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Fractals in Noncommutative Geometry

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Abstract. To any spectral triple $(\mathcal{A}, D, \mathcal{H})$ a dimension d is associated, in analogy with the Hausdorff dimension for metric spaces. Indeed d is the unique number, if any, such that $|D|^{-d}$ has non trivial logarithmic Dixmier trace. Moreover, when $d \in (0, \infty)$, there always exists a singular trace which is finite nonzero on $|D|^{-d}$, giving rise to a noncommutative integration on $\mathcal{A}$.

Such results are applied to fractals in $\mathbb{R}$, using Connes' spectral triple, and to limit fractals in $\mathbb{R}^n$, a class which generalises self-similar fractals, using a new spectral triple. The noncommutative dimension or measure can be computed in some cases. They are shown to coincide with the (classical) Hausdorff dimension and measure in the case of self-similar fractals.

1 Introduction.

This paper is both a survey and an announcement of results concerning singular traces on $\mathcal{B}(\mathcal{H})$, and their application to the study of fractals in the framework of noncommutative geometry.

Alain Connes' noncommutative geometry is a relatively young discipline founded some twenty years ago, but it is rapidly developing both in theory and the applications (see *e.g.* the books by Connes [4], Gracia-Bondia et al. [8], Connes-Moscovici recent papers, [5], etc.). In all of them, Dixmier logarithmic trace (or its companion Wodzicki noncommutative residue [17]) plays an important role, as providing the proper analogue of integration in the noncommutative context.

One aspect of noncommutative geometry, or, more precisely, of the notion of spectral triple, is that it is broad enough to treat also commutative singular spaces,

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which are too irregular to be treated with the instruments of Riemannian geometry, as *e.g.* fractals (cf. Connes' book). We observe that, in this case, a lot of different singular traces, more general than the logarithmic one, naturally appear. Indeed, in [2], completing results previously obtained by Dixmier [6] and Varga [16], singular traces on $\mathcal{B}(\mathcal{H})$ have been studied and classified, and have been discovered singular traces whose natural domain lies inside $\mathcal{L}^1$.

Our approach to the study of fractals by means of noncommutative geometry recovers the known results on Hausdorff dimension and measure for the class of self-similar fractals. Moreover it has motivated the definition of dimension and of Hausdorff (and Hausdorff-Besicovitch) measure in the abstract setting of spectral triples, because of the strong analogies with the classical case.

One virtue of working with general singular traces, and not just with the logarithmic one, is that we can see the dimension of a spectral triple as a number which produces a noncommutative measure, *i.e.* a linear functional on the spectral triple. Such functional is not, in general, based on the logarithmic trace. Moreover, these more general singular traces appear naturally in the study of the class of limit fractals.

The paper is organised as follows. The second section contains a survey of known results on singular traces on $\mathcal{B}(\mathcal{H})$ [2, 6, 16]. Moreover the notions of order and exponent of singular traceability of a compact operator are introduced, and some of their recently found properties are described. Examples illustrating the notion of exponent of singular traceability constitute section three.

In the fourth section, the notion of dimension of a spectral triple is introduced, together with the associated Hausdorff-Besicovitch functionals, which are constructed by means of singular traces.

Then we recall known results on fractal sets, and introduce a new class of fractals, which we call limit fractals.

The sixth section contains an announcement of our results on fractals in $\mathbb{R}$, approached by means of the spectral triple introduced by Connes.

In the last section, we associate a spectral triple to limit fractals in $\mathbb{R}^n$, which is different from Connes' one. In the self-similar case, this triple recovers Hausdorff dimension and measure. Partial results are described in the general case.

2 Singular traces on the compact operators of a Hilbert space.

In this section we present the theory of singular traces on $\mathcal{B}(\mathcal{H})$ as it was developed by Dixmier [6], who first showed their existence, and then in [16], [2] and [10].

A singular trace on $\mathcal{B}(\mathcal{H})$ is a tracial weight vanishing on the finite rank projections. Any tracial weight is finite on an ideal contained in $\mathcal{K}(\mathcal{H})$ and may be decomposed as a sum of a singular trace and a multiple of the normal trace. Therefore the study of (non-normal) traces on $\mathcal{B}(\mathcal{H})$ is the same as the study of singular traces. Moreover, making use of unitary invariance, a singular trace should depend only on the eigenvalue asymptotics, namely, if A and B are positive compact operators on $\mathcal{H}$ and $\mu_n(A) = \mu_n(B) + o(\mu_n(B))$, μ_n denoting the n-th eigenvalue, then $\tau(A) = \tau(B)$ for any singular trace τ. The main problem about singular traces is therefore to detect which asymptotics may be "resummed" by a suitable singular trace, that is to say, which operators are singularly traceable.

In order to state the most general result in this respect we need some notation. Let A be a compact operator. Then we denote by $\{\mu_n(A)\}$ the sequence of the eigenvalues of $|A|$, arranged in non-increasing order and counted with multiplicity. We consider also the (integral) sequence $\{S_n(A)\}$ defined as follows:

$$S_n(A) := \begin{cases} \sum_{k=1}^n \mu_k(A) & A \notin \mathcal{L}^1 \\ \sum_{k=n+1}^\infty \mu_k(A) & A \in \mathcal{L}^1, \end{cases}$$

where $\mathcal{L}^1$ denotes the ideal of trace-class operators. We call a compact operator *singularly traceable* if there exists a singular trace which is finite non-zero on $|A|$. We observe that the domain of such singular trace should necessarily contain the ideal $\mathcal{I}(A)$ generated by A. A compact operator is called *eccentric* if

$$\frac{S_{2n_k}(A)}{S_{n_k}(A)} \to 1 \tag{1}$$

for a suitable subsequence n_k. Then the following theorem holds.

Theorem 2.1 *A positive compact operator A is singularly traceable iff it is eccentric. In this case there exists a sequence n_k such that condition (1) is satisfied and, for any generalised limit* Lim_ω *on ℓ^∞, the positive functional*

$$\tau_\omega(B) = \begin{cases} \mathrm{Lim}_\omega \left(\left\{ \frac{S_{n_k}(B)}{S_{n_k}(A)} \right\} \right) & B \in \mathcal{I}(A)_+ \\ +\infty & B \notin \mathcal{I}(A), \ B > 0, \end{cases}$$

is a singular trace whose domain is the ideal $\mathcal{I}(A)$ generated by A.

Now we give a sufficient condition to ensure eccentricity. It is based on the notion of order of infinitesimal.

Definition 2.2
(i) For $A \in \mathcal{K}(\mathcal{H})$ we define *order of infinitesimal* of A

$$\mathrm{ord}(A) := \liminf_{n\to\infty} \frac{\log \mu_n(A)}{\log(1/n)} ,$$

(ii) $\alpha \in (0,\infty)$ is called an *exponent of singular traceability* for $A \in \mathcal{K}(\mathcal{H})$ if there is a singular trace τ on $\mathcal{B}(\mathcal{H})$ such that $\tau(A^\alpha) = 1$.

