Singular traces, dimensions, and Novikov-Shubin invariants

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Abstract

In Alain Connes noncommutative geometry, the question of the existence of a non-trivial integral can be described in terms of the singular traceability of the compact operator $|D|^{-d}$, D being the Dirac operator.

In this paper we give a non-triviality condition different from the cohomological one used by Connes, namely we show that, under suitable regularity conditions on the eigenvalue sequence of $|D|$, the dimension d can be uniquely determined by imposing that $|D|^{-d}$ is singularly traceable, thus providing a geometric measure theoretic definition for d.

In the second part of the paper we discuss large scale counterparts of this notion of dimension for the case of covering manifolds.

We show that $\Delta_p^{-1/2}$, raised to power α_p, the p-th Novikov-Shubin number, is singularly traceable. As a consequence, Novikov-Shubin numbers can be considered as (asymptotic) dimensions in the sense of geometric measure theory.

Finally we show that the (lower) Novikov-Shubin number $\underline{\alpha}_p$ coincides with (the supremum of) the dimension at ∞ of the semigroup generated by the Laplacian on p-forms introduced in [24].

0 Introduction.

In Alain Connes noncommutative geometry, integration of a function a on a noncommutative manifold is described via the formula $\int a = \tau(a|D|^{-d})$, where D is an unbounded selfadjoint operator playing the role of the Dirac operator, d is the dimension of the manifold, and τ is a singular trace introduced by Dixmier and corresponding to logarithmic divergences.

As a consequence the question of the existence of a non-trivial integral can be described in terms of the singular traceability of the compact operator $|D|^{-d}$, namely of the existence of a finite non-trivial singular trace on the ideal generated by $|D|^{-d}$.

In this paper we show that, under suitable regularity conditions on the eigenvalue sequence of $|D|$, the dimension d can be uniquely determined by imposing that $|D|^{-d}$ is singularly traceable, thus providing a geometric measure theoretic definition for d. We remark that with this choice of d the operator $|D|^{-d}$ always produces a singular trace which gives rise to a non-trivial integration procedure, even when the logarithmic trace fails.

Singular traceability of compact operators has been described in [1] via their eccentricity. We propose here another sufficient condition in terms of the polynomial order of (the eigenvalue sequence of) the given compact operator. More precisely, we prove that if T has polynomial order equal to 1 it is singularly traceable. Since the polynomial order of $|D|^{-d}$ is equal to d times the polynomial order of $|D|^{-1}$, this shows that the geometric measure theoretic dimension d coincides with the inverse of the polynomial order of $|D|^{-1}$.

In the second part of the paper we discuss large scale counterparts of the geometric measure theoretic dimension for the case of covering manifolds.

Le us recall that in the case of von Neumann algebras with a continuous trace singular traces can be constructed on the ideal generated by a trace-compact operator a when the eigenvalue function $\mu_a(t)$ has a suitable asymptotics when $t \to \infty$, as in the discrete case.

However, a new family of singular traces arises in this case, detecting the asymptotic behaviour of $\mu_a(t)$ when $t \to 0$ [13].

We prove here that if the polynomial order at 0, $\mathrm{ord}_0(a)$, is 1, then a is singularly traceable.

In the case of a non-compact covering M of a compact Riemannian manifold, Atiyah considered the von Neumann algebra with continuous trace consisting of the bounded operators on $L^2(M)$ commuting with the action of the covering group [2]. In this case both behaviours at 0 and at ∞ of trace-compact operators may give rise to singular traces.

As far as the behaviour at infinity is concerned, we note that the dimension n of M is twice the inverse of the order at infinity of Δ_k^{-1}, therefore, by the result mentioned above, $\Delta_k^{-n/2}$ is singularly traceable at ∞, and the dimension of M has a geometric measure theoretic interpretation.

We prove here that the large-scale counterpart of this dimension, namely the number $d = 2\mathrm{ord}_0(\Delta_k^{-1})^{-1}$, coincides with the k-th Novikov-Shubin number.

As a consequence, Novikov-Shubin numbers can be considered as (asymptotic) dimensions in the sense of geometric measure theory, since the corresponding power of $\Delta_k^{-1/2}$ produces a singular trace. Let us remark that an asymptotic-dimensional interpretation of the 0th Novikov-Shubin number is given by the fact, proved by Varopoulos [25], that α_0 coincides with the growth of the covering group.

Finally we show that the Novikov-Shubin number $\underline{\alpha}_k$ coincides with (the supremum of) the dimension at ∞ of the semigroup $e^{-t\Delta_k}$ introduced by Varopoulos, Saloff-Coste and Coulhon [24].

Generalizations of the mentioned results on covering manifolds to the case of open manifolds with bounded geometry are studied in [13].

1 Polynomial order of an operator

In this section we give a quick review to some notions pertaining to singular traces [12], and then give some sufficient conditions for the construction of singular traces in terms of the local or asymptotic polynomial order of an operator.

Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a semifinite von Neumann algebra equipped with a normal semifinite faithful trace tr. We refer the reader to [22] for the general theory of von Neumann algebras. Let $\overline{\mathcal{M}}$ be the collection of the closed, densely defined operators x on $\mathcal{H}$ affiliated with $\mathcal{M}$ such that $tr(e_{|x|}(t,\infty)) < \infty$ for some $t > 0$.

$\overline{\mathcal{M}}$, equipped with strong sense operations [19] and with the topology of convergence in measure ([21], [15]), becomes a topological $*$-algebra, called the algebra of tr-measurable operators.

For any $x = \int_0^\infty t\ de_x(t) \in \overline{\mathcal{M}}_+$, $E \in \mathbb{R} \to \nu_x(E) := tr(e_x(E))$ is a Borel measure on $\mathbb{R}$, and $tr(x) := \int_0^\infty t\ d\nu_x(t)$ is a faithful extension of tr to $\overline{\mathcal{M}}_+$.

Definition 1.1. [10] Let $a \in \overline{\mathcal{M}}$, and define, for all $t > 0$,

(i) $\lambda_a(t) := tr(e_{|a|}(t,\infty))$, the distribution function of a w.r.t. tr,

(ii) $\mu_a(t) := \inf\{s \geq 0 : \lambda_a(s) \leq t\}$, the non-increasing rearrangement of a w.r.t. tr, which is a non-increasing and right-continuous function. Moreover $\lim_{t\downarrow 0} \mu_a(t) = \|a\| \in [0,\infty]$.

Remark 1.2. (i) If $\mathcal{M} := L^\infty(X,m)$ and $tr(f) := \int f dm$, then $\overline{\mathcal{M}}$ is the *-algebra of functions that are bounded except on a set of finite m-measure, and, for any $f \in \overline{\mathcal{M}}$, $\mu_f \equiv f^*$ is the classical non-increasing rearrangement of f [3].
(ii) If $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and tr is the usual trace, then $\mu_a = \sum_{n=0}^\infty s_n \chi_{[n,n+1)}$, where $\{s_n\}$ is the sequence of singular values of the operator a, arranged in non-increasing order and counted with multiplicity [20].

Definition 1.3. [12] Let $a \in \overline{\mathcal{M}}$. Then

(i) a is called 0-eccentric if, setting

$$S_a^0(t) := \begin{cases} \int_0^t \mu_a(s)ds & \mu_a \in L^1(0,1) \\ \\ \int_t^1 \mu_a(s)ds & \mu_a \notin L^1(0,1), \end{cases}$$

1 is a limit point of $\{\frac{S_a^0(2t)}{S_a^0(t)}\}$, when $t \to 0$.

(ii) a is called ∞-eccentric if, setting

$$S_a^\infty(t) := \begin{cases} \int_1^t \mu_a(s)ds & \mu_a \notin L^1(1,\infty) \\ \\ \int_t^\infty \mu_a(s)ds & \mu_a \in L^1(1,\infty), \end{cases}$$

1 is a limit point of $\{\frac{S_a^\infty(2t)}{S_a^\infty(t)}\}$, when $t \to \infty$.

