends essentially on the fact that all singular traces are allowed, not only those vanishing on $\mathcal{L}^1$ (cf. [1]).

The quantities $\underline{\delta}$, $\overline{\delta}$ mentioned above exhibit some dimensional behaviour (cf. Theorem 2.10), therefore we shall call them minimal, resp. maximal, dimension of the spectral triple. In order to understand if these quantities, which have been introduced in a purely noncommutative fashion, have a commutative counterpart, we need to pass to the second part of this note, namely to test our definitions on some fractal sets.

In this paper we confine our attention to fractals in $\mathbb{R}$, namely to totally disconnected compact subsets of $\mathbb{R}$ with no isolated points. Even though fractals in $\mathbb{R}$ are not interesting from the point of view of fractal diffusions, they constitute a quite general class for our purposes, allowing very general situations from the point of view of Hausdorff-Besicovitch theory. Indeed this class, or better the class of their complements, the so called fractal strings, constituted the first playground of the analysis of Lapidus and collaborators [29].

The study of fractals in $\mathbb{R}^n$ from the point of view of noncommutative geometry will be the object of a forthcoming paper [21].

A simple spectral triple, which we call the "lacunary" spectral triple, can be associated to fractals in $\mathbb{R}$, following the analysis of Connes in [6], IV.3.ε. We show here some measure theoretic properties of such a triple. In particular, for any fractal in $\mathbb{R}$, its upper box dimension coincides with the Hausdorff dimension d of its lacunary spectral triple. When the fractal is d-Minkowski measurable, a result in [30] implies that the singular traceability exponent is unique and equal to D, and the corresponding functional is the Hausdorff functional. By making use of recent results of He and Lapidus [25], we also prove that for h-Minkowski measurable fractals the singular traceability exponent is unique, the corresponding functional being not necessarily Hausdorff.

However, the lacunary spectral triple does not reconstruct the original metric on the fractal unless the fractal has zero Lebesgue measure. Following an idea of Connes [7], we propose here a new spectral triple for a wide class of fractals in $\mathbb{R}$, which we call limit fractals, and can be seen as a subclass of the so called random fractals [31]. Such a triple has the advantage of retaining all the measure theoretic properties of the lacunary spectral triple for limit fractals, and moreover reconstructs the original metric without any further assumption. This “complete” spectral triple can be described as a direct sum of the lacunary spectral triple above, and a “filled” spectral triple, which we proposed in [20] as a spectral triple for limit fractals in $\mathbb{R}^n$. Indeed, all properties of limit fractals we prove here for the complete spectral triple are valid for the direct summand triples.

The simplest case of a limit fractal is a self-similar fractal. We show that for any self-similar fractal, the singular traceability exponent is unique, and the associated Hausdorff-Besicovitch functional is indeed the Hausdorff functional, and coincides (up to a constant) with the Hausdorff measure. The uniqueness result above in the case of the lacunary triple is not implied by the analogous result in the Minkowski measurable case, since not all self-similar fractals are Minkowski measurable (cf. [14]). The fact that the Hausdorff functional for the lacunary triple reconstructs the Hausdorff measure has been shown in [6] IV.3.ε for Cantor-like fractals, and is proved here in the general case (Theorem 4.15).

Then we show that, for limit fractals, the value of the singular trace on the elements $f|D|^{-\alpha}$, f being a continuous function, does not depend on the generalized limit procedure (measurability in the sense of Connes), namely the Hausdorff-Besicovitch functionals are well defined. We remark, in passing, that an analogous measurability result has been recently proved by Kigami and Lapidus [28]. They consider some class of self-similar fractals, for which a Laplacian on the fractal can be constructed as a generator of a Dirichlet form, and prove that the functional $f \to Tr_\omega(f\Delta^{-\alpha})$, where $\alpha > 0$ is related to the self-similar dimension, does not depend on ω.

Returning to this paper, we show that, in some cases (translation fractals), the non commutative Hausdorff-Besicovitch functional coincides with the restriction to the fractal of a Hausdorff-Besicovitch measure on $\mathbb{R}$ (cf. [27]).

For uniformly generated symmetric fractals, we are able to compute explicitly $\underline{\delta}$ and $\overline{\delta}$, and this provides an evidence of a classical interpretation for these numbers. In fact in [22] we define upper and lower pointwise tangential dimensions for fractals in $\mathbb{R}^n$. These dimensions are defined by means of the box dimensions of the balls of the tangent sets at a point, where a tangent set of F at x is any limit, for $\lambda \to \infty$, of the λ-dilations of F around x, in a suitable topology. We show in [22] that, for the uniformly generated symmetric fractals considered here (cf. Theorem 4.20), the upper, resp. lower, tangential dimension does not depend on x, and coincides with $\overline{\delta}$, resp. $\underline{\delta}$. We notice here that while our motivation for introducing the tangential dimensions was the attempt of finding a classical counterpart of $\underline{\delta}$ and $\overline{\delta}$, the description given above has been largely influenced by the notion of micro-set of Furstenberg, as we heard it in his talk at Graz [13].

We conclude by mentioning that two of the results proved here have an interest in the general study of singular traces. The first is the Hölder inequality, which we prove here for a general singular trace. In contrast with the Cauchy-Schwarz inequality, whose proof is purely algebraic, Hölder inequality requires the characterization of singular traces contained in [17].

The second is the complete description of the singular traceability exponents, which is based, and indeed generalises, the characterization of singular traceability given in [1]. Recently, we became aware of a related result contained in [11], where non-positive trace functionals on $B(H)$ are studied. Put together, these results suggest that the exponents of singular traceability for positive singular traces should coincide with those for non positive ones.

This paper is divided into four sections. The first two sections concern integration for spectral triples, the first containing the necessary technicalities on non increasing infinitesimal functions and the second the relevant results.

The last two sections concern fractals in $\mathbb{R}$, which are described in Section 3 from the classical point of view, while Section 4 contains the results of our noncommutative analysis. Hölder inequality for singular traces is proved in the Appendix.

The results of this work have been presented in several international conferences in the period 2000-2001. Some of them have been announced in the Proceedings of a Conference in Siena, July 2000, [20].

1 Non-increasing infinitesimal functions.

As is well known, we may associate, via non-increasing rearrangement, a non-increasing infinitesimal function μ defined on the interval $[0,\infty)$ to any compact operator on a Hilbert space.

In this section we treat some properties of the functions in this class, which we will extensively use in the following sections in order to get results concerning compact operators and singular traces.

Let M be the class of non-increasing right-continuous infinitesimal functions μ defined on the interval $[0,\infty)$, and F be the class of non-decreasing right-continuous functions f on $\mathbb{R}$, which are bounded from below and unbounded from above. Clearly, the map

$$\mu(x) \mapsto f(t) = -\log \mu(e^t), \tag{1.1}$$

and $\mu(0) := \lim_{x\to 0^+} \mu(x)$, gives a bijection between these two classes.

Given $f \in F$, we consider the following asymptotic indices:

$$\begin{aligned}
\underline{\delta}(f) &= \left(\lim_{h\to\infty} \limsup_{t\to\infty} \frac{f(t+h)-f(t)}{h} \right)^{-1} \\
\overline{\delta}(f) &= \left(\lim_{h\to\infty} \liminf_{t\to\infty} \frac{f(t+h)-f(t)}{h} \right)^{-1} \\
\underline{d}(f) &= \left(\limsup_{t\to\infty} \frac{f(t)}{t} \right)^{-1} \\
\overline{d}(f) &= \left(\liminf_{t\to\infty} \frac{f(t)}{t} \right)^{-1}
\end{aligned}$$

According to the previously mentioned correspondence between M and F, we shall write $\underline{d}(\mu)$, $\overline{d}(\mu)$, $\underline{\delta}(\mu)$, $\overline{\delta}(\mu)$ as well. Let us observe that these last two indices are the Matuszewska indices of the function μ (cf. e.g. [4]). Some of the properties below may be known, but we prove them for the sake of completeness.

1.1 Properties of the asymptotic indices

Proposition 1.1. *For any $f \in F$, the limits in the definitions of $\underline{\delta}$, $\overline{\delta}$ exist, and the following relations hold:*

$$\begin{aligned}\underline{\delta}(f) = \left(\inf_{h>0} \limsup_{t\to\infty} \frac{f(t+h)-f(t)}{h}\right)^{-1} &\leq \underline{d}(f) \\ \leq \overline{d}(f) \leq \left(\sup_{h>0} \liminf_{t\to\infty} \frac{f(t+h)-f(t)}{h}\right)^{-1} &= \overline{\delta}(f).\end{aligned}$$

Proof. Let us set $\overline{g}(h) = \limsup_{t\to\infty} \frac{f(t+h)-f(t)}{h}$. Then we have

$$\begin{aligned}\overline{g}(nh) &= \limsup_{t\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{f(t+kh)-f(t+(k-1)h)}{h} \\ &\leq \frac{1}{n}\sum_{k=1}^{n} \limsup_{t\to\infty} \frac{f(t+kh)-f(t+(k-1)h)}{h} = \overline{g}(h). \qquad (1.2)\end{aligned}$$

Let us denote by $[s]_r$ the r-integer part of s, namely $[s]_r = r[\frac{s}{r}]$. Since f is increasing, for any infinite sequence h_n we get

$$\limsup_n \overline{g}(h_n) = \limsup_n \overline{g}([h_n]_r), \quad \liminf_n \overline{g}(h_n) = \liminf_n \overline{g}([h_n]_r).$$

Now assume, ad absurdum, that $\limsup_{h\to\infty} \overline{g}(h) > \liminf_{h\to\infty} \overline{g}(h)$. Then we can find two sequences h_i and h'_i such that $\overline{g}(h_i) \to \liminf_{h\to\infty} \overline{g}(h)$, $\overline{g}(h'_i) \to \limsup_{h\to\infty} \overline{g}(h)$, and $\overline{g}(h_i) < \overline{g}(h'_j)$ for any i, j. But

$$\lim_j \overline{g}(h'_j) = \lim_j \overline{g}([h'_j]_{h_i}) = \lim_j \overline{g}(h_i \left[\frac{h'_j}{h_i}\right]) \leq \overline{g}(h_i),$$

which is absurd. The proof for $\underline{g}$ is analogous.
Now we prove the relations. By equation (1.2) and the existence of the limit of $\overline{g}(h)$ when $h \to \infty$, we have that $\lim_{h\to\infty} \overline{g}(h) = \inf_{h\geq 0} \overline{g}(h)$, namely the first equation.
Since f is increasing, we get

$$\limsup_{t\to\infty} \frac{f(t)}{t} = \limsup_{t\to\infty} \frac{f([t]_h)}{[t]_h},$$

therefore, setting $a_k = \frac{f(kh)-f((k-1)h)}{h}$, we have, for any $h > 0$,

$$\begin{aligned}\limsup_{t\to\infty} \frac{f(t)}{t} &= \limsup_{n} \frac{f(nh)}{nh} = \limsup_{n} \left(\frac{1}{n} \sum_{k=1}^{n} a_k + \frac{f(0)}{nh} \right) \\ &\leq \limsup_{k} a_k \leq \limsup_{t\to\infty} \frac{f(t+h)-f(t)}{h},\end{aligned}$$

or, equivalently,

$$\limsup_{t\to\infty} \frac{f(t)}{t} \leq \inf_{h\geq 0} \overline{g}(h),$$

which is the first inequality of the statement.
The last equality, resp. inequality of the statement are proved analogously. □

Lemma 1.2.

$$\underline{\delta}(f)^{-1} = \limsup_{t,h\to\infty} \frac{f(t+h)-f(t)}{h}, \tag{1.3}$$

$$\overline{\delta}(f)^{-1} = \liminf_{t,h\to\infty} \frac{f(t+h)-f(t)}{h}. \tag{1.4}$$

Proof. Setting $\varphi(t,h) := \frac{f(t+h)-f(t)}{h}$, we have to show $\limsup_{t,h\to\infty} \varphi(t,h) = \lim_{h\to\infty} \limsup_{t\to\infty} \varphi(t,h)$. Assume $\lim_{h\to\infty} \limsup_{t\to\infty} \varphi(t,h) = L \in \mathbb{R}$. Let $\varepsilon > 0$, then there is $h_\varepsilon > 0$ such that, for any $h > h_\varepsilon$, $\limsup_{t\to\infty} \varphi(t,h) > L - \varepsilon/2$, hence, for any $t_0 > 0$ there is $t = t(h,t_0)$, such that $\varphi(t,h) > L - \varepsilon$. Hence, for any $h_0 > 0$, $t_0 > 0$ there exist $h > h_0$, $t > t_0$ such that $\varphi(t,h) > L - \varepsilon$, namely $\limsup_{t,h\to\infty} \varphi(t,h) \geq L$. Conversely, assume $\limsup_{t,h\to\infty} \varphi(t,h) = L' \in \mathbb{R}$, and choose t_n, h_n such that $\lim_{n\to\infty} \varphi(t_n,h_n) = L'$. For any $r > 0$, let us denote by $\{s\}_r := r\left\{\frac{s}{r}\right\}$, where $\{s\}$ is the least integer no less than s. Then, for any $h > 0$, with p denoting $\frac{\{h_n\}_h}{h}$, we have

$$\begin{aligned}\varphi(t_n,h_n) &\leq \frac{\{h_n\}_h}{h_n} \varphi(t_n,\{h_n\}_h) = \frac{\{h_n\}_h}{h_n} \frac{1}{p} \sum_{j=0}^{p-1} \varphi(t_n + jh, h) \\ &\leq \frac{\{h_n\}_h}{h_n} \max_{j=0\ldots p-1} \varphi(t_n + jh, h) \leq \frac{\{h_n\}_h}{h_n} \sup_{t\geq t_n} \varphi(t,h).\end{aligned}$$

Hence, for $n \to \infty$, we get $L' \leq \limsup_{t\to\infty} \varphi(t,h)$, which implies the equality. The other cases are treated analogously. □

Now we introduce the notion of eccentricity for a function $\mu \in M$. It is motivated by the fact that a positive compact operator (cf. next section) is singularly traceable if and only if its eigenvalue function is eccentric.