Remark 2.3
(i) In [10] we used ord_∞ instead of ord..
(ii) If $A \in \mathcal{K}(\mathcal{H})_+$, then for any $\alpha > 0$, $\mathrm{ord}(A^\alpha) = \alpha\ \mathrm{ord}(A)$.

Theorem 2.4
(i) *Let $A \in \mathcal{K}(\mathcal{H})$ be s.t.* $\mathrm{ord}(A) = 1$. *Then A is eccentric.*
(ii) *If* $\mathrm{ord}(A) \in (0,\infty)$*, then* $\mathrm{ord}(A)^{-1}$ *is an exponent of singular traceability.*

Because of its importance in determining the eccentricity property of an operator, therefore the existence of a non-trivial singular trace, we give alternative ways of computing the order of an operator. Recall

Definition 2.5

$$\mathcal{L}^{1,\infty} := \{A \in \mathcal{K}(\mathcal{H}) : \sum_{k=1}^n \mu_k(A) = O(\log n)\},$$

$$\mathcal{L}^{1,\infty}_0 := \{A \in \mathcal{K}(\mathcal{H}) : \sum_{k=1}^n \mu_k(A) = o(\log n)\}.$$

Theorem 2.6 *Let* $A \in \mathcal{K}(\mathcal{H})_+$. *Then* $\mathrm{ord}(A) = \inf\{\alpha > 0 : A^\alpha \in \mathcal{L}^1\} = \inf\{\alpha > 0 : A^\alpha \in \mathcal{L}^{1,\infty}\} = \sup\{\alpha > 0 : A^\alpha \in \mathcal{L}_0^{1,\infty}\}$.

Now we associate to any compact operator A two numbers, which give bounds for singular traceability. We denote by μ_A the locally constant function defined by $\mu(x) \equiv \mu_A(x) := \mu_n$ when $x \in [n, n+1)$, $n \in \mathbb{N}$, and by $f \equiv f_A$ the increasing, diverging function determined by $f(t) = -\log \mu(e^t)$.

Definition 2.7 Let A be a compact operator, $f \equiv f_A$ the increasing, diverging function defined before. Then we set

$$\begin{aligned} \underline{c}(A) &= \left(\lim_{h\to\infty} \limsup_{t\to\infty} \frac{f(t+h) - f(t)}{h} \right)^{-1}, \\ \overline{c}(A) &= \left(\lim_{h\to\infty} \liminf_{t\to\infty} \frac{f(t+h) - f(t)}{h} \right)^{-1}. \end{aligned}$$

Theorem 2.8 *Let A be a compact operator. Then the two limits above exist, and, if α is an exponent of singular traceability, then necessarily $\alpha \in [\underline{c}(A), \overline{c}(A)]$. In particular $\underline{c}(A) \le \mathrm{ord}(A)^{-1} \le \overline{c}(A)$.*

The first result on singular traceability is due to Dixmier, who showed in [6] that $\frac{S_{2n}(A)}{S_n(A)} \to 1$ is a sufficient condition for singular traceability when $A \notin \mathcal{L}^1$. Then Varga proved that the eccentricity condition is necessary and sufficient when $A \notin \mathcal{L}^1$ [16]. Finally it was observed in [1] that singular traces may be non-trivial on trace-class operators, while Theorem 2.1 in the previous form is contained in [2]. Theorem 2.4 is in [10], while the proof of Theorems 2.6, 2.8 will appear in [11].

3 Examples

This section is devoted to some examples, where the necessary condition in Theorem 2.8 is sufficient. In particular in the first class of examples $\underline{c}$ and $\overline{c}$ are finite non-zero, and the exponents of singular traceability are exactly the elements of $[\underline{c}, \overline{c}]$. In the second class of examples $\underline{c} = 0$ and $\overline{c} = \infty$, and all positive numbers are exponents of singular traceability.

3.1 On a class of operators for which all $\gamma \in [\underline{c}, \overline{c}]$ are indices of singularly traceability. In this subsection we will use the following notation:

$$\begin{aligned} \sigma^{(\gamma)}(x) &:= \int_1^x \mu(y)^\gamma dy, \qquad \sigma(x) = \sigma^{(1)}(x) \\ s^{(\gamma)}(x) &:= \int_x^\infty \mu(y)^\gamma dy, \qquad s(x) = s^{(1)}(x) \end{aligned}$$

and the property (analogous to Theorem 2.1, cf. [9]) that T is singularly traceable if and only if 1 is a limit point of $\frac{\sigma(x)}{\sigma(2x)}$ or $\frac{s(x)}{s(2x)}$ as the case may be.

Let us choose two numbers $0 < \beta \le \alpha$ and a non-decreasing sequence $\{a_n\}_{n\in\mathbb{N}}$, $a_0 = 0$, $a_1 > 0$.

Set

$$b_n = \sum_{k=0}^{n} a_k$$
$$\varphi(t) = \begin{cases} \alpha & t \in [b_n, b_{n+1}), n \text{ even} \\ \beta & t \in [b_n, b_{n+1}), n \text{ odd} \end{cases}$$
$$f(t) = \int_0^t \varphi(s)ds$$

Observe that f is nondecreasing and goes to ∞ as $t \to \infty$; as a consequence, $\mu(x) = e^{-f(\log x)}$ is nonincreasing and goes to 0. We choose a compact operator T such that $\mu_n(T) = \mu(n)$, and prove the following.

Theorem 3.1 *T^γ is singularly traceable iff $\gamma \in [\underline{c}(T), \overline{c}(T)]$*

Lemma 3.2 *If $\sup a_n = \lim a_n = a < \infty$, then $\underline{c} = \overline{c}$.*

Proof If $k > 1$, $\exists n_0$ such that $a \geq a_n > a(1 - 1/k)$ for all $n \geq n_0$. Then, on an interval $[t, t + ka]$, $t \geq n_0$, there are $k/2 \pm 1$ intervals where $\varphi = \alpha$ and $k/2 \pm 1$ intervals where $\varphi = \beta$, hence

$$\begin{aligned} \frac{f(t+ka) - f(t)}{ka} &= \frac{1}{ka} \int_t^{t+ka} \varphi(s)ds \\ &= \frac{1}{ka} \left(a\alpha \left(\frac{k}{2} \pm 1 \right) + a\beta \left(\frac{k}{2} \pm 1 \right) \right) \\ &= \frac{\alpha + \beta}{2} + \frac{\pm\alpha \pm \beta}{k} \end{aligned}$$

which implies $\underline{c} = \overline{c} = \left(\frac{\alpha+\beta}{2} \right)^{-1}$. □

Lemma 3.3 *If $\sup a_n = \infty$ then $\underline{c} = 1/\alpha$, $\overline{c} = 1/\beta$.*

Proof For all $h > 0$ $\exists n_0$ such that $n \geq n_0$ implies $a_n > h$; hence $b_{n+1} - b_n = a_{n+1} > h$ and

$$\frac{f(b_n + h) - f(b_n)}{h} = \begin{cases} \alpha & n \text{ even} \\ \beta & n \text{ odd} \end{cases}$$

which implies $\overline{c}^{-1} \leq \beta$, $\underline{c}^{-1} \geq \alpha$. On the other hand $\beta \geq \frac{f(t+h)-f(t)}{h} \geq \alpha$, which implies the thesis. □

Proof (*of Theorem 3.1.*) If $a_n \to a < \infty$, then $\underline{c} = \overline{c} = \text{ord}(T)^{-1}$ by Theorem 2.8 and Lemma 3.2, hence the thesis follows by Theorem 2.4. The same argument applies if $\alpha = \beta$, therefore we may assume $\alpha < \beta$. Because of Theorem 2.8, the claim is proved if we show that, assuming $a_n \to \infty$ and $\alpha < \beta$, T^γ is singularly traceable for any $\gamma \in [\underline{c}, \overline{c}]$.