In (i) we could replace $(0,1)$ with $(0,c)$ for any $c < \infty$, and in (ii) we could replace $(1,\infty)$ with (c,∞) for any $c < \infty$, without affecting the definitions.

Theorem 1.4. ([12], Theorem 6.4) *Let $(\mathcal{M}, tr)$ be a semifinite von Neumann algebra with a normal semifinite faithful trace, $a \in \overline{\mathcal{M}}_+$ an eccentric operator. Then there exists a singular trace τ, namely a trace vanishing on projections which are finite w.r.t. tr, whose domain is the measurable bimodule generated by a and such that $\tau(a) = 1$.*

Now we give sufficient conditions to ensure eccentricity at 0, or at ∞. It is based on the notions of order of infinite at 0 and of order of infinitesimal at ∞.

Definition 1.5. For $a \in \overline{\mathcal{M}}$ we define

(i) order of infinitesimal of a at ∞

$$\mathrm{ord}_\infty(a) := \liminf_{t\to\infty} \frac{\log \mu_a(t)}{\log(1/t)} ,$$

(ii) order of infinite of a at 0

$$\mathrm{ord}_0(a) := \liminf_{t\to 0} \frac{\log \mu_a(t)}{\log(1/t)} .$$

Remark 1.6. If $a \in \overline{\mathcal{M}}_+$, then for any $\alpha > 0$, $\mathrm{ord}_\infty(a^\alpha) = \alpha\ \mathrm{ord}_\infty(a)$ and $\mathrm{ord}_0(a^\alpha) = \alpha\ \mathrm{ord}_0(a)$.

Theorem 1.7. *Let $a \in \overline{\mathcal{M}}$ be s.t. $\mathrm{ord}_0(a) = 1$. Then a is 0-eccentric.*

Proof. It is a consequence of the following Propositions. □

Proposition 1.8. *Suppose $\mu_a \notin L^1(0,1)$, and $\liminf_{t\to 0^+} \frac{\log \mu_a(t)}{\log(1/t)} = 1$. Then $\limsup_{t\to 0^+} \frac{S_a^0(2t)}{S_a^0(t)} = 1$.*

Proof. Let us set $x := \log(1/t)$, and $g(x) := t\mu_a(t)$. Then $\frac{\log \mu_a(t)}{\log(1/t)} = \frac{\log g(x)}{x} + 1$, so that the hypothesis becomes $\liminf_{x\to\infty} \frac{\log g(x)}{x} = 0$. Besides, as $S(t) \equiv S_a^0(t) = \int_t^1 \mu_a(s)ds$, S is nonincreasing and convex, so that $S(t) \geq S(2t) \geq$

$S(t) - t\mu_a(t)$, that is $1 \geq \frac{S(2t)}{S(t)} \geq 1 - \frac{t\mu_a(t)}{S(t)}$, and the thesis is implied by $\liminf_{t\to 0^+} \frac{t\mu_a(t)}{S(t)} = 0$, but, as $\frac{t\mu_a(t)}{S(t)} = \frac{g(x)}{\int_0^x g(s)ds}$, setting $G(x) := \int_0^x g(s)ds$, the thesis follows from $\liminf_{x\to\infty} \frac{g(x)}{G(x)} = 0$.
So we have to prove that $\liminf_{x\to\infty} \frac{\log g(x)}{x} = 0$ implies $\liminf_{x\to\infty} \frac{g(x)}{G(x)} = 0$. Suppose on the contrary that there are $x_0, \varepsilon > 0$ s.t. $\frac{g(x)}{G(x)} \geq \varepsilon$, for $x \geq x_0$. Then, for $x \geq x_0$, we have $\log \frac{G(x)}{G(x_0)} = \int_{x_0}^x \frac{G'(s)}{G(s)} ds \geq \varepsilon(x - x_0)$, which implies $G(x) \geq G(x_0)\mathrm{e}^{\varepsilon(x-x_0)}$, and $g(x) \geq \varepsilon G(x) \geq \varepsilon G(x_0)\mathrm{e}^{\varepsilon(x-x_0)}$, so that $\frac{\log g(x)}{x} \geq \frac{k}{x} + \varepsilon$ and $\liminf_{x\to\infty} \frac{\log g(x)}{x} \geq \varepsilon > 0$, which is absurd. □

Proposition 1.9. *Suppose* $\mu_a \in L^1(0,1)$ *and* $\liminf_{t\to 0^+} \frac{\log \mu_a(t)}{\log(1/t)} = 1$. *Then* $\liminf_{t\to 0^+} \frac{S_a^0(2t)}{S_a^0(t)} = 1$.

Proof. Recall that $S(t) \equiv S_a^0(t) = \int_0^t \mu_a(s)ds$, is positive, nondecreasing, concave, and $S(0) = 0$, $S(1) = 1$, as we can multiply μ_a by a suitable positive constant without altering our statement.
Then $\log S$ is nondecreasing and concave, and $\lim_{t\to 0^+} \log S = -\infty$, so that $g := \frac{S'}{S}$ is positive, nonincreasing, $\lim_{t\to 0^+} g(t) = \infty$, $\log S(t) = -\int_t^1 g(s)ds$, and $\int_0^1 g(s)ds = -\lim_{t\to 0^+} \log S = \infty$. Finally $\mu_a = S' = gS$, so that $\frac{\log \mu_a(t)}{\log(1/t)} = \frac{\log(tg(t)) - \int_t^1 g(s)ds}{\log(1/t)} + 1$, and the hypothesis becomes $\liminf_{t\to 0^+} \frac{\log(tg(t)) - \int_t^1 g(s)ds}{\log(1/t)} = 0$, that is for any $\varepsilon > 0$ there is $t_\varepsilon > 0$ s.t.

$$\frac{\log(tg(t)) - \int_t^1 g(s)ds}{\log(1/t)} \geq -\varepsilon\ , \tag{1.1}$$

for all $t \in (0, t_\varepsilon)$. Observe now that $\frac{S(2t)}{S(t)} = \mathrm{e}^{\int_t^{2t} g(s)ds}$, so that $\liminf_{t\to 0^+} \frac{S(2t)}{S(t)} = 1$ is equivalent to $\liminf_{t\to 0^+} \int_t^{2t} g(s)ds = 0$, and, as $0 \leq \int_t^{2t} g(s)ds \leq tg(t)$, our thesis will be proved as soon as we can show that $\liminf_{t\to 0^+} tg(t) = 0$. So suppose, on the contrary, there are $t_0, c > 0$ s.t. $tg(t) \geq c$, for all $t \in (0, t_0)$. From inequality (1.1) it follows that

$$\begin{aligned}
\frac{\log(tg(t))}{\log(1/t)} &\geq \frac{\int_t^1 g(s)ds}{\log(1/t)} - \varepsilon \\
&\geq \frac{c\log(t_0/t)}{\log(1/t)} + \frac{\int_{t_0}^1 g(s)ds}{\log(1/t)} - \varepsilon \\
&= c - \varepsilon + \frac{c\log t_0 + \int_{t_0}^1 g(s)ds}{\log(1/t)}
\end{aligned}$$

for all $t \in (0, \min\{t_\varepsilon, t_0\})$. As $\lim_{t\to 0^+} \frac{c\log t_0 + \int_{t_0}^1 g(s)ds}{\log(1/t)} = 0$, we can choose $\varepsilon < c/4$, and $t_1 > 0$ s.t. $\frac{\log(tg(t))}{\log(1/t)} \geq \frac{c}{2}$, for all $t \in (0, t_1)$. Then we have