Definition 1.3. Given a function $\mu \in M$, we define its integral function S as

$$S(x) = \begin{cases} S^{\uparrow}(x) := \int_0^x \mu(y)dy & \mu \notin L^1[0,\infty) \\ S^{\downarrow}(x) := \int_x^\infty \mu(y)dy & \mu \in L^1[0,\infty). \end{cases}$$

A function $\mu \in M$ is eccentric if 1 is a limit point, when $x \to \infty$, of the function

$$\frac{S(\lambda x)}{S(x)},$$

for some $\lambda > 1$. Note that if it is true for one λ, it is indeed true for any $\lambda > 1$, cf. [17].

It was proved in [19] that $\overline{d}(\mu) = 1$ is a sufficient condition for μ to be eccentric.

Theorem 1.4. *Let μ be a function in M. Then the following (possibly infinite) quantities coincide with $\overline{d}(\mu)$:*

$$\begin{aligned}
d_1 &= \sup\{\alpha > 0 : \limsup_{x\to\infty} \frac{\int_0^x \mu(t)^\alpha dt}{\log x} = \infty\} \\
d_2 &= \inf\{\alpha > 0 : \lim_{x\to\infty} \frac{\int_0^x \mu(t)^\alpha dt}{\log x} = 0\} \\
d_3 &= \inf\{\alpha > 0 : \int_0^x \mu(t)^\alpha dt < \infty\} \\
d_4 &= \left(\liminf_{t\to\infty} \frac{\log 1/\mu(t)}{\log t}\right)^{-1}
\end{aligned}$$

Proof. Set

$$\begin{aligned}
\Omega_1 &= \{\alpha > 0 : \limsup_{x\to\infty} \frac{\int_0^x \mu(t)^\alpha dt}{\log x} = \infty\} \\
\Omega_2 &= \{\alpha > 0 : \lim_{x\to\infty} \frac{\int_0^x \mu(t)^\alpha dt}{\log x} = 0\} \\
\Omega_3 &= \{\alpha > 0 : \int_0^x \mu(t)^\alpha dt < \infty\}.
\end{aligned}$$

$d_1 \leq d_2$. If $\alpha \in \Omega_1$, then $(0,\alpha] \supseteq \Omega_1$, if $\beta \in \Omega_2$, then $[\beta,\infty) \supseteq \Omega_2$, and $\Omega_1 \cap \Omega_2 = \emptyset$, hence Ω_1 and Ω_2 are separated classes: $\Omega_1 \leq \Omega_2$.
$d_2 \leq d_3$. If $\alpha \in \Omega_3$ then $\alpha \in \Omega_2$, namely $\Omega_3 \subseteq \Omega_2$.
$d_3 \leq d_4$. Let $a(t) = \frac{\log 1/\mu(t)}{\log(t)}$, namely $\mu(t) = t^{-a(t)}$ and $\liminf_{t\to\infty} a(t) = 1/d_4$. If $\alpha > d_4$, then $\liminf_{t\to\infty} \alpha a(t) = \alpha/d_4 > 1$, hence there exists $\beta > 1$ such that $\alpha a(t) \geq \beta$ for t sufficiently large. Therefore

$$\int_0^\infty \mu(t)^\alpha dt = \int_0^\infty t^{-\alpha a(t)} dt \leq const + \int_0^\infty t^{-\beta} dt < \infty,$$

which implies $\alpha \in \Omega_3$, namely $(d_4, \infty) \subset \Omega_3$.
$d_4 \leq d_1$. We may assume $d_4 > 0$, namely $1/d_4 < \infty$. Now let $\ell \in [0,\infty)$ be a limit point, for $t \to \infty$, of the function $\frac{\log 1/\mu(t)}{\log(t)}$, namely $\ell_k := \frac{\log 1/\mu(t_k)}{\log t_k} \to \ell$, for a suitable sequence $t_k \to \infty$. We have $\mu(t_k) = t_k^{-\ell_k}$. Let now $\alpha < 1/\ell$ (possibly

$1/\ell = \infty$), namely $\alpha\ell_k \to \alpha\ell < 1$, and choose $\varepsilon > 0$ such that $\alpha\ell_k \le 1 - \varepsilon$ eventually. Then

$$\int_0^{t_k} \mu(t)^\alpha dt \ge t_k \mu(t_k)^\alpha = t_k \cdot t_k^{-\alpha\ell_k} \ge t_k^\varepsilon.$$

Therefore

$$\frac{\int_1^{t_k} \mu(t)^\alpha}{\log t_k} \ge \frac{t_k^\varepsilon}{\log t_k} \to \infty,$$

which means that $\alpha \in \Omega_1$, i.e. $(0, 1/\ell) \subseteq \Omega_1$ and $d_1 \ge 1/\ell$.
The equality $d_4 = \overline{d}$ follows immediately from the definitions. □

We want to show that μ^α is eccentric *iff* $\alpha \in [\underline{\delta}, \overline{\delta}] \cap (0, \infty)$, thus giving a meaning to the quantities $\underline{\delta}, \overline{\delta}$.

Theorem 1.5. *Let μ be in M, γ be a positive number. If μ^γ is eccentric, then $\gamma \in [\underline{\delta}, \overline{\delta}]$.*

Proof. Let $\alpha < \underline{\delta}(\mu)$. The first equality in Proposition 1.1 says that $\alpha < \underline{\delta}(\mu)$ if and only if $\alpha < \sup_{h>0} \left(\limsup_{t\to\infty} \frac{f(t+h)-f(t)}{h}\right)^{-1}$, namely if and only if there exists $h > 0$ such that $\alpha < \left(\limsup_{t\to\infty} \frac{f(t+h)-f(t)}{h}\right)^{-1}$, or, equivalently, using the function μ associated with f, if there exists $\lambda > 1$ for which

$$\lambda \liminf_{t\to\infty} \frac{\mu(\lambda t)^\alpha}{\mu(t)^\alpha} > 1.$$

Now observe that, by the inequalities in Proposition 1.1, $\alpha < \overline{d}$, hence, by Theorem 1.4, $\mu^\alpha \notin L^1$. Therefore

$$\frac{S_{\mu^\alpha}(\lambda x)}{S_{\mu^\alpha}(x)} = \frac{\lambda \int_0^x \mu(\lambda t)^\alpha dt}{\int_0^x \mu(t)^\alpha dt} = \lambda \frac{\int_0^x \left(\frac{\mu(\lambda t)}{\mu(t)}\right)^\alpha \mu(t)^\alpha dt}{\int_0^x \mu(t)^\alpha dt},$$

hence

$$\liminf_{x\to\infty} \frac{S_{\mu^\alpha}(\lambda x)}{S_{\mu^\alpha}(x)} \ge \lambda \liminf_{x\to\infty} \left(\frac{\mu(\lambda x)}{\mu(x)}\right)^\alpha > 1,$$

which implies that μ^α is not eccentric.
The proof for $\alpha > \overline{\delta}(\mu)$ is analogous. □

To prove the converse direction, we need some preliminary results.

Proposition 1.6. *1 is a limit point of $\frac{S(x)}{S(2x)}$ if and only if $\liminf \frac{x\mu(x)}{S(x)} = 0$.*

Proof. Assume first μ is not summable, i.e. $S(x) = S^\uparrow(x)$. Then the thesis follows by the following inequalities:

$$\frac{S^\uparrow(2x)}{S^\uparrow(x)} - 1 \le \frac{x\mu(x)}{S^\uparrow(x)}$$
$$\frac{2x\mu(2x)}{S^\uparrow(2x)} \le 2\frac{S^\uparrow(2x) - S^\uparrow(x)}{S^\uparrow(2x)} = 2\left(1 - \frac{S^\uparrow(x)}{S^\uparrow(2x)}\right).$$

When μ is summable, i.e. $S(x) = S^{\downarrow}(x)$, we have, analogously,

$$1 - \frac{S^{\downarrow}(2x)}{S^{\downarrow}(x)} \leq \frac{x\mu(x)}{S^{\downarrow}(x)}$$
$$\frac{2x\mu(2x)}{S^{\downarrow}(2x)} \leq 2\left(\frac{S^{\downarrow}(x)}{S^{\downarrow}(2x)} - 1\right)$$

and the thesis follows. □

Proposition 1.7. *If* $\inf_t \frac{t\mu(t)}{S^{\uparrow}(t)} = k > 0$ *for any* $t > 0$, *then* $\underline{\delta} > 1$.

Proof. Since μ is the derivative of $S^{\uparrow}$, the hypothesis means that

$$\frac{d}{dt}\log S^{\uparrow}(t) \geq \frac{k}{t}, \forall t.$$

Integrating on the interval $[x, \lambda x]$ one gets

$$S^{\uparrow}(\lambda x) \geq \lambda^k S^{\uparrow}(x).$$

Since $\frac{x\mu(x)}{S^{\uparrow}(x)} \leq 1$, one obtains

$$\frac{\mu(x)}{\mu(\lambda x)} \leq \frac{\frac{S^{\uparrow}(x)}{x}}{\frac{kS^{\uparrow}(\lambda x)}{\lambda x}} \leq \frac{\lambda^{1-k}}{k}.$$

As a consequence,

$$\begin{aligned}\underline{\delta}^{-1} &= \lim_{\lambda\to\infty} \frac{1}{\log\lambda} \limsup_{x\to\infty} \log\frac{\mu(x)}{\mu(\lambda x)} \\ &\leq \lim_{\lambda\to\infty} \frac{-\log k + (1-k)\log\lambda}{\log\lambda} = 1 - k < 1\end{aligned}$$

□

Proposition 1.8. *If* $\inf_t \frac{t\mu(t)}{S^{\downarrow}(t)} = k > 0$ *for any* $t > 0$, *then* $\overline{\delta} < 1$.

Proof. Since $-\mu$ is the derivative of $S^{\downarrow}$, we may prove, in analogy with the previous Proposition, that

$$S^{\downarrow}(\lambda x) \leq \lambda^{-k} S^{\downarrow}(x).$$

Since $S^{\downarrow}(t) \geq \int_t^{2t} \mu(s)ds \geq t\mu(2t)$, which implies $\frac{x\mu(2x)}{S^{\downarrow}(x)} \leq 1$, one obtains

$$\frac{\mu(2\lambda x)}{\mu(x)} \leq \frac{\frac{S^{\downarrow}(\lambda x)}{\lambda x}}{\frac{kS^{\downarrow}(x)}{x}} \leq \frac{\lambda^{-1-k}}{k},$$

namely

$$\frac{\mu(x)}{\mu(\lambda x)} \geq k\left(\frac{\lambda}{2}\right)^{1+k},$$

As a consequence,

$$\begin{aligned}\overline{\delta}^{-1} &= \lim_{\lambda\to\infty} \frac{1}{\log\lambda} \liminf_{x\to\infty} \log \frac{\mu(x)}{\mu(\lambda x)} \\ &\geq \lim_{\lambda\to\infty} \frac{\log k + (1+k)\log 2 + (1+k)\log\lambda}{\log\lambda} = 1+k > 1,\end{aligned}$$

□

Theorem 1.9. *The set of the eccentricity exponents of μ is the interval whose endpoints are $\underline{\delta}(\mu)$, $\overline{\delta}(\mu)$ and is relatively closed in $(0,\infty)$.*

Proof. Assume μ^γ is not eccentric.
If $\mu^\gamma \notin L^1$, then $\inf_t \frac{t\mu(t)^\gamma}{\int_0^t \mu(s)^\gamma ds} > 0$ for any $t > 0$. Therefore, by Proposition 1.7, $1 < \underline{\delta}(\mu^\gamma) = \frac{1}{\gamma}\underline{\delta}(\mu)$, i.e. $\gamma < \underline{\delta}(\mu)$.
If $\mu^\gamma \in L^1$, then $\inf_t \frac{t\mu(t)^\gamma}{\int_t^\infty \mu(s)^\gamma ds} > 0$ for any $t > 0$. Therefore, by Proposition 1.8, $1 > \overline{\delta}(\mu^\gamma) = \frac{1}{\gamma}\overline{\delta}(\mu)$, i.e. $\gamma > \overline{\delta}(\mu)$.
The converse implication is contained in Theorem 1.5. □

1.2 Direct sums of infinitesimal functions.

Let f, g be two real valued measurable functions defined on the measure spaces A and B respectively. We say (cf. [3]) that g is a rearrangement of f if there is a measure preserving bijection φ from the support of f to the support of g and $f = g \circ \varphi$. The nonincreasing rearrangement f^* of f is defined as the unique non-increasing, right continuous rearrangement of f on $[0,\infty)$ with the Lebesgue measure.