Observations:

(1) $t \in [b_{2n}, b_{2n+1})$ implies

$$\begin{aligned} f(t) = f(b_{2n}) + \int_{b_{2n}}^t \varphi &= f(b_{2n}) + (t - b_{2n})\alpha \\ = f(b_{2n+1}) - \int_t^{b_{2n+1}} \varphi &= f(b_{2n+1}) - (b_{2n+1} - t)\alpha. \end{aligned}$$

(2) $t \in [b_{2n-1}, b_{2n})$ implies

$$\begin{aligned} f(t) &= f(b_{2n-1}) + (t - b_{2n-1})\beta \\ &= f(b_{2n}) - (b_{2n} - t)\beta. \end{aligned}$$

Setting $x_n = e^{b_n}$, we get

$$\mu(x) = \begin{cases} \mu(x_{2n}) \left(\frac{x_{2n}}{x}\right)^{\alpha} = \mu(x_{2n+1}) \left(\frac{x_{2n+1}}{x}\right)^{\alpha} & x \in [x_{2n}, x_{2n+1}] \\ \mu(x_{2n-1}) \left(\frac{x_{2n-1}}{x}\right)^{\beta} = \mu(x_{2n}) \left(\frac{x_{2n}}{x}\right)^{\beta} & x \in [x_{2n-1}, x_{2n}]. \end{cases}$$

Now assume $\underline{c} < \gamma < \overline{c}$, which is equivalent to $\alpha\gamma - 1 > 0$ and $\beta\gamma - 1 < 0$. We shall show that, for any such γ, 1 is a limit point of $\sigma^{(\gamma)}(\lambda x)/\sigma^{(\gamma)}(x)$ and of $s^{(\gamma)}(x/\lambda)/s^{(\gamma)}(x)$ for any $\lambda > 1$ (but we only need the case $\lambda = 2$). Indeed

$$0 < \frac{\sigma^{(\gamma)}(\lambda x_{2n+1})}{\sigma^{(\gamma)}(x_{2n+1})} - 1 \leq \frac{\int_{x_{2n+1}}^{\lambda x_{2n+1}} \left(\frac{x_{2n+1}}{x}\right)^{\beta\gamma} dx}{\int_{x_{2n}}^{x_{2n+1}} \left(\frac{x_{2n+1}}{x}\right)^{\alpha\gamma} dx} = \frac{\alpha\gamma - 1}{1 - \beta\gamma} \quad \frac{\lambda^{1-\beta\gamma} - 1}{e^{a_{2n+1}(\alpha\gamma-1)} - 1} \to 0$$

$$0 < \frac{s^{(\gamma)}(\frac{x_{2n-1}}{\lambda})}{s^{(\gamma)}(x_{2n-1})} - 1 \leq \frac{\int_{\frac{x_{2n-1}}{\lambda}}^{x_{2n-1}} \left(\frac{x_{2n-1}}{x}\right)^{\alpha\gamma} dx}{\int_{x_{2n-1}}^{x_{2n}} \left(\frac{x_{2n-1}}{x}\right)^{\beta\gamma} dx} = \frac{1 - \beta\gamma}{\alpha\gamma - 1} \quad \frac{\lambda^{\alpha\gamma-1} - 1}{e^{a_{2n}(1-\beta\gamma)} - 1} \to 0$$

Let now $\gamma = \underline{c} = 1/\alpha$; then

$$0 < \frac{\sigma^{(1/\alpha)}(\lambda x_{2n+1})}{\sigma^{(1/\alpha)}(x_{2n+1})} - 1 \leq \frac{\int_{x_{2n+1}}^{\lambda x_{2n+1}} \left(\frac{x_{2n+1}}{x}\right)^{\beta/\alpha} dx}{\int_{x_{2n}}^{x_{2n+1}} \frac{x_{2n+1}}{x} dx} = \frac{1}{1 - \beta/\alpha} \quad \frac{\lambda^{1-\beta/\alpha} - 1}{a_{2n+1}} \to 0$$

$$0 < \frac{s^{(1/\alpha)}(\frac{x_{2n-1}}{\lambda})}{s^{(1/\alpha)}(x_{2n-1})} - 1 \leq \frac{\int_{\frac{x_{2n-1}}{\lambda}}^{x_{2n-1}} \frac{x_{2n-1}}{x} dx}{\int_{x_{2n-1}}^{x_{2n}} \left(\frac{x_{2n-1}}{x}\right)^{\beta/\alpha} dx} = (1 - \beta/\alpha) \quad \frac{\log\lambda}{e^{a_{2n}(1-\beta\gamma)} - 1} \to 0$$

Finally, let $\gamma = \overline{c} = 1/\beta$; then

$$0 < \frac{\sigma^{(1/\beta)}(\lambda x_{2n+1})}{\sigma^{(1/\beta)}(x_{2n+1})} - 1 \leq \frac{\int_{x_{2n+1}}^{\lambda x_{2n+1}} \frac{x_{2n+1}}{x} dx}{\int_{x_{2n}}^{x_{2n+1}} \left(\frac{x_{2n+1}}{x}\right)^{\alpha/\beta} dx} = (\alpha/\beta - 1) \quad \frac{\log\lambda}{e^{a_{2n+1}(\alpha/\beta-1)} - 1} \to 0$$

$$0 < \frac{s^{(1/\beta)}(\frac{x_{2n-1}}{\lambda})}{s^{(1/\beta)}(x_{2n-1})} - 1 \leq \frac{\int_{\frac{x_{2n-1}}{\lambda}}^{x_{2n-1}} \left(\frac{x_{2n-1}}{x}\right)^{\alpha/\beta} dx}{\int_{x_{2n-1}}^{x_{2n}} \frac{x_{2n-1}}{x} dx} = \frac{1}{\alpha/\beta - 1} \quad \frac{\lambda^{\alpha/\beta-1} - 1}{a_{2n}} \to 0$$

□

Remark 3.4 It may happen that $\underline{c} < \underline{d}$ and $\overline{d} < \overline{c}$, where

$$\underline{d} = \left(\limsup_{t\to\infty} \frac{f(t)}{t}\right)^{-1}, \qquad \overline{d} = \left(\liminf_{t\to\infty} \frac{f(t)}{t}\right)^{-1} = \mathrm{ord}(T)^{-1}.$$

Choose $a_n = n$, $\beta < \alpha$. Then $b_n = \sum_{k=0}^{n} k = \frac{n(n+1)}{2}$. If $t \in [b_{2n}, b_{2n+1}]$, then

$$f(b_{2n}) \leq f(t) \leq f(b_{2n+2}) = f(b_{2n}) + \alpha(n+1) + \beta(n+2),$$

Hence

$$\frac{f(b_{2n})}{(n+1)(2n+1)} \leq \frac{f(t)}{t} \leq \frac{f(b_{2n})}{n(2n+1)} + \frac{\alpha(n+1) + \beta(n+2)}{n(2n+1)}$$

Finally,

$$f(b_{2n}) = \sum_{j=1}^{n} \beta(2j) + \sum_{j=1}^{n} \alpha(2j-1) = n(n+1)\beta + n^2\alpha,$$

which implies $\frac{f(b_{2n})}{2n^2} \to \frac{\alpha+\beta}{2}$, therefore $\lim \frac{f(t)}{t} = \frac{\alpha+\beta}{2}$.