$g(t) \geq (1/t)^{c/2+1}$, for all $t \in (0, t_1)$, so that

$$\frac{\int_t^1 g(s)ds}{\log(1/t)} \geq \frac{\int_t^{t_1} (1/s)^{c/2+1} ds + \int_{t_1}^1 g(s)ds}{\log(1/t)}$$

from which it follows

$$\lim_{t\to 0^+} \frac{\int_t^1 g(s)ds}{\log(1/t)} = +\infty. \tag{1.2}$$

As inequality (1.1) it equivalent to $\frac{\log(g(t)) - \int_t^1 g(s)ds}{\log(1/t)} \geq 1 - \varepsilon$, for all $t \in (0, t_\varepsilon)$, it follows that $\frac{\log g(t)}{\int_t^1 g(s)ds} \geq 1 + (1-\varepsilon)\frac{\log(1/t)}{\int_t^1 g(s)ds} \to 1$, when $t \to 0^+$. Therefore, with $G(t) := \int_t^1 g(s)ds$, we have $\frac{\log(-G'(t))}{G(t)} \geq 1 - \varepsilon$, for all $t \in (0, t_2)$. This implies $-G'(t)\mathrm{e}^{-(1-\varepsilon)G(t)} \geq 1$, which we integrate in $(0, t)$ for $t \leq t_2$, to obtain $t \leq \frac{1}{1-\varepsilon}\, \mathrm{e}^{-(1-\varepsilon)G(t)}$ that is $G(t) \leq \frac{1}{1-\varepsilon}[\log(1/t) - \log(1-\varepsilon)]$, and choosing ε small enough, we get $G(t) \leq 2\log(1/t)$, for all t near 0. But this contrasts with equation (1.2), so we are done. $\square$

Theorem 1.10. *Let $a \in \overline{\mathcal{M}}$ be s.t.* $\mathrm{ord}_\infty(a) = 1$. *Then a is ∞-eccentric.*

Proof. It is a consequence of the following Propositions. $\square$

Proposition 1.11. *Suppose $\mu_a \notin L^1(1, \infty)$ and* $\liminf_{t\to\infty} \frac{\log \mu_a(t)^{-1}}{\log t} = 1$. *Then* $\liminf_{t\to\infty} \frac{S_a^\infty(2t)}{S_a^\infty(t)} = 1$.

Proof. Recall that $S(t) \equiv S_a^\infty(t) = \int_1^t \mu_a(s)ds$, is positive, nondecreasing, concave, and $S(1) = 0$, $\lim_{t\to\infty} S(t) = \infty$.
Then $\log S$ is nondecreasing and concave, and $\lim_{t\to\infty} \log S = \infty$, so that $g := \frac{S'}{S}$ is positive, nonincreasing, $\lim_{t\to\infty} g(t) = 0$, $\log S(t) = \int_1^t g(s)ds$, and $\int_1^\infty g(s)ds = \lim_{t\to\infty} \log S = \infty$. Finally $\mu_a = S' = gS$, so that $\frac{\log \mu_a(t)^{-1}}{\log t} = -\frac{\log(tg(t)) + \int_1^t g(s)ds}{\log t} + 1$, and the hypothesis becomes

$$\limsup_{t\to\infty} \frac{\log(tg(t)) + \int_1^t g(s)ds}{\log t} = 0 \tag{1.3}$$

Observe now that $\frac{S(2t)}{S(t)} = \mathrm{e}^{\int_t^{2t} g(s)ds}$, so that $\liminf_{t\to\infty} \frac{S(2t)}{S(t)} = 1$ is equivalent to $\liminf_{t\to\infty} \int_t^{2t} g(s)ds = 0$, and, as $0 \leq \int_t^{2t} g(s)ds \leq tg(t)$, our thesis will be proved as soon as we can show that $\liminf_{t\to\infty} tg(t) = 0$. So suppose, on the contrary, there are $t_0, c > 0$ s.t. $tg(t) \geq c$, for all $t \in (t_0, \infty)$. From equation (1.3) it follows that for all $\varepsilon > 0$ there is $t_\varepsilon > 0$ s.t. $\frac{\log(tg(t)) + \int_1^t g(s)ds}{\log t} \leq \varepsilon$, for all $t > t_\varepsilon$, so that

$$\begin{aligned}\frac{\log(tg(t))}{\log t} &\leq -\frac{\int_1^t g(s)ds}{\log t} + \varepsilon \\ &\leq -\frac{\int_1^{t_0} g(s)ds}{\log t} - \frac{c\log(t/t_0)}{\log t} + \varepsilon \\ &= -c + \varepsilon + \frac{c\log t_0 - \int_1^{t_0} g(s)ds}{\log t}\end{aligned}$$

for all $t > \max\{t_\varepsilon, t_0\}$. As $\lim_{t\to\infty} \frac{c\log t_0 - \int_1^{t_0} g(s)ds}{\log t} = 0$, we can choose $\varepsilon < c/4$, and $t_1 > \max\{t_\varepsilon, t_0\}$ s.t. $\frac{\log(tg(t))}{\log t} \leq -\frac{c}{2}$, for all $t > t_1$. Then we have $g(t) \leq t^{-c/2-1}$, for all $t > t_1$, so that $c/t \leq g(t) \leq t^{-c/2-1}$, for all $t > t_1$, and this is absurd. □

Proposition 1.12. *Suppose* $\mu_a \in L^1(1,\infty)$ *and* $\liminf_{t\to\infty} \frac{\log \mu_a(t)^{-1}}{\log t} = 1$. *Then* $\limsup_{t\to\infty} \frac{S_a^\infty(2t)}{S_a^\infty(t)} = 1$.

Proof. Setting $S := S_a^\infty$, as S is a decreasing and convex function, $1 \geq \frac{S(2t)}{S(t)} \geq 1 - \frac{t\mu(t)}{S(t)}$, so that $\liminf_{t\to\infty} \frac{t\mu(t)}{S(t)} = 0$ implies the thesis. Let us assume on the contrary that there are $c, t_0 > 0$ s.t. $\frac{t\mu(t)}{S(t)} \geq c$, for all $t > t_0$. Then $\frac{S'(t)}{S(t)} = -\frac{\mu(t)}{S(t)} \leq -\frac{c}{t}$, so that $S(t) \leq S(t_0)\left(\frac{t}{t_0}\right)^{-c} = kt^{-c}$, for some positive k. Then $t\mu(2t) \leq \int_t^{2t} \mu(s)ds \leq \int_t^\infty \mu(s)ds \leq kt^{-c}$, for all $t \geq t_0$, therefore $\mu(t) \leq k't^{-c-1}$, for all $t \geq 2t_0$, so that $\liminf_{t\to\infty} \frac{\log \mu(t)}{\log(1/t)} \geq 1 + c$ contrary to the hypothesis. □

Because of their importance in determining the eccentricity properties of an operator a, let us compute the orders of a in terms of the asymptotics of its distribution function λ_a.

Proposition 1.13. $\mathrm{ord}_0(a)$ *and* $\mathrm{ord}_\infty(a)$ *are also given by*

$$\begin{aligned}\mathrm{ord}_0(a) &= \left(\limsup_{s\to\infty} \frac{\log \lambda_a(s)}{\log(1/s)}\right)^{-1}, \\ \mathrm{ord}_\infty(a) &= \left(\limsup_{s\to 0^+} \frac{\log \lambda_a(s)}{\log(1/s)}\right)^{-1}.\end{aligned}$$

Proof. Let us set $\mu := \mu_a$, $\lambda := \lambda_a$, and assume μ is never 0 and $\lim_{t\to 0} \mu(t) = +\infty$, otherwise the proof is obvious.
Let $G_1 = \{(x,y) \in \mathbb{R}_+^2 : y = \mu(x)\}$, $G_2 = \{(x,y) \in \mathbb{R}_+^2 : x = \lambda(y)\}$. Then

$$\liminf_{t\to\infty} \frac{\log \mu(t)}{\log(1/t)} = \lim_{t\to\infty} \inf_{\substack{(x,y)\in G_1 \\ x>t}} \frac{\log y}{\log(1/x)} = \lim_{s\to 0} \inf_{\substack{(x,y)\in G_1 \\ y<s}} \frac{\log y}{\log(1/x)}$$

and

$$\liminf_{s\to 0} \frac{\log s}{\log(1/\lambda(s))} = \lim_{s\to 0} \inf_{\substack{(x,y)\in G_2 \\ y<s}} \frac{\log y}{\log(1/x)} = \lim_{t\to\infty} \inf_{\substack{(x,y)\in G_2 \\ x>t}} \frac{\log y}{\log(1/x)}.$$