It is known that f^* can be defined as $f^*(t) := \inf\{s \geq 0 : \lambda_f(s) \leq t\}$, $t \geq 0$, where $\lambda_f(t)$ is the measure of $\{x : f(x) > t\}$.

Consider now the following binary operation on M: let α, $\beta \in M$, and set $\alpha\oplus\beta$ to be the nonincreasing rearrangement of $\tilde\alpha+\tilde\beta$, where $\tilde\alpha$ and $\tilde\beta$ have disjoint supports and α, resp. β is the nonincreasing rearrangement of $\tilde\alpha$, resp. $\tilde\beta$. This operation is well defined, namely does not depend on the rearrangements $\tilde\alpha$ and $\tilde\beta$. Indeed $\lambda_{\tilde\alpha+\tilde\beta} = \lambda_{\tilde\alpha} + \lambda_{\tilde\beta} = \lambda_\alpha + \lambda_\beta$.

The need for this operation relies on Proposition 2.9 below.

Proposition 1.10. *Let α, β be elements of M. Then*

$$\underline{\delta}(\alpha\oplus\beta) = \underline{\delta}(\alpha\vee\beta) \tag{1.5}$$

$$\overline{\delta}(\alpha\oplus\beta) = \overline{\delta}(\alpha\vee\beta) \tag{1.6}$$

$$\underline{d}(\alpha\oplus\beta) = \underline{d}(\alpha\vee\beta) \tag{1.7}$$

$$\overline{d}(\alpha\oplus\beta) = \overline{d}(\alpha\vee\beta). \tag{1.8}$$

Proof. We have $\alpha = (\tilde\alpha)^* \leq (\tilde\alpha+\tilde\beta)^* = \alpha\oplus\beta$ and analogously for β, therefore we get

$$(\alpha\vee\beta)(x) \leq (\alpha\oplus\beta)(x). \tag{1.9}$$

Moreover, since $(a+b)^*(s+t) \leq a^*(s) + b^*(t)$ (cf. [3]), we have

$$(\alpha \oplus \beta)(2x) \leq 2(\alpha \vee \beta)(x), \tag{1.10}$$

from which equalities (1.5) to (1.8) follow immediately. □

Proposition 1.11. *Let α, β be elements of M. Then*

$$\underline{\delta}(\alpha \oplus \beta) \geq \underline{\delta}(\alpha) \wedge \underline{\delta}(\beta) \tag{1.11}$$
$$\overline{\delta}(\alpha \oplus \beta) \leq \overline{\delta}(\alpha) \vee \overline{\delta}(\beta) \tag{1.12}$$
$$\underline{d}(\alpha \oplus \beta) \geq \underline{d}(\alpha) \vee \underline{d}(\beta) \tag{1.13}$$
$$\overline{d}(\alpha \oplus \beta) = \overline{d}(\alpha) \vee \overline{d}(\beta). \tag{1.14}$$

Proof. We have

$$\begin{aligned}
\overline{d}(\alpha \oplus \beta)^{-1} &= \overline{d}(\alpha \vee \beta)^{-1} \\
&= \liminf_{x\to\infty} \frac{-\log(\alpha(x) \vee \beta(x))}{\log x} \\
&= \liminf_{x\to\infty} \left(\frac{-\log\alpha(x)}{\log x} \wedge \frac{-\log\beta(x)}{\log x} \right) \\
&= \overline{d}(\alpha)^{-1} \wedge \overline{d}(\beta)^{-1},
\end{aligned}$$

which shows equation (1.14). Inequality (1.13) is proved analogously.
Using (1.5), we get

$$\begin{aligned}
\underline{\delta}(\alpha \oplus \beta)^{-1} &= \lim_{\lambda\to\infty} \frac{1}{\log\lambda} \limsup_{x\to\infty} \log \frac{(\alpha\vee\beta)(x)}{(\alpha\vee\beta)(\lambda x)} \\
&= \lim_{\lambda\to\infty} \limsup_{x\to\infty} \frac{(\log\alpha(x) \vee \log\beta(x)) + (-\log\alpha(\lambda x) \wedge -\log\beta(\lambda x))}{\log\lambda},
\end{aligned} \tag{1.15}$$

from which, observing that $(a \vee b) + (c \wedge d) \leq (a+c) \vee (b+d)$, we get (1.11).
Making use of $(a+c) \wedge (b+d) \leq (a \vee b) + (c \wedge d)$, inequality (1.12) is proved. □

Theorem 1.12. *The interval $[\underline{\delta}(\alpha \oplus \beta), \overline{\delta}(\alpha \oplus \beta)]$ is contained in the interval $[\underline{\delta}(\alpha) \wedge \underline{\delta}(\beta), \overline{\delta}(\alpha) \vee \overline{\delta}(\beta)]$. This estimate is optimal, namely the equality may happen, but the interval may shrink to a point ($= \overline{d}(\alpha \oplus \beta)$) in some cases.*

Proof. The first statement immediately follows from Proposition 1.11. The estimate is optimal e.g. in the case of Proposition 1.13. The interval shrinks to a point in the following example.
Making use of the identification (1.1), the two functions may be equivalently described by two non-decreasing functions f, g. By equations (1.5), (1.6), the direct sum now corresponds to $f \wedge g$. Let us choose f and g as follows: choose a sequence of intervals $I_n = (a_n, a_{n+1}]$, $n \in \mathbb{N}$, with increasing length, set $f(t) = t$ for $t \in I_n$, n even, and $f(t) = a_{n+1}$ for $t \in I_n$, n odd. It is easy to check that $\underline{\delta}(f) = 0$ and $\overline{\delta}(f) = +\infty$. Conversely, choose $g(t) = t$ for $t \in I_n$, n odd, and

$g(t) = a_{n+1}$ for $t \in I_n$, n even. Again, $\underline{\delta}(g) = 0$ and $\overline{\delta}(g) = +\infty$. Moreover, $(f \wedge g)(t) = t$, therefore the corresponding interval shrinks to the point $\{1\}$. $\square$

For the application to fractals in Section 4, we need a refinement of the previous Theorem.

Proposition 1.13. *Let α, β be elements of M, and assume there are $A, B \geq 1$ such that $\frac{1}{A}\alpha(Bx) \leq \beta(x) \leq A\alpha(\frac{x}{B})$, for all x large enough. Then*

$$\underline{\delta}(\alpha \oplus \beta) = \underline{\delta}(\alpha) = \underline{\delta}(\beta) \tag{1.16}$$

$$\overline{\delta}(\alpha \oplus \beta) = \overline{\delta}(\alpha) = \overline{\delta}(\beta). \tag{1.17}$$

Proof. From the hypotheses we get, for all x large enough,

$$-\log A + \log \alpha(Bx) \leq \log \alpha(x) \vee \log \beta(x) \leq \log A + \log \alpha(\frac{x}{B}).$$

Therefore, from equation (1.15), we get

$$\begin{aligned}\underline{\delta}(\alpha \oplus \beta)^{-1} &= \lim_{\lambda\to\infty} \limsup_{x\to\infty} \frac{(\log \alpha(x) \vee \log \beta(x)) + (-\log \alpha(\lambda x) \wedge -\log \beta(\lambda x))}{\log \lambda} \\ &\leq \lim_{\lambda\to\infty} \limsup_{x\to\infty} \frac{2\log A + \log \alpha(\frac{x}{B}) - \log \alpha(\lambda B x)}{\log \lambda} = \underline{\delta}(\alpha)^{-1},\end{aligned}$$

and in like manner we obtain the reversed inequality. All the other statements are proved in the same way. $\square$

2 Integration for spectral triples

In this section we apply the results of the previous section in order to study traceability properties of compact operators and then to interpret them in the framework of Alain Connes' Noncommutative Geometry.

2.1 Singular traceability

The theory of singular traces on $\mathcal{B}(\mathcal{H})$, namely positive trace functionals vanishing on the finite rank projections, was developed by Dixmier [9], who first showed their existence, and then in [33], [1]. For the theory of non-positive traces see [11]. For generalizations to von Neumann algebras and C*-algebras see [17, 18, 10, 2, 5].

Any tracial weight is finite on an ideal contained in $\mathcal{K}(\mathcal{H})$ and may be decomposed as a sum of a singular trace and a multiple of the normal trace. Therefore the study of (non-normal) traces on $\mathcal{B}(\mathcal{H})$ is the same as the study of singular traces.

Moreover, because of singularity and unitary invariance, a singular trace depends only on the eigenvalue asymptotics, namely, if a and b are positive compact operators on $\mathcal{H}$ and $\mu_n(a) = \mu_n(b) + o(\mu_n(b))$, μ_n denoting the n-th eigenvalue, then $\tau_\omega(a) = \tau_\omega(b)$ for any singular trace τ_ω.

The main problem about singular traces is therefore to detect which asymptotics may be "summed" by a suitable singular trace, that is to say, which operators are singularly traceable.

In order to state the most general result in this respect we need some notation.

Let a be a compact operator. Then we denote by $\{\mu_n(a)\}_{n=0}^{\infty}$ the sequence of the eigenvalues of $|a|$, arranged in non-increasing order and counted with multiplicity, and by μ_a the corresponding eigenvalue function, which is equal to $\mu_k(a)$ on the interval $[k, k+1)$ for any k. We denote the corresponding integral function S_{μ_a}, defined in the previous section, simply by S_a.

A compact operator is called *singularly traceable* if there exists a singular trace which is finite non-zero on $|a|$. We observe that the domain of such singular trace should necessarily contain the ideal $\mathcal{I}(a)$ generated by a. Then the following theorem holds.

Theorem 2.1. *[1] A positive compact operator a is singularly traceable iff μ_a is eccentric (cf. Definition 1.3). In this case there exists a sequence $x_k \to \infty$ such that, for any generalised limit Lim_ω on ℓ^∞, the positive functional*

$$\tau_\omega(b) = \begin{cases} \mathrm{Lim}_\omega \left(\left\{ \frac{S_b(x_k)}{S_a(x_k)} \right\} \right) & b \in \mathcal{I}(a)_+ \\ +\infty & b \notin \mathcal{I}(a), \; b > 0, \end{cases}$$

is a singular trace whose domain is the ideal $\mathcal{I}(a)$ generated by a.

The best known eigenvalue asymptotics giving rise to a singular trace is $\mu_n \sim \frac{1}{n}$, which implies $S(x) \sim \log x$. The corresponding logarithmic singular trace is generally called Dixmier trace.

Definition 2.2. If $a \in \mathcal{K}(\mathcal{H})$ we define $\underline{\delta}(a) = \underline{\delta}(\mu_a)$, $\overline{\delta}(a) = \overline{\delta}(\mu_a)$, $\underline{d}(a) = \underline{d}(\mu_a)$, $\overline{d}(a) = \overline{d}(\mu_a)$. We say that $\alpha > 0$ is an exponent of singular traceability for a if $|a|^\alpha$ is singularly traceable.

Theorem 2.3. *Let a be a compact operator. Then, the set of singular traceability exponents is the closed interval in $(0, \infty)$ whose endpoints are $\underline{\delta}(a)$ and $\overline{\delta}(a)$. In particular, if $\overline{d}(a)$ is finite nonzero, it is an exponent of singular traceability.*

Proof. The statement follows by Theorems 1.9, 2.1. □

Note that the interval of singular traceability may be $(0, \infty)$, as shown in [20].

In [23] the previous Theorem has been generalised to any semifinite factor, and some questions concerning the domain of a singular trace have been considered.

2.2 Singular traces and spectral triples

In this section we shall discuss some notions of dimension in noncommutative geometry in the spirit of Hausdorff-Besicovitch theory.

As is known, the measure for a noncommutative manifold is defined via a singular trace applied to a suitable power of some geometric operator (e.g. the Dirac operator of the spectral triple of Alain Connes). Connes showed that such procedure recovers the usual volume in the case of compact Riemannian manifolds, and more generally the Hausdorff measure in some interesting examples [6], Section IV.3.

Let us recall that $(\mathcal{A}, \mathcal{H}, D)$ is called a *spectral triple* when $\mathcal{A}$ is an algebra acting on the Hilbert space $\mathcal{H}$, D is a self adjoint operator on the same Hilbert space such that $[D, a]$ is bounded for any $a \in \mathcal{A}$, and D has compact resolvent. In the following we shall assume that 0 is not an eigenvalue of D, the general case being recovered by replacing D with $D|_{\ker(D)^\perp}$. Such a triple is called d^+-summable, $d \in (0, \infty)$, when $|D|^{-d}$ belongs to the Macaev ideal $\mathcal{L}^{1,\infty} = \{a : \frac{S_a^\uparrow(t)}{\log t} < \infty\}$.