3.2 On a class of operators for which all positive numbers are indices of singularly traceability. Choose an increasing sequence b_n such that $e^{b_n} \in \mathbb{N}$, $b_{n+1} - b_n \to \infty$, $b_0 = 0$, and set

$$f(t) = b_n, \quad b_{n-1} < t \leq b_n.$$

As before, set $\mu(x) = e^{-f(\log x)}$, namely

$$\mu(x) = \frac{1}{x_n}, \quad x_{n-1} < x \leq x_n,$$

where $x_n = e^{b_n} \in \mathbb{N}$, hence $\frac{x_n}{x_{n+1}} \to 0$. We choose a compact operator T such that $\mu_n(T) = \mu(n)$, and prove the following.

Theorem 3.5 *T^α is singularly traceable for any $\alpha > 0$.*

The proof of this statement requires some steps. First we observe that since $\forall h > 0 \; \exists n_0$ such that $n > n_0 \Rightarrow b_{n+1} > b_n + h$, we have

$$\frac{f(b_{n+1}) - f(b_{n+1} - h)}{h} = 0, \quad n > n_0 \Rightarrow \liminf_{t\to\infty} \frac{f(t+h) - f(t)}{h} = 0,$$

$$\frac{f(b_n + h) - f(b_n)}{h} = \frac{b_{n+1} - b_n}{h}, \quad n > n_0 \Rightarrow \limsup_{t\to\infty} \frac{f(t+h) - f(t)}{h} = +\infty,$$

namely $\underline{c} = 0$, $\overline{c} = \infty$. Also

$$\mathrm{ord}(T) = \liminf_{t\to\infty} \frac{f(t)}{t} = \lim_{t\to\infty} \frac{f(b_n)}{b_n} = 1.$$

Proposition 3.6 *Let A be a compact operator. If $\liminf \frac{\mu_{n+1}(A)}{\mu_n(A)} = 0$, then A^α is singularly traceable for any $\alpha < (\mathrm{ord}(A))^{-1}$.*

Proof By Theorem 2.6, when $\alpha < (\mathrm{ord}(A))^{-1}$, A^α is not trace class; moreover $\mathrm{ord}(A^\alpha) = \alpha\mathrm{ord}(A)$ and $\mu_n(A^\alpha) = (\mu_n(A))^\alpha$. Therefore we may assume A not to be trace class and $\alpha = 1$. Then, let n_k be such that $\frac{\mu_{n_k+1}(A)}{\mu_{n_k}(A)} \to 0$. We have

$$1 \leq \frac{\sigma_{2n_k}}{\sigma_{n_k}} = 1 + \frac{\sum_{j=n_k+1}^{2n_k} \mu_j}{\sum_{j=1}^{n_k} \mu_j} \leq 1 + \frac{n_k \mu_{n_k+1}}{n_k \mu_{n_k}} \to 1.$$

The thesis follows by Theorem 2.1. ☐

Corollary 3.7 *T^α is singularly traceable for $\alpha \in (0,1)$.*

Proof Indeed

$$\frac{\mu(x_{n+1})}{\mu(x_n)} = e^{-(b_{n+1}-b_n)} \to 0.$$

☐

Lemma 3.8 $\frac{s^{(\alpha)}(x_{n+1})}{x_{n+1}^{1-\alpha}} \to 0$*, for any $\alpha > 1$.*

Proof First we show that

$$\lim_{n\to\infty} \frac{\sum_{k=n+1}^{\infty} x_k^{-\varepsilon}}{x_n^{-\varepsilon}} = 0, \qquad \forall\varepsilon > 0. \tag{2}$$

Indeed

$$\frac{\sum_{k=n+1}^{\infty} x_k^{-\varepsilon}}{\sum_{k=n}^{\infty} x_k^{-\varepsilon}} = \frac{\sum_{k=n}^{\infty} \left(\frac{x_{k+1}}{x_k}\right)^{-\varepsilon} x_k^{-\varepsilon}}{\sum_{k=n}^{\infty} x_k^{-\varepsilon}} \leq \sup_{k\geq n} \left(\frac{x_{k+1}}{x_k}\right)^{-\varepsilon} \to 0.$$

Therefore

$$\frac{x_n^{-\varepsilon}}{\sum_{k=n+1}^{\infty} x_k^{-\varepsilon}} = \frac{\sum_{k=n}^{\infty} x_k^{-\varepsilon}}{\sum_{k=n+1}^{\infty} x_k^{-\varepsilon}} - 1 \to \infty.$$

Now we observe that

$$s^{(\alpha)}(x_{n+1}) = \sum_{k=x_{n+1}+1}^{\infty} \mu_k^{\alpha} = \sum_{p=n+1}^{\infty} x_{p+1}^{(1-\alpha)} \left(1 - \frac{x_p}{x_{p+1}}\right) \leq \sum_{p=n+1}^{\infty} x_{p+1}^{(1-\alpha)}.$$

The thesis follows from equation (2). □

Proposition 3.9 *T^{α} is singularly traceable for $\alpha > 1$.*

Proof

$$\begin{aligned}
\frac{s_{2x_n}^{(\alpha)}}{s_{x_n}^{(\alpha)}} &= \frac{\sum_{k=2x_n+1}^{x_{n+1}} \mu_k + s_{x_{n+1}}^{(\alpha)}}{\sum_{k=x_n+1}^{x_{n+1}} \mu_k + s_{x_{n+1}}^{(\alpha)}} \\
&= \frac{(x_{n+1} - 2x_n)x_{n+1}^{-\alpha} + s_{x_{n+1}}^{(\alpha)}}{(x_{n+1} - x_n)x_{n+1}^{-\alpha} + s_{x_{n+1}}^{(\alpha)}} \\
&= \frac{\left(1 - 2\frac{x_n}{x_{n+1}}\right) + \frac{s_{x_{n+1}}^{(\alpha)}}{x_{n+1}^{1-\alpha}}}{\left(1 - \frac{x_n}{x_{n+1}}\right) + \frac{s_{x_{n+1}}^{(\alpha)}}{x_{n+1}^{1-\alpha}}} \to 1.
\end{aligned}$$

□

4 Some results on noncommutative geometric measure theory

In this section we shall discuss a definition of dimension in noncommutative geometry in the spirit of geometric measure theory.