Also

$$\lim_{t\to\infty} \inf_{\substack{(x,y)\in G_1 \\ x>t}} \frac{\log y}{\log(1/x)} = \lim_{t\to\infty} \inf_{\substack{(x,y)\in \overline{G_1} \\ x>t}} \frac{\log y}{\log(1/x)},$$

$$\lim_{s\to 0} \inf_{\substack{(x,y)\in G_1 \\ y<s}} \frac{\log y}{\log(1/x)} = \lim_{s\to 0} \inf_{\substack{(x,y)\in \overline{G_1} \\ y<s}} \frac{\log y}{\log(1/x)},$$

$$\lim_{t\to\infty} \inf_{\substack{(x,y)\in G_2 \\ x>t}} \frac{\log y}{\log(1/x)} = \lim_{t\to\infty} \inf_{\substack{(x,y)\in \overline{G_2} \\ x>t}} \frac{\log y}{\log(1/x)},$$

and

$$\lim_{s\to 0} \inf_{\substack{(x,y)\in G_2 \\ y<s}} \frac{\log y}{\log(1/x)} = \lim_{s\to 0} \inf_{\substack{(x,y)\in \overline{G_2} \\ y<s}} \frac{\log y}{\log(1/x)}.$$

Define, for any $y > 0$, $[\ell_y, r_y] := \left\{x > 0 : (x,y) \in \overline{G_1}\right\}$, and, for any $x > 0$, $[d_x, u_x] = \left\{y > 0 : (x,y) \in \overline{G_2}\right\}$. Then $(x,y) \in \overline{G_1} \cap \overline{G_2} \iff (x,y) \in \overline{G_1}$ and $x \in \{\ell_y, r_y\} \iff (x,y) \in \overline{G_2}$ and $y \in \{d_x, u_x\}$. Indeed, defining $\lambda^-(y) := \lim_{s\to y^-} \lambda(s)$, and $\mu^-(x) := \lim_{t\to x^-} \mu(t)$, one proves that $\overline{G_1(\mu^-)} = \overline{G_1(\mu)}$ and $\overline{G_2(\lambda^-)} = \overline{G_2(\lambda)}$, and the rest follows easily.
Finally, since $\frac{\log \mu(t)}{\log(1/r_t)} \leq \frac{\log \mu(t)}{\log(1/t)}$ one gets, taking the lim inf,

$$\lim_{t\to\infty} \inf_{\substack{(x,y)\in \overline{G_1} \\ x>t}} \frac{\log y}{\log(1/x)} = \lim_{t\to\infty} \inf_{\substack{(x,y)\in \overline{G_1}\cap\overline{G_2} \\ x>t}} \frac{\log y}{\log(1/x)}.$$

Analogously, since $\frac{\log u_s}{\log(1/\lambda(s))} \leq \frac{\log s}{\log(1/\lambda(s))}$ one gets, taking the lim inf,

$$\lim_{s\to 0} \inf_{\substack{(x,y)\in \overline{G_2} \\ y<s}} \frac{\log y}{\log(1/x)} = \lim_{s\to 0} \inf_{\substack{(x,y)\in \overline{G_1}\cap\overline{G_2} \\ y<s}} \frac{\log y}{\log(1/x)}.$$

Putting all this together one gets

$$\begin{aligned}\liminf_{t\to\infty} \frac{\log \mu(t)}{\log(1/t)} &= \liminf_{s\to 0} \frac{\log s}{\log(1/\lambda(s))} = \left(\limsup_{s\to 0} \frac{\log(1/\lambda(s))}{\log s}\right)^{-1} \\ &= \left(\limsup_{s\to 0} \frac{\log \lambda(s)}{\log(1/s)}\right)^{-1}.\end{aligned}$$

The equality

$$\liminf_{t\to 0} \frac{\log \mu(t)}{\log(1/t)} = \left(\limsup_{s\to\infty} \frac{\log \lambda(s)}{\log(1/s)}\right)^{-1}$$

is proved in the same way (using ℓ_t and d_s). $\square$

2 Some results on noncommutative geometric measure theory

In this section we shall discuss some definitions of dimension in noncommutative geometry in the spirit of geometric measure theory.

As it is known, the measure for a noncommutative manifold is defined via a singular trace applied to a suitable power of some geometric operator (e.g. the Dirac operator of the spectral triple of Alain Connes). Connes showed that such procedure recovers the usual volume in the case of compact Riemannian manifolds, and more generally the Hausdorff measure in some interesting examples [6].

Let us recall that $(\mathcal{A}, D, \mathcal{H})$ is called a spectral triple when $\mathcal{A}$ is an algebra acting on the Hilbert space $\mathcal{H}$, D is a self adjoint operator on the same Hilbert space such that $[D, a]$ is bounded for any $a \in \mathcal{A}$, and D has compact resolvent. In the following we shall assume that 0 is not an eigenvalue of D, the general case being recovered by replacing D with $D|_{\ker(D)^\perp}$. Such a triple is called d^+-summable, $d \in (0, \infty)$, when $|D|^{-d}$ belongs to the ideal

$$\mathcal{L}^{1,\infty} := \{a \in \mathcal{K}(\mathcal{H}) : \sum_{k=1}^{n} \mu_k(a) = O(\log n)\}, \tag{2.1}$$

where, in case of compact operators, we denote the non-increasing rearrangement of a by $\mu_k(a)$, instead of $\mu_a(k)$, to conform with tradition.
The noncommutative version of the integral on functions is given by the formula $\tau_\omega(a|D|^{-d})$, where τ_ω is a Dixmier trace, i.e. a singular trace summing logarithmic divergences. Of course the preceding formula does not guarantee the non-triviality of the integral, and in fact cohomological assumptions in this direction have been considered [6]. We are interested in different conditions for non-triviality. In this connection, introducing the space $\mathcal{L}_0^{1,\infty}$, where the $O(\log n)$ in (2.1) is replaced by $o(\log n)$, we observe that the previous noncommutative integration is always trivial when $|D|^{-d}$ belongs to $\mathcal{L}_0^{1,\infty}$.

Lemma 2.1. *Let $(\mathcal{A}, D, \mathcal{H})$ be a spectral triple. Then*

$$\inf\{d > 0 : |D|^{-d} \in \mathcal{L}_0^{1,\infty}\} = \sup\{d > 0 : |D|^{-d} \notin \mathcal{L}^{1,\infty}\}.$$

Proof. Let $d^+ = \inf\{d > 0 : |D|^{-d} \in \mathcal{L}_0^{1,\infty}\}$, $d < d^+$. Then $|D|^{-d'} \notin \mathcal{L}_0^{1,\infty}$ for $d' = \frac{d+d^+}{2}$, i.e. there exists a subsequence n_k such that

$$\lim_{k\to\infty} \frac{1}{\log n_k} \sum_{j=1}^{n_k} \mu_j(|D|^{-d'}) = \ell > 0.$$

Then, setting $\varepsilon = 1 - \frac{d}{d'} > 0$, for any $m \in \mathbb{N}$ we have

$$\begin{aligned}\lim_{k\to\infty} \frac{1}{\log n_k} \sum_{j=1}^{n_k} \mu_j(|D|^{-d}) &= \lim_{k\to\infty} \frac{1}{\log n_k} \sum_{j=m}^{n_k} \mu_j(|D|^{-d}) \\ &= \lim_{k\to\infty} \frac{1}{\log n_k} \sum_{j=m}^{n_k} \frac{\mu_j(|D|^{-d'})}{\mu_j(|D|^{-d'})^{\varepsilon}} \\ &\geq \frac{1}{\mu_m(|D|^{-d'})^{\varepsilon}} \lim_{k\to\infty} \frac{1}{\log n_k} \sum_{j=m}^{n_k} \mu_j(|D|^{-d'}) \\ &= \frac{\ell}{\mu_m(|D|^{-d'})^{\varepsilon}}.\end{aligned}$$

By the arbitrariness of m, the limit is ∞, i.e. $|D|^{-d} \notin \mathcal{L}^{1,\infty}$, which implies the thesis. □

These results, together with the examples by Connes and Sullivan [6], justify the following definition.