The noncommutative version of the integral on functions is given by the formula $\mathrm{Tr}_\omega(a|D|^{-d})$, where Tr_ω is the Dixmier trace, i.e. a singular trace summing logarithmic divergences. By the arguments below, such integral can be non-trivial only if d is the Hausdorff dimension of the spectral triple, but even this choice does not guarantee non-triviality. However, if d is finite non-zero, we may always find a singular trace giving rise to a non-trivial integral.

Theorem 2.4. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple. If s is an exponent of singular traceability for $|D|^{-1}$, namely there is a singular trace τ_ω which is non-trivial on the ideal generated by $|D|^{-s}$, then the functional $a \mapsto \tau_\omega(a|D|^{-s})$ is a trace state (Hausdorff-Besicovitch functional) on the algebra $\mathcal{A}$.*

Proof. It is the same as the proof of Theorem 1.3 in [8], by making use of the Hölder inequality for singular traces proved in the Appendix. □

Remark 2.5. When $(\mathcal{A}, \mathcal{H}, D)$ is associated to an n-dimensional compact manifold M, or to the fractal sets considered in [6], the singular trace is the Dixmier trace, and the associated functional corresponds to the Hausdorff measure. This fact, together with the previous theorem, motivates the following definition.

Definition 2.6. Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple, Tr_ω the Dixmier trace.

(i) We call α-dimensional Hausdorff functional the map $a \mapsto Tr_\omega(a|D|^{-\alpha})$;

(ii) we call (Hausdorff) dimension of the spectral triple the number

$$d(\mathcal{A}, \mathcal{H}, D) = \inf\{d > 0 : |D|^{-d} \in \mathcal{L}_0^{1,\infty}\} = \sup\{d > 0 : |D|^{-d} \notin \mathcal{L}^{1,\infty}\},$$

where $\mathcal{L}_0^{1,\infty} = \{a : \frac{S_a^\uparrow(t)}{\log t} \to 0\}$.

(iii) we call minimal, resp. maximal dimension of the spectral triple the quantity $\underline{\delta}(|D|^{-1})$, resp. $\overline{\delta}(|D|^{-1})$.

(iv) For any s between the minimal and the maximal dimension, we call the corresponding trace state on the algebra $\mathcal{A}$ a Hausdorff-Besicovitch functional on $(\mathcal{A}, \mathcal{H}, D)$.

Theorem 2.7.

(i) $d(\mathcal{A}, \mathcal{H}, D) = \overline{d}(|D|^{-1})$.

(ii) $d := d(\mathcal{A}, \mathcal{H}, D)$ *is the unique exponent, if any, such that* $\mathcal{H}_d$ *is non-trivial.*

(iii) If $d \in (0, \infty)$, *it is an exponent of singular traceability.*

Proof. (i) The equality directly follows from Theorem 1.4.
(ii) It follows easily from the definition.
(iii) It is a direct consequence of (i) and of Theorem 1.9. □

Let us observe that the α-dimensional Hausdorff functional depends on the generalized limit procedure ω, however its value is uniquely determined on the operators $a \in \mathcal{A}$ such that $a|D|^{-d}$ is measurable in the sense of Connes [6]. By an abuse of language we call measurable such operators.

As in the commutative case, the dimension is the supremum of the α's such that the α-dimensional Hausdorff measure is everywhere infinite and the infimum of the α's such that the α-dimensional Hausdorff measure is identically zero. Concerning the non-triviality of the d-dimensional Hausdorff functional, we have the same situation as in the classical case. Indeed, according to the previous result, a non-trivial Hausdorff functional is unique (on measurable operators) but does not necessarily exist. In fact, if the eigenvalue asymptotics of D is e.g. $n \log n$, the Hausdorff dimension is one, but the 1-dimensional Hausdorff measure gives the null functional.

However, if we consider all singular traces, not only the logarithmic ones, and the corresponding trace functionals on $\mathcal{A}$, as we said, there exists a non trivial trace functional associated with $d(\mathcal{A}, \mathcal{H}, D) \in (0, \infty)$, but $d(\mathcal{A}, \mathcal{H}, D)$ is not characterized by this property. In fact this is true if and only if the minimal and the maximal dimension coincide. A sufficient condition is the following.

Proposition 2.8. *Let* $(\mathcal{A}, \mathcal{H}, D)$ *be a spectral triple with finite non-zero dimension* d. *If there exists* $\lim \frac{\mu_n(D^{-1})}{\mu_{2n}(D^{-1})} \in (1, \infty)$, d *is the unique exponent of singular traceability of* D^{-1}.

Proof. It is a consequence of Theorem 1.9, since the existence of the limit above implies $\underline{\delta} = \overline{\delta} = d = \frac{1}{\log 2} \log \left(\lim \frac{\mu_n}{\mu_{2n}} \right)$. □

2.3 Direct sums and tensor products of spectral triples

We study here the behaviour of noncommutative dimensions under direct sum and tensor product.

Proposition 2.9. *Let* A, B *be compact operators. Then*

$$\mu_{A \oplus B} = \mu_A \oplus \mu_B.$$

Proof. In the definition of $\mu_A \oplus \mu_B$, choose $\tilde{\mu}_A$ to be the function defined on two copies of $\mathbb{R}_+$ which is equal to μ_A on the first copy and to zero on the second. Analogously, set $\tilde{\mu}_B$ to be equal to μ_B on the second copy and to zero on the first. Recall that the distribution function of μ is $\lambda_\mu(t) := meas\{x > 0 : \mu(x) > t\}$ (cf. [3, 12]). We clearly have $\lambda_{\tilde{\mu}_A + \tilde{\mu}_B} = \lambda_{\mu_A} + \lambda_{\mu_B}$ and also $\lambda_{A \oplus B} = \lambda_{\mu_A} + \lambda_{\mu_B}$. The thesis follows. □

Corollary 2.10. *Let $\mathbb{A}_i = (\mathcal{A}_i, \mathcal{H}_i, D_i)$, $i = 1, 2$, and $\mathbb{A} = (\mathcal{A}, \mathcal{H}_1 \oplus \mathcal{H}_2, D_1 \oplus D_2)$, be spectral triples. Then $d(\mathbb{A}) = d(\mathbb{A}_1) \vee d(\mathbb{A}_2)$. The interval $[\underline{\delta}(\mathbb{A}), \overline{\delta}(\mathbb{A})]$ is contained in the interval $[\underline{\delta}(\mathbb{A}_1) \wedge \underline{\delta}(\mathbb{A}_2), \overline{\delta}(\mathbb{A}_1) \vee \overline{\delta}(\mathbb{A}_2)]$.*

Proof. Immediately follows by Propositions 1.11, 2.9 and Theorem 1.12 □

Let $\mathbb{A}_i = (\mathcal{A}_i, \mathcal{H}_i, D_i)$, $i = 1, 2$, be spectral triples. Then their tensor product is the spectral triple $\mathbb{A}_1 \otimes \mathbb{A}_2 = (\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A} := \mathcal{A}_1 \otimes \mathcal{A}_2$, $\mathcal{H} := \mathcal{H}_1 \otimes \mathcal{H}_2$, and D is defined in different ways according to the parity of the two triples, but D^2 is, up to a finite multiplicity, always equal to $D_1^2 \otimes 1 + 1 \otimes D_2^2$.

Proposition 2.11. *With notation as above,*

$$d(\mathbb{A}_1 \otimes \mathbb{A}_2) \leq d(\mathbb{A}_1) + d(\mathbb{A}_2).$$

Proof. Let $\zeta_D(\alpha) := \sum_{n=0}^{\infty} \mu_n(D)^\alpha$, $\alpha \in \mathbb{R}$, denote the "zeta" function of the spectral triple $(\mathcal{A}, \mathcal{H}, D)$, and analogously for $(\mathcal{A}_i, \mathcal{H}_i, D_i)$, $i = 1, 2$. Then, if $c \in \mathbb{N}$ denotes the multiplicity, and $\alpha_i > d(\mathcal{A}_i, \mathcal{H}_i, D_i)$, $i = 1, 2$, we have

$$\begin{aligned}
\zeta_D(\alpha_1 + \alpha_2) &= \sum_{n=0}^{\infty} \mu_n(D^2)^{-(\alpha_1+\alpha_2)/2} = c \sum_{n=0}^{\infty} \mu_n(D_1^2 \otimes 1 + 1 \otimes D_2^2)^{-(\alpha_1+\alpha_2)/2} \\
&= c \sum_{i,j=0}^{\infty} \{\mu_i(D_1)^2 + \mu_j(D_2)^2\}^{-(\alpha_1+\alpha_2)/2} \\
&\leq c \sum_{i,j=0}^{\infty} \mu_i(D_1)^{-\alpha_1} \mu_j(D_2)^{-\alpha_2} = c\zeta_{D_1}(\alpha_1)\zeta_{D_2}(\alpha_2),
\end{aligned}$$

which converges. Therefore, by Theorem 1.4, we get the thesis. □

3 Fractals in ℝ. Classical aspects

3.1 Preliminaries

Let (X, ρ) be a metric space, and let $h : [0, \infty) \to [0, \infty)$ be non-decreasing and right-continuous, with $h(0) = 0$. When $E \subset X$, define, for any $\delta > 0$, $\mathcal{H}_\delta^h(E) := \inf\{\sum_{i=1}^{\infty} h(\operatorname{diam} A_i) : \cup_i A_i \supset E, \operatorname{diam} A_i \leq \delta\}$. Then the *Hausdorff-Besicovitch (outer) measure* of E is defined as

$$\mathcal{H}^h(E) := \lim_{\delta \to 0} \mathcal{H}_\delta^h(E).$$

If $h(t) = t^\alpha$, $\mathcal{H}^\alpha$ is called *Hausdorff (outer) measure* of order $\alpha > 0$.

The number

$$d_H(E) := \sup\{\alpha > 0 : \mathcal{H}^\alpha(E) = +\infty\} = \inf\{\alpha > 0 : \mathcal{H}^\alpha(E) = 0\}$$

is called *Hausdorff dimension* of E.

Let $N_\varepsilon(E)$ be the least number of closed balls of radius $\varepsilon > 0$ necessary to cover E. Then the numbers

$$\overline{d_B}(E) := \limsup_{\varepsilon\to 0^+} \frac{\log N_\varepsilon(E)}{-\log\varepsilon}, \quad \underline{d_B}(E) := \liminf_{\varepsilon\to 0^+} \frac{\log N_\varepsilon(E)}{-\log\varepsilon}$$

are called upper and lower *box dimensions* of E.

In case $X = \mathbb{R}^N$, setting $S_\varepsilon(E) := \{x \in \mathbb{R}^N : \rho(x, E) \leq \varepsilon\}$, it is known that $\overline{d_B}(E) = N - \liminf_{\varepsilon\to 0^+} \frac{\log \mathrm{vol}\, S_\varepsilon(E)}{\log\varepsilon}$ and $\underline{d_B}(E) = N - \limsup_{\varepsilon\to 0^+} \frac{\log \mathrm{vol}\, S_\varepsilon(E)}{\log\varepsilon}$. E is said *d-Minkowski measurable* if the following limit exists:

$$\mathcal{M}_d(E) := \lim_{\varepsilon\to 0^+} \frac{\mathrm{vol}\, S_\varepsilon(E)}{\varepsilon^{N-d}} \in (0, \infty).$$

This implies that the upper and lower box dimensions coincide. The quantity $\mathcal{M}_d(E)$ is called *d-Minkowski content* of E. He and Lapidus [25] have recently generalised that as follows. If $h : [0, \infty) \to [0, \infty)$ is non-decreasing and $h(0) = 0$, E is said *h-Minkowski measurable* if the following limit exists:

$$\mathcal{M}_h(E) := \lim_{\varepsilon\to 0} \mathrm{vol}\, S_\varepsilon(E)\ \frac{h(\varepsilon)}{\varepsilon^N} \in (0, \infty).$$

The quantity $\mathcal{M}_h(E)$ is called *h-Minkowski content* of E.

3.2 Fractals in ℝ

By a fractal in $\mathbb{R}$ we mean a compact, totally disconnected subset of $\mathbb{R}$, without isolated points. Let F be such a set, and denote by $[a, b]$ the least closed interval containing F. Then $[a, b]\backslash F$ is the disjoint union of open intervals (a_n, b_n), which we assume ordered in such a way that $\{b_n - a_n\}_{n\in\mathbb{N}}$ is a decreasing sequence. Notice that F is determined by the sequence of intervals $\{(a_n, b_n)\}_{n\in\mathbb{N}}$. Then F has Lebesgue measure zero iff $\sum_{n=1}^\infty (b_n - a_n) = b - a$, and, in that case, (cf. e.g. [32])

$$\overline{d_B}(F) = \limsup_{n\to\infty} \frac{\log n}{|\log(b_n - a_n)|}. \tag{3.1}$$

We will be interested in fractals constructed out of a family $\{w_{ni} : i = 1, \dots p_n, n \in \mathbb{N}\}$ of contracting similarities of $\mathbb{R}$, with dilation parameters λ_{ni}, such that

(*i*) $w_{ni}([a, b]) \subset [a, b]$

(*ii*) $w_{ni}([a, b]) \cap w_{nj}([a, b]) = \emptyset, i \neq j,\ n \in \mathbb{N}$

(*iii*) $\bigcup_{i=1}^{p_n} w_{ni}(\{a, b\}) \supset \{a, b\}$, $n \in \mathbb{N}$.