As it is known, the geometric measure for a noncommutative manifold is defined via a singular trace applied to a suitable power of some geometric operator (e.g. the Dirac operator of the spectral triple of Alain Connes). Connes showed that such procedure recovers the usual volume in the case of compact Riemannian manifolds, and more generally the Hausdorff measure in some interesting examples [4].

Let us recall that $(\mathcal{A}, D, \mathcal{H})$ is called a *spectral triple* when $\mathcal{A}$ is an algebra acting on the Hilbert space $\mathcal{H}$, D is a self adjoint operator on the same Hilbert space such that $[D, a]$ is bounded for any $a \in \mathcal{A}$, and D has compact resolvent. In the following we shall assume that 0 is not an eigenvalue of D, the general case being recovered by replacing D with $D|_{\ker(D)^{\perp}}$. Such a triple is called d^+-summable, $d \in (0, \infty)$, when $|D|^{-d} \in \mathcal{L}^{1,\infty}$.

The noncommutative version of the integral on functions is given by the formula $a \mapsto \mathrm{Tr}_{\omega}(a|D|^{-d})$, where Tr_{ω} is a (logarithmic) Dixmier trace, i.e. a singular trace summing logarithmic divergences. Of course the preceding formula does not guarantee the non-triviality of the integral, and in fact cohomological assumptions in this direction have been considered [4]. We are interested in different conditions for non-triviality. In this connection, we observe that the previous noncommutative integration is always trivial when $|D|^{-d}$ belongs to $\mathcal{L}_0^{1,\infty}$.

Proposition 4.1 *Let $(\mathcal{A}, D, \mathcal{H})$ be a spectral triple. If d is an exponent of singular traceability for $|D|^{-1}$, namely there is a singular trace τ which is non-trivial on the ideal generated by $|D|^{-d}$, then the functional $a \mapsto \tau(a|D|^{-d})$ is a non-trivial trace state on the algebra $\mathcal{A}$.*
We call it a Hausdorff-Besicovitch functional on $(\mathcal{A}, D, \mathcal{H})$. Under suitable conditions (see [3]*) it gives rise to a trace on $\Omega\mathcal{A}$.*

Remark 4.2 Any such trace is a candidate for a geometric measure in noncommutative geometry. Indeed, when $(\mathcal{A}, D, \mathcal{H})$ is associated to an n-dimensional compact manifold M, or to the fractal sets in [4], the singular trace is the logarithmic Dixmier trace, and the associated functional corresponds to the Hausdorff measure. Therefore the following definition is natural.

Definition 4.3 Let $(\mathcal{A}, D, \mathcal{H})$ be a spectral triple, Tr_ω the logarithmic Dixmier trace.
(i) We call α-dimensional Hausdorff functional the map $a \mapsto Tr_\omega(a|D|^{-\alpha})$;
(ii) we call (Hausdorff) dimension of the spectral triple the number

$$d(\mathcal{A}, D, \mathcal{H}) = \inf\{d > 0 : |D|^{-d} \in \mathcal{L}_0^{1,\infty}\} = \sup\{d > 0 : |D|^{-d} \notin \mathcal{L}^{1,\infty}\}.$$

Theorem 4.4 *Assume $d := d(\mathcal{A}, D, \mathcal{H}) \in (0, \infty)$. Then*
(i) d is the unique exponent, if any, such that $\mathcal{H}_d$ is non-trivial;
(ii) $d = \mathrm{ord}(D^{-1})^{-1} = \sup\{\gamma > 0 : |D|^{-\gamma} \notin \mathcal{L}^1\}$; as a consequence it is an exponent of singular traceability;
(iii) let $\zeta(s) = \mathrm{Tr}(|D|^{-s})$, $s > d$. If $(s - d)\zeta(s) \to L \in \mathbb{R}$ as $s \to d^+$, then the Hausdorff-Besicovitch functional associated with $|D|^{-d}$ is indeed the Hausdorff functional (up to the multiplicative constant L).

Proof (i) is in [10], (ii) follows from Theorems 2.4 and 2.6, (iii) follows by the Hardy-Littlewood Theorem [12], cf. Proposition 4, p. 306, [4]. □

Let us observe that the α-dimensional Hausdorff functional depends on the generalised limit procedure ω, however all such functionals coincide on the elements a of $\mathcal{A}$ such that $a|D|^{-d}$ is a measurable operator in the sense of Connes [4]. As in the commutative case, the dimension is the supremum of the α's such that the α-dimensional Hausdorff measure is everywhere infinite and the infimum of the α's such that the α-dimensional Hausdorff measure is identically zero.

Concerning the non-triviality of the d-dimensional Hausdorff functional, we have the same situation as in the classical case. Indeed, according to the previous result, a non-trivial Hausdorff functional is unique but does not necessarily exist. In fact, if the eigenvalue asymptotics of D is e.g. $n \log n$, the Hausdorff dimension is one, but the 1-dimensional Hausdorff measure gives the null functional.

However, if we consider all singular traces, not only the logarithmic ones, and the corresponding functionals on $\mathcal{A}$, as we said, there exists a non trivial functional associated with such a dimension, but such property does not characterize this dimension, in general, namely the exponent of singular traceability is not necessarily unique, cf. the examples in Section 2.

Proposition 4.5 *If* $\underline{c}(|D|^{-1}) = \overline{c}(|D|^{-1}) \in (0,\infty)$*, then* $d(\mathcal{A}, D, \mathcal{H}) = \underline{c} = \overline{c}$ *is the unique exponent of singular traceability of* D^{-1}*. This is the case, in particular, if there exists* $\lim \frac{\mu_n(D^{-1})}{\mu_{2n}(D^{-1})} \in (1,\infty)$.

Proof The first statement follows from 2.8. Since $\lim \frac{\mu_n(D^{-1})}{\mu_{2n}(D^{-1})} = 2^{\underline{c}} = 2^{\overline{c}}$ if it exists, the second statement follows from the first, however it has been proved directly in [10]. □

Remark 4.6 For the spectral triples whose Dirac operator has a spectral asymptotics like $n^\alpha(\log n)^\beta$, we have $d(\mathcal{A}, D, \mathcal{H}) = 1/\alpha$, and the uniqueness result of Proposition 4.5 applies. However, the nontrivial singular trace associated with $|D|^{-1/\alpha}$ by Theorem 2.4 is a logarithmic trace if and only if $\beta = 1$. In this sense, the singular traces associated with a generic eccentric operator generalize the logarithmic Dixmier trace in the same way in which the Besicovitch measure theory generalizes the Hausdorff measure theory.

Remark 4.7 Contrary to the classical case, where there are sets with non-trivial Hausdorff dimension but no non-trivial geometric (*i.e.* Hausdorff or Hausdorff-Besicovitch) measure [15], p. 73, in the noncommutative context, if $d(\mathcal{A}, D, \mathcal{H}) \in (0,\infty)$, there is always a non-trivial geometric measure, whether Hausdorff or (the more general) Hausdorff-Besicovitch.