Definition 2.2. Let $(\mathcal{A}, D, \mathcal{H})$ be a spectral triple. We shall call the functional $a \mapsto \tau_\omega(a|D|^{-\alpha})$ the α-dimensional Hausdorff measure, and the number

$$d_H(\mathcal{A}, D, \mathcal{H}) = \inf\{d > 0 : |D|^{-d} \in \mathcal{L}_0^{1,\infty}\} = \sup\{d > 0 : |D|^{-d} \notin \mathcal{L}^{1,\infty}\}$$

the Hausdorff dimension of the spectral triple.

Let us observe that the d-dimensional Hausdorff measure depends on the generalized limit procedure ω, however all such functionals coincide on measurable operators in the sense of Connes [6]. As in the commutative case, the Hausdorff dimension is the supremum of the d's such that the d-dimensional Hausdorff measure is everywhere infinite and the infimum of the d's such that the d-dimensional Hausdorff measure is identically zero.

Concerning the non-triviality of the d_H-dimensional Hausdorff measure, we have the same situation as in the classical case.

Proposition 2.3. *Let $(\mathcal{A}, D, \mathcal{H})$ be a spectral triple with finite non-zero Hausdorff dimension d_H. Then the d_H-dimensional Hausdorff measure is the only possibly non-trivial functional on $\mathcal{A}$ among the Hausdorff measures.*

Proof. The result obviously follows by Lemma 2.1 and the definition of the Hausdorff measures. □

According to the previous result, a non-trivial Hausdorff measure is unique but does not necessarily exist. In fact, if the eigenvalue asymptotics of D is e.g. $n \log n$, the Hausdorff dimension is one, but the 1-dimensional Hausdorff measure gives the null functional.

We shall now propose another spectral dimension, for which the situation is somewhat the opposite. If we consider all singular traces, not only the logarithmic ones, and the corresponding functionals on $\mathcal{A}$, we shall show that there exists a non trivial functional associated with such a dimension, but such property does not characterize this dimension.

Definition 2.4. Let $(\mathcal{A}, D, \mathcal{H})$ be a spectral triple. We shall call the number

$$d_B(\mathcal{A}, D, \mathcal{H}) = \mathrm{ord}_\infty(D^{-1})^{-1} = \left(\liminf_{n\to\infty} \frac{\log \mu_n(D)}{\log n} \right)^{-1}$$

the box dimension of the spectral triple.

Proposition 2.5. *Let $(\mathcal{A}, D, \mathcal{H})$ be a spectral triple with finite non-zero box dimension d. Then $|D|^{-d}$ is singularly traceable, namely it gives rise to a singular trace τ which is non-trivial on the ideal generated by $|D|^{-d}$. In particular the functional $a \mapsto \tau(a|D|^{-d})$ is a non-trivial trace state on the algebra $\mathcal{A}$.*

Proof. By definition $\mathrm{ord}_\infty(D^{-1}) = d^{-1}$ hence (cf. Remark 1.6) $\mathrm{ord}_\infty(|D|^{-d}) = 1$, therefore, by Theorem 1.10, $|D|^{-d}$ is eccentric and finally, by Theorem 1.4, we get the existence of a singular trace τ. The trace property for the functional $a \mapsto \tau(a|D|^{-d})$ is proved as in [5]. □

Remark 2.6. We call the number d_B a dimension since it is related to the existence of a non-trivial geometric measure. Proposition 2.7 shows that under suitable regularity conditions of the eigenvalue sequence $\mu_n(D)$ such request determines d_B uniquely, and d_B coincides with d_H. However this is not true in general. In a following example we describe some selfadjoint operators D for which the numbers d_H and d_B are different and both $|D|^{-d_H}$ and $|D|^{-d_B}$ are singularly traceable.

Proposition 2.7. *Let $(\mathcal{A}, D, \mathcal{H})$ be a spectral triple with finite non-zero box dimension.*

(a) If there exists $\lim \frac{\log \mu_n(D^{-1})}{\log 1/n}$, *then* $d_B = d_H$.

(b) If there exists $\lim \frac{\mu_n(D^{-1})}{\mu_{2n}(D^{-1})}$, d_B *is characterized by the property that* $|D|^{-d_B}$ *is singularly traceable, and* $d_B = d_H$.

Proof. (a). As $d_H(|D|^\alpha) = \frac{1}{\alpha} d_H(|D|)$, and $d_B(|D|^\alpha) = \frac{1}{\alpha} d_B(|D|)$, for any $\alpha > 0$, we may restrict to the case $d_B = 1$. By hypothesis we have that for any $\varepsilon > 0$

$$\left(\frac{1}{n}\right)^{1+\varepsilon} \le \mu_n(D^{-1}) \le \left(\frac{1}{n}\right)^{1-\varepsilon}.$$

for sufficiently large n. As a consequence, if $\lambda > 1$,

$$\mu_n(|D|^{-\lambda}) \le \left(\frac{1}{n}\right)^{\frac{\lambda+1}{2}}$$

hence it is a summable sequence, which implies $\frac{\sum_{k=1}^n \mu_n(|D|^{-\lambda})}{\log n} = 0$, i.e. $d_H \le 1$. Conversely, if $\lambda < 1$,

$$\mu_n(|D|^{-\lambda}) \ge \left(\frac{1}{n}\right)^{\frac{\lambda+1}{2}}$$

and this implies that $\frac{\sum_{k=1}^{n}\mu_n(|D|^{-\lambda})}{\log n} = \infty$, i.e. $d_H \geq 1$. The thesis follows.
(b). For any $n \in \mathbb{N}$, let $k_n \in \mathbb{N}$ be such that $2^{k_n} \leq n < 2^{k_n+1}$, and write μ_n for $\mu_n(D^{-1})$ and assume for simplicity that $\mu_1 = 1$. Then

$$
\begin{aligned}
-\frac{1}{k_n \log 2}\sum_{j=0}^{k_n} \log \frac{\mu_{2^{j+1}}}{\mu_{2^j}} &= -\frac{\log \mu_{2^{k_n+1}}}{k_n \log 2} \\
&\geq \frac{\log \mu_n}{\log 1/n} \\
&\geq -\frac{\log \mu_{2^{k_n}}}{(k_n+1)\log 2} \\
&= -\frac{1}{(k_n+1)\log 2}\sum_{j=0}^{k_n-1} \log \frac{\mu_{2^{j+1}}}{\mu_{2^j}}.
\end{aligned}
$$

Taking the limit for $n \to \infty$ one gets that $\lim \frac{\log \mu_n(D^{-1})}{\log 1/n}$ exists, hence $d_B = d_H$ by (a), and also

$$
d_B^{-1} = -\frac{\log\left(\lim_k \frac{\mu_{2k}}{\mu_k}\right)}{\log 2},
$$

namely

$$
\lim_n \frac{\mu_{2n}(D^{-1})}{\mu_n(D^{-1})} = 2^{-1/d_B}.
$$

Assume for the moment that $\mu_n \notin \ell^1$, and denote by $s_n(|D|^{-d}) := \sum_{k=1}^{n}\mu_k(|D|^{-d})$ (the same as $S_a^\infty(n)$ of section 1). Then, by a Cesaro theorem,

$$
\begin{aligned}
\lim_n \frac{s_{2n}(|D|^{-d})}{s_n(|D|^{-d})} &= \lim_n \frac{\sum_{k=1}^{2n}\mu_k(|D|^{-d})}{\sum_{k=1}^{n}\mu_k(|D|^{-d})} = \lim_n \frac{2\mu_{2n}(|D|^{-d})}{\mu_n(|D|^{-d})} \\
&= \lim_n \frac{2\mu_{2n}(|D|^{-1})^d}{\mu_n(|D|^{-1})^d} = 2\left(\lim_n \frac{\mu_{2n}(|D|^{-1})}{\mu_n(|D|^{-1})}\right)^d = 2^{1-d/d_B}.
\end{aligned}
$$