For any $n \in \mathbb{N}$, set $w_n(\Omega) := \bigcup_{i=1}^{p_n} w_{ni}(\Omega)$, $\Omega \subset \mathbb{R}$, and $W_n := w_1 \circ w_2 \circ \dots \circ w_n$. Then $\{W_n([a, b])\}$ is a decreasing sequence of compact sets, containing $\{a, b\}$. Denote by F its intersection. Then

Proposition 3.1. *F is a fractal in $\mathbb{R}$. It has Lebesgue measure zero iff*

$$\prod_n \left(\sum_{i=1}^{p_n} \lambda_{ni} \right) = 0.$$

We call the fractals described above *limit fractals* (cf. [20, 21] for alternate, more general definitions). If $p_n = p$, for all $n \in \mathbb{N}$, and the similarity parameters λ_{ni} do not depend on n, F is a self-similar fractal [26]. If the similarity parameters λ_{ni} do not depend on i, F is called a *translation fractal* (cf. [27]). Observe that for a translation fractal the condition (ii) above implies $p_n\lambda_n < 1$

The fractal is called *symmetric*, if $w_n([a,b]) = \bigcup_{i=1}^{p_n} [a + (i-1)d_n, a + (i-1)d_n + \lambda_n]$, where $d_n := \frac{(b-a)-p_n\lambda_n}{p_n - 1}$. In this case F is uniquely determined by the sequences $\{p_n\}$, $\{\lambda_n\}$.

3.3 Symmetries of limit fractals

Let us denote by F_n the set $\bigcap_{k=0}^{\infty} w_{n+1} \circ w_{n+2} \circ \cdots \circ w_{n+k}([a,b])$. We clearly have $F = w_1 \circ w_2 \circ \cdots \circ w_n(F_n)$. Therefore, if σ denotes a multiindex of length $|\sigma| = n$, and $w_\sigma := w_{1\sigma(1)} \circ w_{2\sigma(2)} \circ \cdots \circ w_{n\sigma(n)}$, we have $F = \bigcup_{|\sigma|=n} w_\sigma(F_n)$, with disjoint union.

We call the similarity maps $w_{\sigma'} \circ w_\sigma^{-1} : w_\sigma(F_n) \mapsto w_{\sigma'}(F_n)$, $|\sigma| = |\sigma'| = n$, $n \in \mathbb{N}$, *generating symmetries* of the limit fractal F.

Observe that if the fractal is a translation fractal the generating symmetries are indeed isometries.

Let us consider a triple (Ω_1, Ω_2, S) where Ω_1, Ω_2 are (relatively) open subsets of F and S is a one-to-one similarity with scaling parameter λ between Ω_1 and Ω_2. We say that a measure μ on F is homogeneous of order $\alpha > 0$ for the triple (Ω_1, Ω_2, S) if $\mu(\Omega_2) = \lambda^\alpha \mu(\Omega_1)$.

Proposition 3.2. *Let F be a limit fractal. Then, for any $\alpha \in (0,1)$, there is a unique probability measure μ_α, with support F, homogeneous of order α w.r.t. the generating symmetries of the fractal. All these measures are distinct, unless F is a translation fractal, in which case they all coincide.*

Proof. For any n, the homogeneity condition uniquely determines the measure of the sets $w_\sigma(F_n)$, $|\sigma| = n$. Indeed, if w_σ has similarity parameter λ_σ,

$$\begin{aligned}
1 = \mu_\alpha(F) &= \sum_{|\sigma'|=n} \mu_\alpha(w_{\sigma'}(F_n)) \\
&= \sum_{|\sigma'|=n} \mu_\alpha(w_{\sigma'} \circ w_\sigma^{-1}(w_\sigma(F_n))) \\
&= \sum_{|\sigma'|=n} (\lambda_{\sigma'}\lambda_\sigma^{-1})^\alpha \mu_\alpha(w_\sigma(F_n)) \\
&= \lambda_\sigma^{-\alpha} \mu_\alpha(w_\sigma(F_n)) \sum_{|\sigma'|=n} (\lambda_{\sigma'})^\alpha
\end{aligned}$$

namely

$$\mu_\alpha(w_\sigma(F_n)) = \lambda_\sigma^\alpha \left(\sum_{|\sigma'|=n} (\lambda_{\sigma'})^\alpha \right)^{-1}.$$

The measure uniquely extends to the sigma-algebra generated by these sets, which clearly coincides with the family of Borel subsets of F. The second statement is obvious. □

Remark 3.3. It has been proved in [27] that, when F is a translation fractal in $\mathbb{R}$, there is a gauge function h such that the corresponding Hausdorff-Besicovitch measure $\mathcal{H}^h$ is non-trivial on F. Since any Hausdorff-Besicovitch measure is isometry invariant, it satisfies the hypotheses of the previous proposition, hence $\mathcal{H}^h|_F$ coincides (up to a constant) with the homogeneous measure μ.

4 Fractals in $\mathbb{R}$. Noncommutative aspects.

4.1 The lacunary spectral triple

Let F be a fractal in $\mathbb{R}$, namely a compact, totally disconnected subset of $\mathbb{R}$, without isolated points. Now we introduce a "lacunary" spectral triple for the fractal F, namely a spectral triple completely determined by the "lacunae" of F, hence in particular canonically associated to F. Amendments to this spectral triple will be discussed below. Let a, b, a_n, b_n be as in subsection 3.2, and denote by I_n the lacuna (a_n, b_n). Set $\mathcal{H}_\ell = \oplus_{n=1}^\infty \mathcal{H}(I_n)$, $D_\ell = \oplus_{n=1}^\infty D(I_n)$, where

$$\mathcal{H}(I) := \ell^2(\partial I), \tag{4.1}$$

$$D(I) := \frac{1}{|I|} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.2}$$

Consider the action of $C(F)$ on $\mathcal{H}_\ell$ by left multiplication: $(f\xi)(x) = f(x)\xi(x)$, $x \in \mathcal{D}_\ell := \{a_n, b_n : n \in \mathbb{N}\}$, and define $\mathcal{A} := Lip(F)$. Then

Theorem 4.1.

(i) $(\mathcal{A}, \mathcal{H}_\ell, D_\ell)$ *is a spectral triple*

(ii) the characteristic values of D_ℓ^{-1} *are the numbers* $b_n - a_n$, $n \in \mathbb{N}$, *each with multiplicity 2.*

If F *is Minkowski measurable, and has box dimension* $d \in (0,1)$, *then*

(iii) $|D_\ell|^{-d} \in \mathcal{L}^{1,\infty}$

(iv) $\mathrm{Tr}_\omega(|D_\ell|^{-d}) = 2^d(1-d)\mathcal{M}_d(F)$.

Proof. It is due to Connes [6], using results of Lapidus and Pomerance, [30]. □

Making use of recent results of He and Lapidus [25], we can improve on the previous Theorem. Recall from [25] that the family of gauge functions G_d, for

$d \in (0,1)$, consists of the functions $h : (0,\infty) \to (0,\infty)$ which are continuous, strictly increasing, with $\lim_{x\to 0} h(x) = 0$, $\lim_{x\to\infty} h(x) = \infty$, and satisfy

$$\lim_{x\to 0} \frac{h(tx)}{h(x)} = t^d$$

uniformly in t on any compact subset of $(0,\infty)$, and one more condition (H3), which won't be needed in the following. Then, setting $g(x) := h^{-1}(1/x)$, $x > 0$, we have

Theorem 4.2. *Let $d \in (0,1)$, and $h \in G_d$, and assume F is h-Minkowski measurable. Then*

(i) the function g^d is eccentric, so it gives rise to a singular trace

$$\tau_{h,\omega}(a) = \mathrm{Lim}_\omega \left(\frac{S_a(n)}{S_{g^d}(n)} \right)$$

(ii) $d = d(\mathcal{A}, \mathcal{H}_\ell, D_\ell)$ and is the unique exponent of singular traceability of D_ℓ^{-1}

(iii) $\tau_{h,\omega}(|D_\ell|^{-d}) = 2^d(1-d)\mathcal{M}_h(F)$, and is therefore independent of the state ω.

Proof. (i) Recall from [25], Theorems 2.4 and 2.5, that F is h-Minkowski measurable iff there is $L > 0$ such that $b_n - a_n \sim Lg(n)$, $n \to \infty$, and in this case $\mathcal{M}_h(F) = \frac{2^{1-d}L^d}{1-d}$.

Besides, it follows from [25], Lemma 3.1 that

$$\lim_{z\to\infty} \frac{g(tz)}{g(z)} = \frac{1}{t^{1/d}},$$

for any $t > 0$. Therefore

$$\lim_{z\to\infty} \frac{\log g(z)}{\log 1/z} = \frac{1}{d}$$

which shows that g^d is eccentric.

(ii) First observe that

$$\lim_{z\to\infty} \frac{g(tz+a)}{g(z)} = \frac{1}{t^{1/d}}, \tag{4.3}$$

for any $t > 0$, $a \in \mathbb{R}$. Indeed, for any $\varepsilon \in (0,t)$, there is $z_\varepsilon > 0$ such that $(t-\varepsilon)z \le tz + a \le (t+\varepsilon)z$, $z > z_\varepsilon$, so that

$$\frac{1}{(t+\varepsilon)^{1/d}} = \lim_{z\to\infty} \frac{g((t+\varepsilon)z)}{g(z)} \le \lim_{z\to\infty} \frac{g(tz+a)}{g(z)} \le \lim_{z\to\infty} \frac{g((t-\varepsilon)z)}{g(z)} = \frac{1}{(t-\varepsilon)^{1/d}},$$

and the thesis follows from the arbitrariness of ε.

Let us now denote by μ_n the n-th characteristic value of D_ℓ^{-1}. Because of the previous Theorem, $\mu_{2n-1} = \mu_{2n} = b_n - a_n$, so that

$$\lim_{n\to\infty} \frac{\mu_{2n}}{\mu_{4n}} = \lim_{n\to\infty} \frac{g(n)}{g(2n)} = 2^{1/d}$$
$$\lim_{n\to\infty} \frac{\mu_{2n-1}}{\mu_{4n-2}} = \lim_{n\to\infty} \frac{g(n)}{g(2n-1)} = 2^{1/d},$$

where the last equality follows from (4.3). Therefore $\lim_{n\to\infty} \frac{\mu_n}{\mu_{2n}} = 2^{1/d}$, and, by Proposition 2.8 and its proof, we conclude.

(iii) Let us first observe that

$$\exists \lim_{t\to\infty} \frac{\mu(t)}{g(t)} = \alpha \in [0,\infty] \iff \exists \lim_{n\to\infty} \frac{\mu(2n)}{g(2n)} = \alpha \in [0,\infty].$$

Indeed, for any $t > 0$, there is $n \in \mathbb{N}$ such that $t \in (2n-2, 2n]$, so that

$$\frac{g(2n)}{g(2n-2)} \frac{\mu(2n)}{g(2n)} = \frac{\mu(2n)}{g(2n-2)} \leq \frac{\mu(t)}{g(t)} \leq \frac{\mu(2n-2)}{g(2n)} = \frac{\mu(2n-2)}{g(2n-2)} \frac{g(2n-2)}{g(2n)}$$

and the thesis follows from (4.3).

Now assume that $\int_0^\infty g(t)^d \, dt = \infty$. Then

$$\lim_{t\to\infty} \frac{\int_0^t \mu(s)^d \, ds}{\int_0^t g(s)^d \, ds} = \left(\lim_{t\to\infty} \frac{\mu(t)}{g(t)}\right)^d = \left(\lim_{n\to\infty} \frac{\mu(2n)}{g(2n)}\right)^d$$
$$= \left(\lim_{n\to\infty} \frac{b_n - a_n}{g(n)} \frac{g(n)}{g(2n)}\right)^d 2L^d = 2^d(1-d)\mathcal{M}_h(F).$$

We can proceed in a similar way if $\int_0^\infty g(t)^d dt < \infty$. □

Even if F is not h-Minkowski measurable, we have that, by Theorem 2.4, any singular traceability exponent gives rise to a trace state on the C*-algebra of continuous functions on the fractal, namely to a probability measure on the fractal. In particular,

Theorem 4.3. (i) *For any singular traceability exponent s for $|D_\ell|^{-1}$ we get a Hausdorff-Besicovitch functional on the spectral triple, giving rise to a probability measure μ on F.*
(ii) *Let F have zero Lebesgue measure. Then $d(\mathcal{A}, \mathcal{H}_\ell, D_\ell) = \overline{d_B}(F)$. Therefore, if $\overline{d_B}(F) \neq 0$, we get a corresponding measure on F.*

Proof. (i) follows from Theorem 2.4 (cf. Definition 2.6) and Riesz Theorem.
(ii) follows by equation (3.1) and Theorem 4.1 $(i), (ii)$. □

4.2 The reconstruction of the metric

First we discuss the "lacunary" metric on the fractal, namely the metric on F determined *à la Connes* via the lacunary spectral triple. As explained below, such metric does not coincide in general with the original one.