5 Fractals in $\mathbb{R}^n$. Classical aspects

Let (X,ρ) be a metric space, and let $h : [0,\infty) \to [0,\infty)$ be non-decreasing and right-continuous, with $h(0) = 0$. When $E \subset X$, define, for any $\delta > 0$, $\mathcal{H}^h_\delta(E) := \inf\{\sum_{i=1}^\infty h(\operatorname{diam} A_i) : \cup_i A_i \supset E, \operatorname{diam} A_i \le \delta\}$. Then the *Hausdorff-Besicovitch (outer) measure* of E is defined as

$$\mathcal{H}^h(E) := \lim_{\delta\to 0} \mathcal{H}^h_\delta(E).$$

If $h(t) = t^\alpha$, $\mathcal{H}^\alpha$ is called *Hausdorff (outer) measure* of order $\alpha > 0$.

The number

$$d_H(E) := \sup\{\alpha > 0 : \mathcal{H}^\alpha(E) = +\infty\} = \inf\{\alpha > 0 : \mathcal{H}^\alpha(E) = 0\}$$

is called *Hausdorff dimension* of E.

Let $N_\varepsilon(E)$ be the least number of closed balls of radius $\varepsilon > 0$ necessary to cover E. Then the numbers

$$\overline{d_B}(E) := \limsup_{\varepsilon\to 0^+} \frac{\log N_\varepsilon(E)}{-\log\varepsilon}, \quad \underline{d_B}(E) := \liminf_{\varepsilon\to 0^+} \frac{\log N_\varepsilon(E)}{-\log\varepsilon}$$

are called upper and lower *box dimensions* of E.

In case $X = \mathbb{R}^N$, setting $S_\varepsilon(E) := \{x \in \mathbb{R}^N : \rho(x,E) \le \varepsilon\}$, it is known that $\overline{d_B}(E) = N - \liminf_{\varepsilon\to 0^+} \frac{\log \operatorname{vol} S_\varepsilon(E)}{\log\varepsilon}$ and $\underline{d_B}(E) = N - \limsup_{\varepsilon\to 0^+} \frac{\log \operatorname{vol} S_\varepsilon(E)}{\log\varepsilon}$. E is said *Minkowski measurable* if

$$\lim_{\varepsilon\to 0^+} \frac{\log \operatorname{vol} S_\varepsilon(E)}{\log\varepsilon} = N - d \quad \text{and} \quad \mathcal{M}_d(E) := \lim_{\varepsilon\to 0^+} \frac{\operatorname{vol} S_\varepsilon(E)}{\varepsilon^{N-d}} \in (0,\infty).$$

$\mathcal{M}_d(E)$ is called *Minkowski content* of E.

5.1 Selfsimilar fractals. Let $\{w_j\}_{j=1,\dots,p}$ be contracting similarities of $\mathbb{R}^N$, *i.e.* there are $\lambda_j \in (0,1)$ such that $\|w_j(x) - w_j(y)\| = \lambda_j\|x-y\|$, $x,y \in \mathbb{R}^N$. Denote by $\mathcal{K}(\mathbb{R}^N)$ the family of all non-empty compact subsets of $\mathbb{R}^N$, endowed with the Hausdorff metric, which turns it into a complete metric space. Then $W : K \in \mathcal{K}(\mathbb{R}^N) \to \cup_{j=1}^p w_j(K) \in \mathcal{K}(\mathbb{R}^N)$ is a contraction.

Definition 5.1 The unique non-empty compact subset F of $\mathbb{R}^N$ such that

$$F = W(F) = \bigcup_{j=1}^{p} w_j(F)$$

is called the *self-similar fractal* defined by $\{w_j\}_{j=1,\dots,p}$.

If we denote by $Prob_{\mathcal{K}}(\mathbb{R}^N)$ the set of probability measures on $\mathbb{R}^N$ with compact support endowed with the Hutchinson metric, *i.e.* $d(\mu,\nu) := \sup\{|\int f d\mu - \int f d\nu| : \|f\|_{Lip} \le 1\}$, then the map

$$\begin{array}{rccc} T: & Prob_{\mathcal{K}}(\mathbb{R}^N) & \to & Prob_{\mathcal{K}}(\mathbb{R}^N) \\ & \mu & \mapsto & \sum_{j=1}^p \lambda_j^s \mu \circ w_j^{-1} \end{array}$$

is a contraction, where $s > 0$ is the unique real number, called similarity dimension, satisfying $\sum_{j=1}^p \lambda_j^s = 1$. We then observe that if μ has support K, then $T\mu$ has support $W(K)$. Since the sequence $W^n(K)$ is convergent, it turns out that it is bounded, namely there exists a compact set K_0 containing the supports of all the measures $T^n\mu$. But on the space $Prob(K_0)$ the Hutchinson metric induces the weak* topology, and this space is compact in such topology, hence complete in the Hutchinson metric. Therefore there exists a fixed point of T in $Prob_{\mathcal{K}}(\mathbb{R}^N)$, which is of course unique.

Open Set Condition. The similarities $\{w_j\}_{j=1,\dots,p}$ are said to satisfy the open set condition if there is a non-empty bounded open set $V \subset \mathbb{R}^N$ such that $\cup_{j=1}^p w_j(V) \subset V$ and $w_i(V) \cap w_j(V) = \emptyset$, $i \ne j$. In this case $d_H(F) = \underline{d_B}(F) = \overline{d_B}(F) = s$, and the Hausdorff measure $\mathcal{H}^s$ is non-trivial on F. Therefore $\mathcal{H}^s|_F$ is the unique (up to a constant factor) Borel measure μ, with compact support, such that $\mu(A) = \sum_{j=1}^p \lambda_j^s \mu(w_j^{-1}(A))$, for any Borel subset A of $\mathbb{R}^N$.

It has recently been proved [7] that, if the similarities $\{w_j\}_{j=1,\dots,p}$ satisfy the open set condition and $\log\lambda_1, \dots \log\lambda_p$ generate $(\mathbb{R},+)$ as a minimal closed subgroup, then F is Minkowski measurable.

5.2 Limit fractals. Several generalisations of the class of self-similar fractals have been studied. Here we propose a new one, that we call the class of limit fractals. For its construction we need the following theorem.

Theorem 5.2 *Let (X,ρ) be a complete metric space, $T_n : X \to X$ be such that there are $\lambda_n \in (0,1)$ for which $\rho(T_n x, T_n y) \le \lambda_n \rho(x,y)$, for $x,y \in X$. Assume $\sum_{n=1}^\infty \prod_{j=1}^n \lambda_j < \infty$, and there is $x \in X$ such that $\sup_{n\in\mathbb{N}} \rho(T_n x, x) < \infty$. Then*
(i) $\sup_{n\in\mathbb{N}} \rho(T_n y, y) < \infty$, for any $y \in X$,
(ii) $\lim_{n\to\infty} T_1 \circ T_2 \circ \dots \circ T_n x = x_0 \in X$ for any $x \in X$.