Therefore $|D|^{-d}$ is eccentric if and only if such limit is one, i.e. when $d = d_B$. If $\mu_n \in \ell^1$, then denoting by $s_n(|D|^{-d}) := \sum_{k=n}^{\infty}\mu_k(|D|^{-d})$ (the same as $S_a^\infty(n)$ of section 1), the calculation above, suitably modified, shows that $|D|^{-d}$ is eccentric if and only $d = d_B$. □

Example 2.8. Let us construct a family of Dirac operators D_λ, $\lambda > 1$, s.t. $d_B(D_\lambda) = 1$, $d_H(D_\lambda) = \lambda$, and the λ-dimensional Hausdorff measure is non-trivial. Since the dimensions and the singular traceability property depend only on the eigenvalue sequence $\mu_n(\lambda) := \mu_n(|D_\lambda|^{-1})$, we shall concentrate only on the construction of the sequence $\mu_n(\lambda)$.
Let a_k be any increasing diverging sequence, $a_1 = 0$, and set $\mu_n = \mathrm{e}^{-a_k}$ when $\mathrm{e}^{a_k} \leq n < \mathrm{e}^{a_{k+1}}$. Then

$$
d_B^{-1} = \liminf_n \frac{\log \mu_n}{\log 1/n} = \lim_k \frac{\log \mu_{[\mathrm{e}^{a_k}]+1}}{\log 1/([\mathrm{e}^{a_k}])} = 1
$$

where $[\cdot]$ denotes the integer part. If, setting $\sigma_{n,\lambda} = \sigma_n(|D_\lambda|^{-\lambda}) = \sum_{k=1}^n \mu_k(\lambda)^\lambda$, we show that

$$\limsup_n \frac{\sigma_{n,\lambda}}{\log n}$$

is finite non-zero, this shows at once that $d_H(D_\lambda) = \lambda$ and that there exists a non trivial logarithmic singular trace on $|D_\lambda|^{-\lambda}$ [23, 1].
Now, for any $\lambda > 1$, set $a_k := \lambda^k - \frac{\log\lambda}{\lambda-1}k$ and observe that, with this choice, $a_{j+1} - \lambda a_j = j\log\lambda - \log\lambda/(\lambda-1)$. Then

$$\begin{aligned}
\limsup_n \frac{\sigma_{n,\lambda}}{\log n} &= \lim_k \frac{\sigma_{[\mathrm{e}^{a_k}],\lambda}}{\log[\mathrm{e}^{a_k}]} = \lim_k \frac{\sigma_{[\mathrm{e}^{a_k}],\lambda}}{a_k} \\
&= \lim_k a_k^{-1} \sum_{j=1}^{k-1} (\mathrm{e}^{a_{j+1}} - \mathrm{e}^{a_j})\mathrm{e}^{-\lambda a_j} \\
&= \lim_k \lambda^{-k} \sum_{j=1}^{k-1} \mathrm{e}^{a_{j+1}-\lambda a_j} = \lim_k \lambda^{-k} \sum_{j=1}^{k-1} \lambda^j \lambda^{\frac{1}{1-\lambda}} \\
&= \lim_k \lambda^{-k}\lambda^{\frac{1}{1-\lambda}} \frac{\lambda^k - 1}{\lambda - 1} = \frac{\lambda^{\frac{1}{1-\lambda}}}{\lambda - 1}.
\end{aligned}$$

Remark 2.9. Example 2.8 describes situations where the two geometric spectral dimensions considered here are different, and give rise to different (non trivial) geometric integrations.
For the spectral triples whose Dirac operator has a spectral asymptotics like $n^\alpha(\log n)^\beta$ instead, we have $d_B = d_H = 1/\alpha$, namely the two dimensions coincide, and the uniqueness result of the preceding Proposition applies. However, the nontrivial singular trace associated with $|D|^{-d_B}$ by Theorem 1.4 is a logarithmic trace if and only if $\beta = 1$. In this sense, the singular traces associated with a generic eccentric operator generalize the logarithmic trace in the same way in which the Besicovitch measure theory generalizes the Hausdorff measure theory.

3 Novikov-Shubin invariants as asymptotic dimensions

In this section we apply the theory of polynomial orders of operators introduced in section 1 to a geometric operator (the Laplacian on k-forms) and show that these orders give topological information on the manifold.

3.1 Weyl's asymptotics via Atiyah's trace on covering manifolds

Let M be a complete connected Riemannian n-manifold, and G an infinite discrete group of isometries of M. Suppose that G acts freely (*i.e.* any $g \in G$, $g \neq e$, acts without fixed points), properly discontinuously, and that $X :=$

M/G is a compact manifold. Let $\mathcal{F}$ be a fundamental domain for G, that is ([2], page 52) an open subset of M, disjoint from all its translates by G, and such that $M \setminus \cup_{g\in G} g(\mathcal{F})$ has measure zero. Let $L^2\Lambda^k(M)$ be the Hilbert space of square-integrable k-forms on M, w.r.t. the volume measure, then $L^2\Lambda^k(M) \cong \ell^2(G) \otimes L^2\Lambda^k(X)$. G acts on $L^2\Lambda^k(M)$ as left translation operators $(L_g u)(x) := u(g^{-1}x)$. Let $\mathcal{M}_k \equiv \mathcal{M}_k(M,G)$ be the von Neumann algebra of bounded G-invariant operators, so that $\mathcal{M}_k \cong \mathcal{R}(G) \otimes \mathcal{B}(L^2\Lambda^k(X))$ and $\mathcal{M}_k' = \{L_g : g \in G\}'' \cong \mathcal{L}(G) \otimes \mathbb{C}$, where $\mathcal{R}(G)$, $\mathcal{L}(G)$ are the right, resp. left, regular representations of G. Any self-adjoint G-invariant operator on $L^2\Lambda^k(M)$ is affiliated with $\mathcal{M}_k$.

By the previous isomorphism, $\mathcal{M}_k$ inherits a trace $Tr_G = \tau_G \otimes Tr$, and we quote a result in [2], which gives a more explicit description of Tr_G.

Proposition 3.1. *Let $A \in \mathcal{M}_k$ be a positive self-adjoint operator, with a C^∞ kernel $A(x,y)$. Then $A \in \mathcal{L}^1(\mathcal{M}_k, Tr_G)$, and $Tr_G(A) = \int_{\mathcal{F}} trA(x,x)dvol(x)$, where tr is the usual matrix trace.*

The Laplacian Δ_k acting on exterior k-forms on M is essentially self-adjoint as an operator on $L^2\Lambda^k(M)$ [4], and we use the same notation for its closure. Let $\Delta_k = \int t dE_k(t)$, be its spectral decomposition; then $e_k(t,\cdot,\cdot)$, the Schwartz kernel of $E_k(t)$, belongs to $C^\infty(M\times M)$, and we have for the spectral distribution function $N_k(t) := Tr_G(E_k(t)) = \int_{\mathcal{F}} tre_k(t,x,x)dvol(x)$. N_k is an increasing function on $\mathbb{R}$ which vanishes on $(-\infty, 0)$.

We are now in a position to make explicit the topological information contained in $\mathrm{ord}_\infty(\Delta_k^{-1})$. We need a lemma.

Lemma 3.2. $\lambda_{\Delta_k^{-1}}(t) = N_k(1/t) - b_k$.

Proof.

$$\begin{aligned}\lambda_{\Delta_k^{-1}}(t) &= Tr_G(E_{(t,\infty)}(\Delta_k^{-1})) = Tr_G(\chi_{(t,\infty)}(\Delta_k^{-1})) \\ &= Tr_G(\chi_{(0,1/t)}(\Delta_k)) = Tr_G(E_{(0,1/t)}(\Delta_k)) = N_k(1/t) - b_k.\end{aligned}$$

□

Theorem 3.3. $dim(M) = 2(\mathrm{ord}_\infty(\Delta_k^{-1}))^{-1}$. *As a consequence $\Delta_k^{-n/2}$ is ∞-eccentric, and gives rise to a singular trace on $\mathcal{M}_k$.*

Proof. Recall from [9], equation (4.5), that $N_k(t) \sim \beta t^{n/2}$, as $t \to \infty$, where $\beta \neq 0$. Then from Lemma 3.2 it follows $\lambda_{\Delta_k^{-1}}(s) \sim \beta s^{-n/2}$, as $s \to 0$, so that the thesis follows from Proposition 1.13. □

3.2 Novikov-Shubin invariants as asymptotic dimensions

Novikov and Shubin [16], [17] have studied the asymptotic behaviour of $N_k(t)$ as $t \to 0$, which, through the efforts of Efremov-Shubin [9], Lott [14], and Gromov-Shubin [11], has been proved to be a homotopy invariant. We want to show that the Novikov-Shubin invariants are asymptotic dimensions, so we need some notation.