Let us first compute $\|[D_\ell, f]\|$. Observe that, setting $I = (x, y)$,

$$\|[D(I), f|_{\partial I}]\| = \frac{1}{|I|} \left\| \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} f(x) & 0 \\ 0 & f(y) \end{pmatrix} \right] \right\| = \left| \frac{f(y) - f(x)}{y - x} \right|.$$

Therefore we have

$$\|[D_\ell, f]\| = \sup_n \|[D(I_n), f|_{\partial I_n}]\| = \sup_n \left| \frac{f(b_n) - f(a_n)}{b_n - a_n} \right| \leq \|f\|_{Lip(F)}. \tag{4.4}$$

Theorem 4.4. *Let F be a compact, totally disconnected subset of $\mathbb{R}$ with no isolated points. Then the lacunary metric*

$$d_\ell(x, y) = \sup\{|f(y) - f(x)| : f \in C(F), \|[D_\ell, f]\| \leq 1\} \tag{4.5}$$

coincides with the one induced by the metric on $\mathbb{R}$ if and only if F has Lebesgue measure zero.

Proof. Assume F has Lebesgue measure zero. Now, for any $f \in C(F)$, we denote by $\tilde{f}$ the continuous function on $[a, b]$ coinciding with f on F and linear on any interval $[a_n, b_n]$. For any pair of points $x < y$ in F, let us denote by $\mathcal{I}(x, y)$ the family of lacunary intervals I_n which are subsets of $[x, y]$. Then

$$\begin{aligned} |f(y) - f(x)| &= \left| \sum_{I \in \mathcal{I}(x,y)} \int_I \tilde{f}'(t) dt \right| \\ &\leq \sum_{I \in \mathcal{I}(x,y)} |I| \|[D(I), f|_{\partial I}]\| \leq |y - x| \|[D_\ell, f]\|. \end{aligned}$$

Comparing the previous inequality with (4.4) we get $\|f\|_{Lip(F)} = \|[D_\ell, f]\|$, namely the equality $d_\ell = d$.
Conversely, assuming F has positive Lebesgue measure, let f be the restriction to F of the primitive of the characteristic function of F. Clearly $\|[D_\ell, \lambda f]\| = 0$ for any $\lambda > 0$, hence $d_\ell(x, y) = +\infty$ for any pair x, y in F which are not boundary of the same lacuna. □

Connes proposed us an emendation of the lacunary spectral triple in order to reproduce the original distance also in the case of positive Lebesgue measure. In the case of the Cantor middle third set, the idea is to add to the lacunary intervals also the images of the interval $[0, 1]$ under the similarity maps [7].

For a general compact, totally disconnected fractal F, the idea of Connes may be generalized as follows:

Assign the family $\mathcal{F}_n$ of (closed) filled intervals of level n and the family $\mathcal{L}_n$ of (open) lacunary intervals of level n in such a way that

- $F = \cap_n \cup_{I \in \mathcal{F}_n} I$,
- $\mathcal{L}_n \subseteq \mathcal{L}_{n+1}$,
- $\mathcal{I}_n := \mathcal{F}_n \cup \mathcal{L}_n$ form a finite partition of $[a, b]$ for any n.

Then, setting $\mathcal{I} = \cup_n \mathcal{I}_n$, and, according to the notation in subsections 3.2 and 3.3, $\mathcal{H} = \oplus_{I\in\mathcal{I}}\mathcal{H}(I)$, $D = \oplus_{I\in\mathcal{I}}D(I)$, we get that $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple, $\mathcal{A}$ being the *-algebra of Lipschitz functions. Moreover,

Theorem 4.5. *Let F be a compact, totally disconnected subset of $\mathbb{R}$ with no isolated points. Then the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ reconstructs the original distance on F.*

Proof. For any pair $x < y$ in the boundary of some lacunae, there exists a k such that both x and y belong to the boundary of some interval of level k. Therefore, setting $\mathcal{I}_k(x, y)$ for the family of intervals of level k which are subsets of $[x, y]$, we have

$$|f(y) - f(x)| \le \sum_{I\in\mathcal{I}_k(x,y)} |I| \| [D(I), f|_{\partial I}] \| \le |y - x| \| [D, f] \|.$$

Since F is totally disconnected, x and y vary in a dense subset of F, therefore, by continuity, the previous inequality holds for any pair $x, y \in F$, giving $\|f\|_{Lip(F)} \le \|[D, f]\|$. On the other hand, as in (4.4), the converse inequality holds too, hence the result follows. □

4.3 A spectral triple for limit fractals

The spectral triple described in the previous subsection depends on the choice of the filled and lacunary intervals of level n. Of course, one may either select the filled intervals of level n first, and then the lacunae as the connected components of the complement, or the converse. The first choice appears very natural in the case of limit fractals, therefore we shall adopt this point of view, limiting our further analysis to this family.

Definition 4.6. Let F be a limit fractal, with similarities $w_{n,i}$. We set $\mathcal{F}_n$ to be $w_\sigma[a, b]$, where σ varies in the set of multi-indices of length n.

The Dirac operator is a direct sum of the lacunary Dirac D_ℓ and the Dirac $D_f = \oplus_{n\in\mathbb{N}, I\in\mathcal{F}_n} D(I)$ acting on the Hilbert space $\mathcal{H}_f = \oplus_{n\in\mathbb{N}, I\in\mathcal{F}_n}\mathcal{H}(I)$.

We choose $\mathcal{A}$ to be the *-algebra of Lipschitz functions, acting on $\mathcal{H}$ by pointwise multiplication.

Remark 4.7.

(1) The spectral triple $(\mathcal{A}, \mathcal{H}, D)$ reconstructs the original distance on F, by Theorem 4.5. Any singular traceability exponent, namely any number between $\underline{\delta}(\mathcal{A}, \mathcal{H}, D)$ and $\overline{\delta}(\mathcal{A}, \mathcal{H}, D)$ gives rise to a Hausdorff-Besicovitch functional on $\mathcal{A}$, by Theorems 2.3 and 2.4.

(2) $(\mathcal{A}, \mathcal{H}_f, D_f)$ is a spectral triple, indeed it is exactly the spectral triple we proposed in [20] for limit fractals in $\mathbb{R}^n$.

(3) Concerning the second choice, namely defining the intervals $\mathcal{I}$ selecting the lacunae first, one may e.g. call λ_n the values of the lengths of the lacunae arranged in decreasing order, and then $\mathcal{L}_n$ the lacunae of length lower equal than

λ_n. While this choice is completely canonical, even the analysis of self-similar fractals is far less obvious than the corresponding one for the lacunary spectral triple. However, the spectral triple corresponding to such a choice coincides with the one in Definition 4.6 in the case of uniformly generated symmetric fractals.

In the following we shall prove some results on the dimensions and measures associated with the spectral triple of a limit fractal. The results are generally stated for the triple $(\mathcal{A}, \mathcal{H}, D)$, but hold also for the "lacunary" and "filled" spectral triples. Indeed we shall prove such properties for the latter triples, then showing that they remain valid for the direct sum of the Dirac operators.

Theorem 4.8. *Let F be a limit fractal, s a singular traceability exponent for $|D|^{-1}$, τ_ω a corresponding singular trace. Then, for any continuous function f on F,*

$$\tau_\omega(f|D|^{-s}) = \int_F f \; d\mu_s, \tag{4.6}$$

where μ_s is the measure introduced in Proposition 3.2. In particular continuous functions are measurable, namely the integral of continuous functions does not depend on the generalized limit procedure ω.

Before proving the theorem, we give a corollary which follows immediately from Remark 3.3

Corollary 4.9. *If F is a translation fractal, all the Hausdorff-Besicovitch functionals on $\mathcal{A}$ described above give rise to the same measure, which is indeed the restriction to F of the Hausdorff-Besicovitch measure constructed in [27].*

Let us first discuss the statement above for the lacunary Dirac operator.

Lemma 4.10. *The probability measure μ on F associated with a singular traceability exponent s for $|D_\ell|^{-1}$ has the following property:*

$$\mu(\Omega_2) = \lambda^s \mu(\Omega_1) \tag{4.7}$$

where Ω_1, Ω_2 are relatively open subsets of F related by a similarity of parameter λ.

Proof. Observe that given any (Ω_1, Ω_2, S), where Ω_1, Ω_2 are clopen sets in F and $S : \Omega_1 \to \Omega_2$ is a one-to-one similarity of parameter λ, we have that the sizes of the lacunae in Ω_1 are multiples of the sizes of the lacunae in Ω_2 with scaling λ. Therefore, by Theorem 4.1 (ii), the eigenvalues of $\chi_{\Omega_1}|D_\ell|^{-s}$ are multiples of the eigenvalues of $\chi_{\Omega_2}|D_\ell|^{-s}$ with scaling λ^s. As a consequence the measure of Ω_1 is equal to λ^s times the measure of Ω_2. This clearly extends to pairs of open sets Ω_1, Ω_2.
□

Previous lemma can be void for general fractals, namely there may be nontrivial triples (Ω_1, Ω_2, S). However, if limit fractals are concerned, the generating symmetries determine the measure μ_s introduced in Proposition 3.2, hence we have proved

Proposition 4.11. *Equation (4.6) holds for the lacunary Dirac.*

The proof for the filled Dirac operator is analogous. Indeed the following lemma is a direct consequence of the definition of D_f.

Lemma 4.12. *The eigenvalues of D_f are the numbers $(b-a)\lambda_\sigma$, each with multiplicity two, where we have set $\lambda_\sigma = \prod_{i=1,\dots,|\sigma|} \lambda_{i,\sigma_i}$.*

Then, if Ω_1, Ω_2 are clopen sets in F related by a generating symmetry of F, the eigenvalues of $\chi_{\Omega_1}|D_f|^{-s}$ are multiples of the eigenvalues of $\chi_{\Omega_2}|D_f|^{-s}$ with scaling λ^s. Reasoning as above, we have

Proposition 4.13. *Equation (4.6) holds for the filled Dirac.*

Proof. (of Theorem 4.8). If Ω_1, Ω_2 are clopen sets in F related by a generating symmetry of F, the eigenvalues of $\chi_{\Omega_1}|D|^{-s}$ are multiples of the eigenvalues of $\chi_{\Omega_2}|D|^{-s}$ with scaling λ^s, since this property holds for the two direct summands D_ℓ and D_f. The result then follows. □

Remark 4.14. Let us note that in spite of the fact that equation (4.6) holds for the three Dirac operators, the singularity exponents for the different Dirac's are different in general. They will coincide however for uniformly generated symmetric fractals.

Theorem 4.15. *Let F be a self-similar fractal, and $d \in (0,1)$ its Hausdorff dimension. Then d is the unique exponent of singular traceability for D^{-1}, and the Hausdorff functional on the spectral triple gives rise to the d-dimensional Hausdorff measure on F, up to a multiplicative constant. In particular the commutative and noncommutative Hausdorff dimensions coincide.*

Proof. Set $\mathcal{D}_\ell := \{a_n, b_n : n \in \mathbb{N}\}$, $\mathcal{D}_f := \bigsqcup_{\sigma\in\Sigma^*}\{\sigma(a), \sigma(b)\}$, where $\bigsqcup$ denotes disjoint union, and Σ^* is the set of all multi-indices. Define the following operators on $\ell^2(\mathcal{D}_\ell)$:

$$S_{\ell,j}\xi(b) := \begin{cases} \xi(w_j^{-1}(b)) & b \in w_j\mathcal{D}_\ell \\ 0 & b \notin w_j\mathcal{D}_\ell, \end{cases}$$

and analogously for $S_{f,j}$ on $\ell^2(\mathcal{D}_f)$, $j = 1, \dots, p$. Then $S_{\ell,j}$ and $S_{f,j}$ are isometries and $|D_\ell|^{-s} = \sum_{j=1}^p \lambda_j^s S_{\ell,j}|D_\ell|^{-s}S_{\ell,j}^*$, and an analogous formula for $|D_f|^{-s}$. Hence, with $S_j := S_{\ell,j} \oplus S_{f,j}$, we obtain $|D|^{-s} = \sum_{j=1}^p \lambda_j^s S_j|D|^{-s}S_j^*$. Therefore, if s is an exponent of singular traceability for $|D|^{-1}$, the corresponding Hausdorff-Besicovitch functional is homogeneous of order s. This implies that s coincides with d, namely d is the unique exponent of singular traceability. We now prove that the d-dimensional Hausdorff functional corresponds to the d-dimensional Hausdorff measure. Let us compute the zeta functions of D_ℓ and