Proof *(i)* $\rho(T_n y, y) \le \rho(T_n y, T_n x) + \rho(T_n x, x) + \rho(x,y) \le (1+\lambda_n)\rho(x,y) + \rho(T_n x, x)$, so that $\sup_{n\in\mathbb{N}} \rho(T_n y, y) \le 2\rho(x,y) + \sup_{n\in\mathbb{N}} \rho(T_n x, x) < \infty$.
(ii) Set $M := \sup_{n\in\mathbb{N}} \rho(T_n x, x) < \infty$, and $S_n := T_1 \circ T_2 \circ \dots \circ T_n$, $n \in \mathbb{N}$. As $\rho(S_{n+1}x, S_n x) \le \lambda_1\lambda_2\cdots\lambda_n \rho(T_{n+1}x, x) \le M\lambda_1\lambda_2\cdots\lambda_n$, there follows, for any $p \in \mathbb{N}$, $\rho(S_{n+p}x, S_n x) \le \rho(S_{n+p}x, S_{n+p-1}x) + \ldots + \rho(S_{n+1}x, S_n x) \le M\sum_{k=n}^{n+p-1} \prod_{j=1}^k \lambda_k \le$

$M \sum_{k=n}^{\infty} \prod_{j=1}^{k} \lambda_k \to 0$, as $n \to \infty$, that is $\{S_n x\}$ is Cauchy in X. Therefore there is $x_0 \in X$ such that $S_n x \to x_0$.
Let us prove that x_0 is independent of x. Indeed, if $y \in X$, then $\rho(S_n x, S_n y) \leq \lambda_1 \lambda_2 \cdots \lambda_n \rho(x, y) \to 0$, as $n \to \infty$, so that $S_n x$ and $S_n y$ have the same limit. □

Remark 5.3 A sufficient condition for $\sum_{n=1}^{\infty} \prod_{j=1}^{n} \lambda_j < \infty$ to hold is

$$\sup_{n \in \mathbb{N}} \lambda_n < 1.$$

We now describe the class of limit fractals. Let $\{w_{nj}\}$, $n \in \mathbb{N}$, $j = 1, \ldots, p_n$, be contracting similarities of $\mathbb{R}^N$, with contraction parameter $\lambda_{nj} \in (0, 1)$. Set $\Sigma := \cup_{n \in \mathbb{N}} \{\sigma : \{1, \ldots, n\} \to \mathbb{N} : \sigma(k) \in \{1, \ldots, p_k\}, k = 1, \ldots, n\}$, and write $w_\sigma := w_{1\sigma(1)} \circ w_{2\sigma(2)} \circ \cdots \circ w_{n\sigma(n)}$, for any $\sigma \in \Sigma$, $|\sigma| = n$. Assume $\sup_{n,j} \lambda_{nj} < 1$ and $\{w_\sigma(x) : \sigma \in \Sigma\}$ is bounded, for some (hence any) $x \in \mathbb{R}^N$. Then, by Theorem 5.2, the sequence of maps $W_n : K \in \mathcal{K}(\mathbb{R}^N) \to \cup_{j=1}^{p_n} w_{nj}(K) \in \mathcal{K}(\mathbb{R}^N)$ is such that $\{W_1 \circ W_2 \circ \cdots \circ W_n(K)\}$ has a limit in $\mathcal{K}(\mathbb{R}^N)$, which is independent of $K \in \mathcal{K}(\mathbb{R}^N)$.

Definition 5.4 The unique compact set F which is the limit of $\{W_1 \circ W_2 \circ \cdots \circ W_n(K)\}_{n \in \mathbb{N}}$ is called the limit fractal defined by $\{w_{nj}\}$. In the particular case that $\lambda_{nj} = \lambda_n$, $j = 1, \ldots, p_n$, $n \in \mathbb{N}$, F is called a translation (limit) fractal. The limit fractal F is said to satisfy the *countably ramified open set condition* if there exists a nonempty bounded open set V in $\mathbb{R}^n$ for which $w_{nj}(V) \subset V$ and $\overline{w_{ni}(V)} \cap \overline{w_{nj}(V)}$ is at most countable, for any n, $i \neq j$.

As before, we may consider the action of the similarities on measures, besides that on sets. Given $s > 0$ we set

$$\begin{array}{rccc} T_n : & Prob_{\mathcal{K}}(\mathbb{R}^N) & \to & Prob_{\mathcal{K}}(\mathbb{R}^N) \\ & \mu & \mapsto & \frac{1}{\sum_{j=1}^{p} \lambda_{nj}^s} \sum_{j=1}^{p} \lambda_{nj}^s \mu \circ w_{nj}^{-1} \end{array}$$

and consider the sequence $\{T_1 \circ T_2 \circ \cdots \circ T_n \mu\}_{n \in \mathbb{N}}$. As before the supports of all such measures are contained in a common compact set, therefore Theorem 5.2 applies and we get a unique limit measure μ_s, depending on the chosen s.

If the countably ramified open set condition holds, the sets $w_{\sigma \cdot i} \overline{V}$, $w_{\sigma \cdot j} \overline{V}$ are essentially disjoint when $i \neq j$, where $\sigma \cdot i$ is the concatenation of strings, and are related by the similarity $w_{ni} \circ w_{nj}^{-1}$, $n = |\sigma| + 1$, therefore the measure μ_s is the unique probability measure with support the limit fractal F which is homogeneous with parameter s. If F is a translation fractal, $w_{ni} \circ w_{nj}^{-1}$ is indeed an isometry, hence μ_s is independent of s and is the unique probability measure on F which is invariant under the mentioned isometries.

5.3 Fractals in $\mathbb{R}$. We now specialise to subsets of $\mathbb{R}$ and survey some known results on compact, totally disconnected subsets of $\mathbb{R}$, without isolated points. Let F be such a set, and denote by $[a, b]$ the least closed interval containing F. Then $[a, b] \setminus F$ is the disjoint union of open intervals (a_n, b_n), where $b_n - a_n \leq b_{n-1} - a_{n-1}$, $n \geq 2$. If F has Lebesgue measure zero, *i.e.* if $\sum_{n=1}^{\infty} (b_n - a_n) = b - a$, then

$$\overline{d_B}(F) = \limsup_{n \to \infty} \frac{\log n}{|\log(b_n - a_n)|}. \tag{3}$$

It has been proved in [13] that, when F is a translation fractal in $\mathbb{R}$, there is a gauge function h such that the corresponding Hausdorff-Besicovitch measure $\mathcal{H}^h$ is non-trivial on F. Moreover, if $\lim_{t \to 0} \frac{\log h(t)}{\log t} = \alpha$, then $d_H(F) = \alpha$. As a consequence, if the countably ramified open set condition holds, $\mathcal{H}^h|_F$ coincides (up to a constant) with the limit measure μ of the previous subsection.