Let $\vartheta_k(t) := Tr_G(\mathrm{e}^{-t\Delta_k}) = \int \mathrm{e}^{-st} dN_k(s)$ be the Laplace-Stieltjes transform of $N_k(t)$. ϑ_k is a decreasing positive function on $(0,\infty)$.

Gromov and Shubin introduced (weak) Novikov-Shubin numbers, which, using Lott normalization [14], are defined as follows

(*i*) $\underline{\alpha}_k \equiv \underline{\alpha}_k(M,G) := 2\liminf_{t\to 0} \frac{\log(N_k(t)-b_k)}{\log t} = 2\liminf_{t\to\infty} \frac{-\log(\vartheta_k(t)-b_k)}{\log t}$

(*ii*) $\alpha_k \equiv \alpha_k(M,G) := 2\limsup_{t\to 0} \frac{\log(N_k(t)-b_k)}{\log t}$

(*iii*) $\alpha'_k \equiv \alpha'_k(M,G) := 2\limsup_{t\to\infty} \frac{-\log(\vartheta_k(t)-b_k)}{\log t}$

where $b_k := \lim_{t\to 0} N_k(t)$ are the so-called L^2-Betti numbers, and are homotopy invariant [8]. Gromov and Shubin showed that these numbers are G-homotopy invariants of M.

In analogy with the definition given in [13], we call asymptotic spectral dimension of the covering manifold M with structure group G the number

$$d_\infty(M,G,\Delta_k) := 2(\mathrm{ord}_0(\Delta_k^{-1}))^{-1}.$$

Then

Theorem 3.4. *Let k be s.t. $0 < \alpha_k < \infty$. Then $\alpha_k = d_\infty(M,G,\Delta_k)$. Therefore $\Delta_k^{-\alpha_k/2}$ is 0-eccentric, and gives rise to a non-trivial singular trace on $\mathcal{M}_k$.*

Proof. From Proposition 1.13 and Lemma 3.2 it follows that

$$d_\infty(M,G,\Delta_k) = 2\mathrm{ord}_0(\Delta_k^{-1})^{-1} = 2\limsup_{t\to 0} \frac{\log(N_k(t)-b_k)}{\log t} = \alpha_k.$$

Therefore, if $0 < \alpha_k < \infty$, $\mathrm{ord}_0(\Delta_k^{-\alpha_k/2}) = 1$ and the thesis follows from Theorem 1.7. □

Remark 3.5. If $k = 0$, a result by Varopoulos [25] shows that $\alpha_0(M,G) =$ growth G. Since the growth of G coincides with the asymptotic metric dimension of M [13], we obtain that the 0-th Novikov-Shubin number coincides with the asymptotic metric dimension.

3.3 Relation between Novikov-Shubin invariants and the asymptotic dimension of the heat semigroups

Based on the notion of dimension at infinity due to Varopoulos, Saloff-Coste, Coulhon [24], see also [7], we define the asymptotic dimension of a semigroup of bounded operators on a measure space.

Let $(X,\mathcal{M},\mu)$ be a measure space, V a finite dimensional (real or complex) vector space, $L^p(X,\mathcal{M},\mu;V)$ the Lebesgue space of V-valued functions.

Definition 3.6. Let $T_t : L^1(X, \mathcal{M}, \mu; V) \to L^\infty(X, \mathcal{M}, \mu; V)$ be a semigroup of bounded operators. Then we set

$$d_\infty(T) := \liminf_{t\to\infty} \frac{2\log \|T_t\|_{1\to\infty}}{\log\left(\frac{1}{t}\right)}.$$

Theorem 3.7. ([24], Theorem II.4.3)
Let $T_t \in \mathcal{B}(L^1(X, \mathcal{M}, \mu; V) \cap L^\infty(X, \mathcal{M}, \mu; V))$ and assume it extends to a semigroup on L^p, for any $p \in [1, \infty]$, of class C^0 if $p < \infty$. Suppose moreover that T_t is equicontinuous on L^1 and L^∞, bounded analytic on L^2, and $\|T_1\|_{1\to\infty} < \infty$. Denote by A the generator of the semigroup, and by $\mathcal{D} := \operatorname{span}\{\int_0^\infty \varphi(t)T_t f dt :$ $\varphi \in C_0^\infty(0,\infty),\ f \in\ L^\infty(X, \mathcal{M}, \mu; V),\ \mu\{f \neq 0\} < \infty\}$. Then for any $n > 0$, and $0 < \alpha < \frac{n}{2}$, the following are equivalent

(i) $\|f\|_{2n/(n-2\alpha)} \leq C(\|A^{\alpha/2} f\|_2 + \|A^{\alpha/2} f\|_{2n/(n-2\alpha)})$, $f \in \mathcal{D}$

(ii) $\|T_1 f\|_{2n/(n-2\alpha)} \leq C\|A^{\alpha/2} f\|_2$, $f \in \mathcal{D}$

(iii) $\|T_t\|_{1\to\infty} \leq Ct^{-n/2}$, $t \in [1, \infty)$.

Proposition 3.8. *Let $T_t \in \mathcal{B}(L^1(X, \mathcal{M}, \mu; V) \cap L^\infty(X, \mathcal{M}, \mu; V))$ and assume it extends to a semigroup on L^1 of class C^0, and that $\|T_1\|_{1\to\infty} < \infty$. Then the following are equivalent*

(i) $\|T_t\|_{1\to\infty} \leq Ct^{-n/2}$, $t \geq 1$

(ii) $\|T_t\|_{1\to\infty} \leq Ct^{-n/2}$, $t \geq t_0 > 1$.

Proof. *(ii)* $\Rightarrow$ *(i)* Let $t \in (1, t_0]$ and observe that $\|T_t\|_{1\to\infty} = \|T_1 T_{t-1}\|_{1\to\infty} \leq \|T_1\|_{1\to\infty}\|T_{t-1}\|_{1\to1} \leq k\|T_1\|_{1\to\infty} =: M$, where $k := \sup_{t\in[0,t_0]} \|T_t\|_{1\to1} < \infty$ because T_t is a semigroup of class C^0 on L^1. So that, with $C_0 := \max\{C, Mt_0^{n/2}\}$, we get the thesis. ☐

Proposition 3.9. $d_\infty(T) = \sup\{n > 0 : \|T_t\|_{1\to\infty} \leq Ct^{-n/2},\ t \geq 1\}$.

Proof. Set d for the supremum. Then for all $\varepsilon > 0$, there is $t_0 > 1$ s.t. $\|T_t\|_{1\to\infty} \leq t^{-(d_\infty(T)-\varepsilon)/2}$, for all $t \geq t_0$, and, by previous proposition, $d_\infty(T) - \varepsilon \leq d$. Conversely $\|T_t\|_{1\to\infty} \leq t^{-(d-\varepsilon)/2}$, for all $t \geq 1$ implies $d - \varepsilon \leq d_\infty(T)$. ☐

Remark 3.10. Varopoulos, Saloff-Coste and Coulhon call dimension at ∞ of the semigroup any of the numbers n verifying the equivalent conditions of Theorem 3.7. Clearly such dimensions form a left half line, and the previous Proposition shows that $d_\infty(T)$ coincides with its upper bound.

We want to give a formula for the computation of the asymptotic dimension of a semigroup, in the special case of a semigroup of integral operators with continuous kernel. We need some preliminary results. So let X be a Hausdorff topological space, and μ a Borel measure on it with $\operatorname{supp} \mu = X$.