D_f separately:

$$\zeta_f(s) := Tr(|D_f|^{-s}) = 2(b-a)\sum_\sigma \lambda_\sigma^s = 2(b-a)\sum_{n=0}^\infty \sum_{|\sigma|=n}\prod_{j=1}^n \lambda_{\sigma(j)}^s$$
$$= 2(b-a)\sum_{n=0}^\infty \left(\sum_{j=1}^p \lambda_j^s\right)^n = \frac{2(b-a)}{1-\sum_{j=1}^p \lambda_j^s},$$

so that

$$\lim_{s\to d}(s-d)\zeta_f(s) = \frac{2(b-a)}{\sum_{j=1}^p \lambda_j^d \log(1/\lambda_j)},$$

which, using [6], Proposition IV.2.β.4, implies that the Hausdorff functional is non-trivial. As for D_ℓ, denoting by $c_1,\ldots,c_{p-1}$ the lengths of the connected components of $[a,b]\setminus\cup_{j=1}^p w_j([a,b])$, we obtain

$$\zeta_\ell(s) := Tr(|D_\ell|^{-s}) = 2\sum_{j=1}^{p-1} c_j^s \sum_\sigma \lambda_\sigma^s = \frac{2\sum_{j=1}^{p-1} c_j^s}{1-\sum_{j=1}^p \lambda_j^s},$$

so that

$$\lim_{s\to d}(s-d)\zeta_\ell(s) = \frac{2\sum_{j=1}^{p-1} c_j^d}{\sum_{j=1}^p \lambda_j^d \log(1/\lambda_j)},$$

which, using [6], Proposition IV.2.β.4, implies that the Hausdorff functional is non-trivial. The thesis follows from the fact that $Tr(|D|^{-s}) = \zeta_f(s)+\zeta_\ell(s)$, and Theorem 4.8. □

Also in the case of symmetric fractals we have a formula for the noncommutative Hausdorff dimension. We recall that symmetric fractals (with convex hull $[0,1]$) are determined by two sequences $\{p_n\}$, $\{\lambda_n\}$, where p_n is a natural number greater or equal than 2 and $p_n\lambda_n < 1$. We say that the fractal is uniformly generated if $\sup_n p_n < \infty$ and $\sup_n p_n\lambda_n < 1$.

Theorem 4.16. *Let $(\mathcal{A},\mathcal{H},D)$ be the spectral triple associated with a uniformly generated symmetric fractal F, where the similarities $w_{n,i}$, $i=1,\ldots,p_n$ have scaling parameter λ_n. Then*

$$d(\mathcal{A},\mathcal{H},D) = \limsup_n \frac{\sum_1^n \log p_k}{\sum_1^n \log 1/\lambda_k},$$

and such dimension coincides with the upper box dimension of F.

Proof. The thesis will follow from the next two Propositions, Corollary 2.10 and Theorem 4.3, (ii). □

Proposition 4.17. *Let F be a uniformly generated symmetric fractal as before. Then*

$$d(\mathcal{A},\mathcal{H}_\ell,D_\ell) = \limsup_n \frac{\sum_1^n \log p_k}{\sum_1^n \log 1/\lambda_k}.$$

Proof. It is not restrictive to assume $a = 0$, $b = 1$. Then the eigenvalues of $|D_\ell|^{-1}$ are given by

$$\tilde{\Lambda}_k = \frac{1 - p_{k+1}\lambda_{k+1}}{p_{k+1} - 1} \prod_{j=1}^{k} \lambda_j \tag{4.8}$$

with multiplicity $2\tilde{P}_k = 2(p_{k+1} - 1)\prod_{j=1}^k p_j$, $k \in \mathbb{N} \cup \{0\}$. Therefore,

$$Tr(|D_\ell|^{-\alpha}) = 2\sum_{k=0}^{\infty}(1 - p_{k+1}\lambda_{k+1})^\alpha (p_{k+1} - 1)^{1-\alpha} \prod_{i=1}^{k} p_i(\lambda_i)^\alpha .$$

Setting $\Lambda_n := \prod_{k=1}^n \lambda_k$, $P_n := \prod_{k=1}^n p_k$, we obtain

$$\begin{aligned} Tr(|D_\ell|^{-\alpha}) &= 2\sum_{k=0}^{\infty}(1 - p_{k+1}\lambda_{k+1})^\alpha (p_{k+1} - 1)^{1-\alpha} P_k \Lambda_k^\alpha \\ &= 2\sum_{k=0}^{\infty}(1 - p_{k+1}\lambda_{k+1})^\alpha (p_{k+1} - 1)^{1-\alpha} \exp\left(\log P_k \left(1 - \alpha\frac{\log 1/\Lambda_k}{\log P_k}\right)\right). \end{aligned}$$

By the n-th root criterion for series, the series diverges/converges if

$$\limsup_k \left(\frac{(1 - p_{k+1}\lambda_{k+1})^\alpha}{(p_{k+1} - 1)^{\alpha-1}} \exp\left(\log P_k \left(1 - \alpha\frac{\log 1/\Lambda_k}{\log P_k}\right)\right)\right)^{1/k} \gtrless 1,$$

namely, since by the uniform generation assumption $\lim_k (1 - p_k\lambda_k)^{1/k} = 1$ and $\lim_k (p_k - 1)^{1/k} = 1$, if

$$\limsup_k \frac{\log P_k}{k}\left(1 - \alpha\frac{\log 1/\Lambda_k}{\log P_k}\right) \gtrless 0,$$

and, finally, if

$$\limsup_k \frac{\log P_k}{\log 1/\Lambda_k} \gtrless \alpha,$$

which implies that $\limsup_k \frac{\log P_k}{\log 1/\Lambda_k}$ is the abscissa of convergence of the zeta function of $|D_\ell|^{-1}$, hence the spectral dimension by Theorem 2.7 (i). □

Proposition 4.18. *Let F be a uniformly generated symmetric fractal as before. Then*

$$d(\mathcal{A}, \mathcal{H}_f, D_f) = \limsup_n \frac{\sum_1^n \log p_k}{\sum_1^n \log 1/\lambda_k}.$$

Proof. It is not restrictive to assume $a = 0$, $b = 1$. Then the eigenvalues of $|D_f|^{-1}$ are given by $\Lambda_k = \prod_{j=0}^k \lambda_j$, with multiplicity $2P_k = 2\prod_{j=0}^k p_j$, $k \in \mathbb{N} \cup \{0\}$, where we have set $\lambda_0 := 1$, $p_0 := 1$. Therefore,

$$\begin{aligned} Tr(|D_f|^{-\alpha}) &= 2\sum_{k=0}^{\infty}\prod_{i=1}^{k} p_i(\lambda_i)^\alpha = 2\sum_{k=0}^{\infty} P_k\Lambda_k^\alpha \\ &= 2\sum_{k=0}^{\infty}\exp\left(\log P_k\left(1 - \alpha\frac{\log 1/\Lambda_k}{\log P_k}\right)\right). \end{aligned}$$

As in the proof of the previous Theorem we conclude. □

Remark 4.19. In [22] we define pointwise tangential upper and lower dimensions for subspaces of $\mathbb{R}^n$. It turns out that for the uniformly generated symmetric fractals, such dimensions are constant and equal respectively to the maximal and minimal dimension computed below.

Theorem 4.20. *Let $(\mathcal{A}, \mathcal{H}, D)$ be the spectral triple associated with a uniformly generated symmetric fractal F, where the similarities $w_{n,i}$, $i = 1, \dots, p_n$ have scaling parameter λ_n. Then*

$$\underline{\delta}(\mathcal{A}, \mathcal{H}, D) = \liminf_{n,k} \frac{\sum_{j=n}^{n+k} \log p_j}{\sum_{j=n}^{n+k} \log 1/\lambda_j},$$
$$\overline{\delta}(\mathcal{A}, \mathcal{H}, D) = \limsup_{n,k} \frac{\sum_{j=n}^{n+k} \log p_j}{\sum_{j=n}^{n+k} \log 1/\lambda_j}.$$

As before, we first discuss the lacunary case.

Proposition 4.21. *Let F be a uniformly generated symmetric fractal, with the notations above. Then*

$$\underline{\delta}(\mathcal{A}, \mathcal{H}_\ell, D_\ell) = \liminf_{n,k} \frac{\sum_{j=n}^{n+k} \log p_j}{\sum_{j=n}^{n+k} \log 1/\lambda_j},$$
$$\overline{\delta}(\mathcal{A}, \mathcal{H}_\ell, D_\ell) = \limsup_{n,k} \frac{\sum_{j=n}^{n+k} \log p_j}{\sum_{j=n}^{n+k} \log 1/\lambda_j}.$$

Proof. Making use of the definitions in (4.8), one gets

$$\mu_{|D_\ell|^{-1}}(x) = \tilde{\Lambda}_k, \qquad \sum_{m=0}^{k-1} \tilde{P}_m < x \leq \sum_{m=0}^{k} \tilde{P}_m.$$

Because of Lemma 1.2, $\underline{\delta}^{-1}$, resp. $\overline{\delta}^{-1}$, is equal to the lim sup, resp. lim inf when t and h go to ∞, of the quantity $\frac{1}{h}(\log 1/\mu(e^{t+h}) - \log 1/\mu(e^t))$, which may be rewritten as

$$\frac{\log 1/\tilde{\Lambda}_k - \log 1/\tilde{\Lambda}_m}{\log\left(\sum_{j=0}^{k} \tilde{P}_j - \vartheta_k \tilde{P}_k\right) - \log\left(\sum_{j=0}^{m} \tilde{P}_j - \vartheta'_m \tilde{P}_m\right)} \tag{4.9}$$

for suitable constants ϑ_k, ϑ'_k in $[0, 1)$. Since the denominator goes to infinity, additive perturbations of the numerator and of the denominator by bounded sequences do not alter the lim sup, resp. lim inf, therefore the uniform generation hypotheses imply that the ratio (4.9) can be replaced by

$$\frac{\log 1/\Lambda_k - \log 1/\Lambda_m}{\log P_k - \log P_m}. \tag{4.10}$$

Finally, since the denominator $\log P_k - \log P_m$ goes to infinity if and only if $k - m \to \infty$, the thesis follows. $\square$

Then we discuss the filled case. Indeed for such spectral triple the result holds in more generality, namely for any symmetric fractal.

Proposition 4.22. *Let F be a symmetric fractal, with the notations above. Then*

$$\underline{\delta}(\mathcal{A}, \mathcal{H}_f, D_f) = \liminf_{n,k} \frac{\sum_{j=n}^{n+k} \log p_j}{\sum_{j=n}^{n+k} \log 1/\lambda_j},$$
$$\overline{\delta}(\mathcal{A}, \mathcal{H}_f, D_f) = \limsup_{n,k} \frac{\sum_{j=n}^{n+k} \log p_j}{\sum_{j=n}^{n+k} \log 1/\lambda_j}.$$

Proof. In this case, the eigenvalues of $|D_f|^{-1}$ are the numbers Λ_k, each with multiplicity $2P_k$. Therefore, the quantity $\frac{1}{h}(\log 1/\mu(e^{t+h}) - \log 1/\mu(e^t))$, may be rewritten as

$$\frac{\log 1/\Lambda_k - \log 1/\Lambda_m}{\log\left(\sum_{j=0}^{k} P_j - \vartheta_k P_k\right) - \log\left(\sum_{j=0}^{m} P_j - \vartheta'_m P_m\right)} \tag{4.11}$$

for suitable constants ϑ_k, ϑ'_k in $[0, 1)$.

Let us observe that, since $p_i \geq 2$,

$$\begin{aligned}\log\left(\sum_{j=0}^{k} P_j - \vartheta_k P_k\right) - \log P_k &\leq \log \frac{\sum_{j=0}^{k} P_j}{P_k} \\ &= \log\left(\sum_{j=0}^{k} \prod_{i=j+1}^{k} \frac{1}{p_i}\right) \leq \log 2,\end{aligned}$$

therefore, as before, the ratio above may be replaced by

$$\frac{\log 1/\Lambda_k - \log 1/\Lambda_m}{\log P_k - \log P_m}. \tag{4.12}$$

The thesis follows as before. $\square$

Now we turn to the triple $(\mathcal{A}, \mathcal{H}, D)$.