6 Fractals in $\mathbb{R}$. Noncommutative aspects.

Let F be a compact, totally disconnected subset of $\mathbb{R}$, without isolated points, and let a, b, a_n, b_n be as in subsection 5.3. Set $\mathcal{H}_n := \ell^2(\{a_n, b_n\})$, $\mathcal{H} := \oplus_{n=1}^{\infty} \mathcal{H}_n$,

$$D_n := \frac{1}{b_n - a_n} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$D := \oplus_{n=1}^{\infty} D_n$. Consider the action of $C(F)$ on $\mathcal{H}$ by left multiplication: $(f\xi)(x) = f(x)\xi(x)$, $x \in \mathcal{D} := \{a_n, b_n : n \in \mathbb{N}\}$, and define $\mathcal{A} := Lip(F)$. Then

Theorem 6.1 [4]
(i) $(\mathcal{A}, D, \mathcal{H})$ *is a spectral triple*
(ii) the characteristic values of D^{-1} *are the numbers* $b_n - a_n$, $n \in \mathbb{N}$, *each with multiplicity 2.*
If F *is Minkowski measurable, and has box dimension* $d \in (0, 1]$, *then*
(iii) $|D|^{-d} \in \mathcal{L}^{1,\infty}$
(iv) $\mathrm{Tr}_\omega(|D|^{-d}) = 2^d(1 - d)\mathcal{M}_d(F)$.

Statements *(iii)* and *(iv)* follow from results of Lapidus and Pomerance, [14].

Even if F is not Minkowski measurable, we have

Theorem 6.2 $d(\mathcal{A}, D, \mathcal{H}) = \overline{d_B}(F)$. *Therefore, if* $\overline{d_B}(F) \neq 0$, *we get a Hausdorff-Besicovitch functional on the spectral triple, giving rise to a non-trivial measure* μ *on* F.

Proof Follows by equation (3), Theorem 4.4 and Theorem 6.1.(ii). □

Remark 6.3 If F is a limit fractal with countably ramified open set condition, then μ can be explicitely computed, in particular it only depends on F. If F is a translation limit fractal, then μ coincides with the measure in subsection 4.3.

Theorem 6.4 *Let* F *be a self-similar fractal, and* $s \in [0, 1]$ *its Hausdorff dimension. Then* s *is the unique exponent of singular traceability for* D^{-1}, *and the Hausdorff-Besicovitch functional on the spectral triple corresponds to the* s*-dimensional Hausdorff measure on* F.

Proof Define

$$S_j\xi(b) := \begin{cases} \xi(w_j^{-1}(b)) & b \in w_j\mathcal{D} \\ 0 & b \notin w_j\mathcal{D}. \end{cases}$$

Then S_j is an isometry and $|D|^{-d} = \sum_{j=1}^{p} \lambda_j^d S_j |D|^{-d} S_j^*$. Therefore, if d is an exponent of singular traceability for $|D|^{-1}$, the corresponding Hausdorff-Besicovitch functional is homogeneous of order d. This implies that d coincides with s, namely s is the unique exponent of singular traceability, and the Hausdorff-Besicovitch functional corresponds to the d-dimensional Hausdorff measure. □

Remark 6.5 One can show that the noncommutative s-dimensional Hausdorff functional is nontrivial on $C(F)$.

7 Fractals in $\mathbb{R}^n$

In this Section we treat the case of self-similar and limit fractals in $\mathbb{R}^n$. Here the construction of the spectral triple due to Connes does not apply, and we propose a new construction for the Dirac operator. This construction is similar to Connes' in that it is based on a discrete approximation of the fractal. On the other hand it differs from that of Connes since the eigenvalues are not proportional to the size of the "holes" of the fractal, but to the size of the remaining parts. So in general the two constructions do not agree on fractals in $\mathbb{R}$, even though we shall show in many cases that they give rise to the same measure on the fractal.

7.1 Self-similar fractals. Assume that F is a self-similar fractal, constructed via the similarities w_n, $n = 1, \dots p$, satisfying open set condition w.r.t the bounded open set V. Choose two points $x, y \in V$ and consider the points $x_\sigma := w_\sigma x$, $y_\sigma := w_\sigma y$, $\sigma \in \Sigma$. Define the space $\mathcal{H}_\sigma := \ell^2(\{x_\sigma, y_\sigma\})$ with the operator

$$D_\sigma := \frac{1}{d(x_\sigma, y_\sigma)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and set $\mathcal{H} := \oplus_{\sigma\in\Sigma}\mathcal{H}_\sigma$, $D := \oplus_{\sigma\in\Sigma} D_\sigma$. Finally introduce the algebra $\mathcal{A}$ of Lipschitz functions on V, acting by left multiplication on $\mathcal{H}$:

$$(f\xi)(x) = f(x)\xi(x), \quad x \in \{x_\sigma, y_\sigma : \sigma \in \Sigma\}.$$

In this way

$$[D, f] = \bigoplus_{\sigma\in\Sigma} \frac{f(x_\sigma) - f(y_\sigma)}{d(x_\sigma, y_\sigma)} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

namely $(\mathcal{A}, D, \mathcal{H})$ is a spectral triple.

The following theorem holds:

Theorem 7.1 *Let $(\mathcal{A}, D, \mathcal{H})$ be the spectral triple associated with a self-similar fractal F with open set condition as above. Then the dimension d of the triple coincides with the Hausdorff dimension of F, and the noncommutative d-dimensional Hausdorff functional corresponds to the classical Haudorff measure.*

Proof The proof will appear in [11] □

Remark 7.2 We note that Theorem 7.1 implies that the noncommutative dimension and measure do not depend on the starting points x, y. Moreover one can replace the pair with any finite family of pairs without affecting the result.

For translation self-similar fractals in $\mathbb{R}$, namely fractals where all similarity parameters coincide, the starting pairs can be chosen in such a way that the spectral triple of Connes coincides with ours.

In general however this is not the case, and the distance induced by our Dirac operator is different from the original one.

7.2 Limit fractals. Let F be a limit fractal as in Section 5. The spectral triple can be defined exactly as for the self-similar case. Here we shall assume, besides open set condition, also countable ramification, namely $w_{\sigma\cdot i}\overline{V} \cap w_{\sigma\cdot j}\overline{V}$ is at most countable when $|\sigma| = n - 1$, $i, j = 1, \dots, p_n$, $i \neq j$.

Then the following holds:

Theorem 7.3 *Let $(\mathcal{A}, D, \mathcal{H})$ be the spectral triple associated with a limit fractal F with countably ramified open set condition as above. Then for any exponent α of singular traceability for $|D|^{-1}$ the corresponding noncommutative Hausdorff-Besicovitch measure coincides with the limit measure on F with scaling parameter α.*

Proof The proof will appear in [11] □

Now we restrict to the class of translation fractals, namely limit fractals for which $\lambda_{n,i} = \lambda_n$. In this case we have a formula for the spectral dimension:

Theorem 7.4 *Let $(\mathcal{A}, D, \mathcal{H})$ be the spectral triple associated with a translation fractal F with countably ramified open set condition, where the similarities $w_{n,i}$, $i = 1, \ldots, p_n$ have scaling parameter λ_n. Then the spectral dimension is given by the formula*

$$d = \limsup_n \frac{\sum_1^n \log p_k}{\sum_1^n \log 1/\lambda_k}.$$

Moreover the measure corresponding to the associated Hausdorff-Besicovitch functional is the unique probability measure on F invariant under the internal isometries of F.

Proof The proof will appear in [11] □

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