Proof. (of Theorem 4.20). The result will follow from Propositions 4.21 and 4.22 if we show that the assumptions of Proposition 1.13 are satisfied. Let us observe that

$$A := \max\left(\sup_k \frac{\tilde{\Lambda}_k}{\Lambda_k}, \sup_k \frac{\Lambda_k}{\tilde{\Lambda}_k}\right) = \sup_k \frac{p_{k+1} - 1}{1 - p_{k+1}\lambda_{k+1}}$$

is finite by hypothesis. Also

$$B := \max\left(\sup_m \frac{\sum_{m=0}^{k+1} \tilde{P}_m}{\sum_{m=0}^{k} P_m}, \sup_m \frac{\sum_{m=0}^{k+1} P_m}{\sum_{m=0}^{k} \tilde{P}_m}\right)$$

is finite. Indeed, setting $p = \sup_m p_m$,

$$\begin{aligned}\frac{\sum_{m=0}^{k+1} \tilde{P}_m}{\sum_{m=0}^{k} P_m} &= \frac{\sum_{m=0}^{k+1}(p_{m+1}-1)P_m}{\sum_{m=0}^{k} P_m} \\ &\le (p-1)\left(1+\frac{P_{k+1}}{\sum_{m=0}^{k} P_m}\right) \\ &= (p-1)\left(1+\frac{1}{\sum_{m=0}^{k}\prod_{j=m+1}^{k+1} p_j^{-1}}\right) \le p^2-1,\end{aligned}$$

and the other bound is obtained in the same way. Therefore, if $\sum_{m=0}^{k-1} \tilde{P}_m < x \le \sum_{m=0}^{k} \tilde{P}_m$ and $\sum_{m=0}^{k-1} P_m < y \le \sum_{m=0}^{k} P_m$, then $x/y \le B$, hence

$$\mu_{|D_\ell|^{-1}}(x) = \tilde{\Lambda}_k \le A\Lambda_k = A\mu_{|D_f|^{-1}}(y) \le A\mu_{|D_f|^{-1}}\left(\frac{x}{B}\right).$$

The inequality in the other direction is proved in the same way. $\square$

We conclude this section with a corollary of the theorems above and of Theorem 4.2

Corollary 4.23. *Assume F is a uniformly generated symmetric fractal, which is h-Minkowski measurable, $h \in G_d$. Then $\underline{\delta} = d = \overline{\delta} = \overline{d}_B(F)$.*

5 Appendix. Hölder inequalities for singular traces

For the reader's convenience, we recall some notions from [17] that will be needed in this section.

Let τ_ω be a singular trace on $\mathcal{B}(\mathcal{H})$. Then there is a unique positive linear functional φ on the cone of positive non-increasing right-continuous functions on $[0,\infty)$, which is dilation-invariant (*i.e.* $\alpha\varphi(D_\alpha\mu) = \varphi(\mu)$, where $D_\alpha\mu(t) := \mu(\alpha t)$, $\alpha, t > 0$) and such that $\tau_\omega(a) = \varphi(\mu_a)$, for any positive compact operator a. In particular the domain of the singular trace consists of the elements a for which $\varphi(\mu_a)$ is finite.

It is not known if every positive linear dilation-invariant functional φ gives rise to a singular trace τ_ω on $\mathcal{B}(\mathcal{H})$ via the formula $\tau_\omega(a) := \varphi(\mu_a)$. But this is true if the functional φ is monotone, *i.e.* increasing (which means $S_\mu^\uparrow \le S_\nu^\uparrow$ implies $\varphi(\mu) \le \varphi(\nu)$) or decreasing (which means $S_\mu^\downarrow \le S_\nu^\downarrow$ implies $\varphi(\mu) \le \varphi(\nu)$). This means in particular that all known formulas for singular traces are given by a monotone functional.

We can now state the main result of this appendix.

Theorem 5.1. *For any a, b in the domain of τ_ω, $p, q \in [1,\infty]$ conjugate exponents, there holds*

$$|\tau_\omega(ab)| \le \tau_\omega(|ab|) \le C_p \tau_\omega(|a|^p)^{1/p}\ \tau_\omega(|b|^q)^{1/q},$$

where $C_p := 1 + 2\frac{\sqrt{p-1}}{p} \in [1,2]$.
If τ_ω *is generated by a monotone* φ*, then one can choose* $C_p = 1$*, for any* $p \in [1,\infty]$.

Proof. It is a consequence of the following propositions. □

Proposition 5.2. *For any* a, b *in the domain of* τ_ω*,* $p, q \in [1,\infty]$ *conjugate exponents, there holds*

$$|\tau_\omega(ab)| \le \tau_\omega(|ab|) \le C_p \tau_\omega(|a|^p)^{1/p}\ \tau_\omega(|b|^q)^{1/q},$$

where $C_p := 1 + 2\frac{\sqrt{p-1}}{p} \in [1,2]$.

Proof. Let $\alpha, \beta > 0$. Then, for any $t > 0$, one has $\mu_{|ab|}((\alpha+\beta)t) \le \mu_a(\alpha t)\mu_b(\beta t)$, *i.e.*, $D_{\alpha+\beta}\mu_{|ab|} \le D_\alpha \mu_a D_\beta \mu_b$, so that

$$\begin{aligned}
\tau_\omega(|ab|) &= \varphi(\mu_{|ab|}) = (\alpha+\beta)\varphi(D_{\alpha+\beta}\mu_{|ab|}) \\
&\le (\alpha+\beta)\varphi(D_\alpha\mu_a D_\beta \mu_b) \\
&\le (\alpha+\beta)\left\{ \frac{1}{p}\varphi((D_\alpha\mu_a)^p) + \frac{1}{q}\varphi((D_\beta\mu_b)^q) \right\} \\
&= (\alpha+\beta)\left\{ \frac{1}{\alpha p}\varphi(\mu_a^p) + \frac{1}{\beta q}\varphi(\mu_b^q) \right\} \\
&= (\alpha+\beta)\left\{ \frac{1}{\alpha p}\tau_\omega(|a|^p) + \frac{1}{\beta q}\tau_\omega(|b|^q) \right\},
\end{aligned}$$

where we used Young's inequality and the dilation invariance of φ. Therefore, substituting $a/\tau_\omega(|a|^p)^{1/p}$ for a, and $b/\tau_\omega(|b|^q)^{1/q}$ for b, we get

$$\tau_\omega(|ab|) \le \frac{(\alpha+\beta)(\alpha p + \beta q)}{\alpha\beta pq} \tau_\omega(|a|^p)^{1/p}\tau_\omega(|b|^q)^{1/q}.$$

Set $g(x) := 1 + \frac{x}{p} + \frac{1}{xq}$, so that $g(\beta/\alpha) = \frac{(\alpha+\beta)(\alpha p+\beta q)}{\alpha\beta pq}$. Minimizing g over $(0,\infty)$, we obtain $\min_{\alpha,\beta>0} \frac{(\alpha+\beta)(\alpha p+\beta q)}{\alpha\beta pq} = 1 + 2\frac{\sqrt{p-1}}{p}$, which is easily seen to belong to $[1,2]$. □

For monotone φ's the result is contained in the following propositions, but we need a preliminary result, which is interesting on its own.

Proposition 5.3. *Let* $a, b \in \mathcal{K}(\mathcal{H})$*. Then, for any* $n \in \mathbb{N} \cup \{0\}$,

$$\sum_{i=2n}^{\infty} \mu_i(ab) \le \sum_{i=n}^{\infty} \mu_i(a)\mu_i(b).$$

Proof. Let us first assume that $a, b \ge 0$, and let $ab = v|ab|$ be the polar decomposition of ab. Let $\mathcal{H}'$ be a separable infinite dimensional Hilbert space, and let $v' \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}')$ be a partial isometry with initial space $\ker|ab| \oplus \mathcal{H}'$ and final space $(\operatorname{ran}|ab|)^\perp \oplus \mathcal{H}'$. Set $u^* := (v \oplus 0) + v'$, which is a unitary operator of

$\mathcal{H}\oplus\mathcal{H}'$. Then $(a\oplus 0)(b\oplus 0) = u^*|(a\oplus 0)(b\oplus 0)|$. As $\mu_n(a\oplus 0) = \mu_n(a)$, as long as they are nonzero, we can assume that $|ab| = uab$, with a unitary operator u.

Let us now recall that

$$\begin{aligned} S^{\downarrow}_{2n}(ab) &:= \sum_{i=2n}^{\infty} \mu_i(ab) = \inf_{p\in\mathcal{P}_{2n}} \mathrm{Tr}(|ab|p^{\perp}) \\ &= \inf_{p,q\in\mathcal{P}_n} \mathrm{Tr}((p\vee q)^{\perp} uab(p\vee q)^{\perp}), \end{aligned}$$

where $\mathcal{P}_n := \{p \in Proj(\mathcal{H}\oplus\mathcal{H}') : p \leq \ker(ab)^{\perp},\ \mathrm{Tr}(p) \leq n\}$. Let us denote by $e_n(x)$ the orthogonal projection on the eigenspace of the compact operator x associated to the largest n eigenvalues. Choose $p := e_n(uau^*) \in \mathcal{P}_n$ and $q := e_n(b) \in \mathcal{P}_n$, and denote by $a_n := ae_n(a)^{\perp}$, $b_n := bq^{\perp}$, so that $p^{\perp}uau^* = ua_nu^*$. Then

$$\begin{aligned} S^{\downarrow}_{2n}(ab) &\leq \mathrm{Tr}((p\vee q)^{\perp} p^{\perp} uau^* ubq^{\perp}(p\vee q)^{\perp}) \\ &= \mathrm{Tr}((p\vee q)^{\perp} ua_nu^* ub_n(p\vee q)^{\perp}) \\ &= |\,\mathrm{Tr}((p\vee q)^{\perp} ua_nb_n)| \leq \mathrm{Tr}(|(p\vee q)^{\perp} ua_nb_n|) \\ &\leq \mathrm{Tr}(|ua_nb_n|) = \mathrm{Tr}(|a_nb_n|) \\ &= \sum_{i=0}^{\infty} \mu_i(a_nb_n) \leq \sum_{i=0}^{\infty} \mu_i(a_n)\mu_i(b_n) \\ &= \sum_{i=0}^{\infty} \mu_{n+i}(a)\mu_{n+i}(b), \end{aligned}$$

where the last inequality is Weyl's inequality ([15], page 49).

In the case where a, b are arbitrary compact operators, let $a = u|a|$, $b = |b^*|v$, $ab = w|ab|$ be polar decompositions. Then

$$\begin{aligned} S^{\downarrow}_{2n}(ab) &= S^{\downarrow}_{2n}(|ab|) = S^{\downarrow}_{2n}(w^*ab) = S^{\downarrow}_{2n}(w^*u|a||b^*|v) \\ &\leq \|w^*u\|\|v\| S^{\downarrow}_{2n}(|a||b^*|) \\ &\leq S^{\downarrow}_{2n}(|a||b^*|) \\ &\leq \sum_{i=n}^{\infty} \mu_i(|a|)\mu_i(|b^*|) \\ &= \sum_{i=n}^{\infty} \mu_i(a)\mu_i(b), \end{aligned}$$

where we used $\mu_i(|x|) = \mu_i(x) = \mu_i(x^*)$, and, in the last inequality, the thesis already proved for positive compact operators. □

Proposition 5.4. *Let φ be a positive linear dilation invariant monotone functional, and let $\tau_\omega(a) = \varphi(\mu_a)$ be the singular trace it defines. Then, for any a, b in the domain of τ_ω, $p, q \in [1,\infty]$ conjugate exponents, there holds*

$$|\tau_\omega(ab)| \leq \tau_\omega(|ab|) \leq \tau_\omega(|a|^p)^{1/p}\ \tau_\omega(|b|^q)^{1/q}.$$

Proof. (i) Let us first assume that φ is increasing. Let us introduce the functions $\mu_a(t) := \mu_n(a), t \in [n, n+1)$, and analogously for μ_b. Then Weyl's inequality reads as follows

$$\int_0^t \mu_{ab}(s)ds \leq \int_0^t \mu_a(s)\mu_b(s)ds.$$

As φ is increasing, we get

$$\begin{aligned}\tau_\omega(|ab|) &= \varphi(\mu_{ab}) \leq \varphi(\mu_a\mu_b) \\ &\leq \frac{1}{p}\varphi(\mu_{a^p}) + \frac{1}{q}\varphi(\mu_{b^q}) \\ &= \frac{1}{p}\tau_\omega(a^p) + \frac{1}{q}\tau_\omega(b^q),\end{aligned}$$

where we used Young's inequality for real numbers and the properties of φ. Therefore, substituting $a/\tau_\omega(|a|^p)^{1/p}$ for a, and $b/\tau_\omega(|b|^q)^{1/q}$ for b, we get

$$\tau_\omega(|ab|) \leq \tau_\omega(|a|^p)^{1/p}\tau_\omega(|b|^q)^{1/q}.$$

(ii) Assume now that φ is decreasing. Then the inequality in Proposition 5.3 reads as follows

$$2\int_t^\infty \mu_{ab}(2s)ds = \int_{2t}^\infty \mu_{ab}(s)ds \leq \int_t^\infty \mu_a(s)\mu_b(s)ds.$$

As φ is decreasing and dilation invariant, we get

$$\begin{aligned}\tau_\omega(|ab|) &= \varphi(\mu_{ab}) = 2\varphi(D_2\mu_{ab}) \leq \varphi(\mu_a\mu_b) \\ &\leq \frac{1}{p}\varphi(\mu_{a^p}) + \frac{1}{q}\varphi(\mu_{b^q}) \\ &= \frac{1}{p}\tau_\omega(a^p) + \frac{1}{q}\tau_\omega(b^q),\end{aligned}$$

where we used Young's inequality for real numbers and the properties of φ. Therefore, the thesis follows as in (i). $\square$

Acknowledgement . We thank Michel L. Lapidus for discussions, and for pointing out the relevance of some results in [25] for our lines of research. We also thank Alain Connes for suggestions.

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