Lemma 3.11. *Let K be an integral operator with kernel $k \in C(X \times X) \cap L^\infty(X \times X, \mathcal{M} \otimes \mathcal{M}, \mu \otimes \mu; End(V))$, where V is endowed with a scalar product. Then $\|K\|_{1\to\infty} = \sup_{x\in X} \|k(x, x)\|$.*

Proof. We begin by proving that $\|K\|_{1\to\infty} = M := \sup_{x,y\in X} \|k(x,y)\|$. Recall that

$$\|K\|_{1\to\infty} = \sup_{f,g\in\Omega} \left\{ \left| \int_X \int_X \langle f(x), k(x,y)g(y)\rangle d\mu(x)d\mu(y) \right| \right\},$$

where $\Omega := \{f \in L^1(X, \mathcal{M}, \mu; V), \|f\| = 1\}$. Then it is easy to see that $\|K\|_{1\to\infty} \le M$. For the reversed inequality, let $\varepsilon > 0$, $(x_\varepsilon, y_\varepsilon) \in X \times X$ be s.t. $M - \varepsilon < \|k(x_\varepsilon, y_\varepsilon)\| \le M$, and $v_\varepsilon, w_\varepsilon \in V$ be s.t. $\langle v_\varepsilon, k(x_\varepsilon, y_\varepsilon)w_\varepsilon\rangle \ge \|k(x_\varepsilon, y_\varepsilon)\| - \varepsilon$. Let $A_\varepsilon, B_\varepsilon \subset X$ be open neighbourhoods of x_ε, respectively y_ε, of finite measure s.t. $M - \varepsilon < \|k(x,y)\| \le M$, for any $(x,y) \in A_\varepsilon \times B_\varepsilon$, and let $f_\varepsilon(x) := \frac{\chi_{A_\varepsilon}(x)}{\mu(A_\varepsilon)} v_\varepsilon$, $g_\varepsilon(y) := \frac{\chi_{B_\varepsilon}(y)}{\mu(B_\varepsilon)} w_\varepsilon$. Then

$$\begin{aligned}
&\left| \int_X \int_X \langle f_\varepsilon(x), k(x,y)g_\varepsilon(y)\rangle d\mu(x)d\mu(y) \right| \\
&= \frac{1}{\mu(A_\varepsilon)\mu(B_\varepsilon)} \left| \int_{A_\varepsilon} d\mu(x) \int_{B_\varepsilon} d\mu(y) \langle v_\varepsilon, k(x,y)w_\varepsilon\rangle \right| \\
&\ge \frac{1}{\mu(A_\varepsilon)\mu(B_\varepsilon)} \int_{A_\varepsilon} d\mu(x) \int_{B_\varepsilon} d\mu(y) \langle v_\varepsilon, k(x_\varepsilon, y_\varepsilon)w_\varepsilon\rangle + \\
&- \frac{1}{\mu(A_\varepsilon)\mu(B_\varepsilon)} \int_{A_\varepsilon} d\mu(x) \int_{B_\varepsilon} d\mu(y)\, |\langle v_\varepsilon, [k(x,y) - k(x_\varepsilon, y_\varepsilon)]w_\varepsilon\rangle| \\
&\ge \|k(x_\varepsilon, y_\varepsilon)\| - \varepsilon - \frac{1}{\mu(A_\varepsilon)\mu(B_\varepsilon)} \int_{A_\varepsilon} d\mu(x) \int_{B_\varepsilon} d\mu(y) \|k(x,y) - k(x_\varepsilon, y_\varepsilon)\| \\
&\ge \|k(x_\varepsilon, y_\varepsilon)\| - 3\varepsilon \ge M - 4\varepsilon.
\end{aligned}$$

So that $\|K\|_{1\to\infty} = M$ follows. Therefore to prove the thesis it suffices to show that $M_0 := \sup_{x\in X} \|k(x,x)\| = M$. As $M_0 \le M$ is obvious, we show the opposite inequality. Let $\varepsilon > 0$, and $x_\varepsilon, y_\varepsilon \in X$, $v_\varepsilon, w_\varepsilon \in V$, $A_\varepsilon, B_\varepsilon \subset X$ be as above. Then

$$\begin{aligned}
|\langle f_\varepsilon, Kf_\varepsilon\rangle| &= \frac{1}{\mu(A_\varepsilon)^2} \left| \int_{A_\varepsilon} d\mu(x) \int_{A_\varepsilon} d\mu(y) \langle v_\varepsilon, k(x,y)v_\varepsilon\rangle \right| \\
&\le \frac{1}{\mu(A_\varepsilon)^2} \int_{A_\varepsilon} d\mu(x) \int_{A_\varepsilon} d\mu(y) \langle v_\varepsilon, k(x_\varepsilon, x_\varepsilon)v_\varepsilon\rangle + \\
&+ \frac{1}{\mu(A_\varepsilon)^2} \int_{A_\varepsilon} d\mu(x) \int_{A_\varepsilon} d\mu(y)\, |\langle v_\varepsilon, [k(x,y) - k(x_\varepsilon, x_\varepsilon)]v_\varepsilon\rangle| \\
&\le \|k(x_\varepsilon, x_\varepsilon)\| + \sup_{x,y\in A_\varepsilon} \|k(x,y) - k(x_\varepsilon, x_\varepsilon)\| \le M_0 + \varepsilon,
\end{aligned}$$

where the last inequality follows from the continuity of k, if we choose A_ε small enough. Analogously $|\langle g_\varepsilon, Kg_\varepsilon\rangle| \le M_0 + 2\varepsilon$. Then using the estimates proved above and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
M - 4\varepsilon &\le |\langle f_\varepsilon, Kg_\varepsilon\rangle| \\
&\le |\langle f_\varepsilon, Kf_\varepsilon\rangle|^{1/2} |\langle g_\varepsilon, Kg_\varepsilon\rangle|^{1/2} \\
&\le M_0 + 2\varepsilon,
\end{aligned}$$

and from the arbitrariness of ε we get the thesis. □

Lemma 3.12. *If $\langle f, Kf\rangle \geq 0$ for any $f \in L^2(X, \mathcal{M}, \mu; V)$, then $\langle v, k(x,x)v\rangle \geq 0$ for any $x \in X$, $v \in V$.*

Proof. Assume on the contrary that there are $x_0 \in X$, $v \in V$ s.t. $\langle v, k(x_0, x_0)v\rangle < 0$. Then, by continuity, there is an open neighbourhood U of x_0 s.t. Re $\langle v, k(x,y)v\rangle < 0$, for any $x, y \in U$. Let $f_U(x) := \frac{\chi_U(x)}{\mu(U)} v$, so that

$$\begin{aligned} 0 \leq \langle f_U, Kf_U\rangle &= \mathrm{Re}\, \langle f_U, Kf_U\rangle \\ &= \frac{1}{\mu(U)^2} \int_U d\mu(x) \int_U d\mu(y) \mathrm{Re}\, \langle v, k(x,y)v\rangle < 0 \end{aligned}$$

which is absurd. □

Theorem 3.13. *Let X be a Hausdorff space, μ a Borel measure on it, $T_t : L^1(X, \mathcal{M}, \mu; V) \to L^\infty(X, \mathcal{M}, \mu; V)$ a semigroup of integral operators with continuous kernels $k(t,x,y)$, satisfying the hypotheses of Proposition 3.8, and assume T_t is a positive bounded operator on $L^2(X, \mathcal{M}, \mu; V)$. Then*

$$d_\infty(T) = \liminf_{t\to\infty} \frac{-2\log(\sup_{x\in X} Tr(k(t,x,x)))}{\log t}.$$

Proof. In the following we use the notation $f \bowtie g$, where $f, g : [0,\infty) \to [0,\infty)$ to say that there are $t_0 > 0$, $C > 0$ s.t. $C^{-1} \leq \frac{f(t)}{g(t)} \leq C$, for any $t \geq t_0$. As $\|\cdot\|_\infty$ and $\|\cdot\|_1$ are equivalent on $End(V)$, and using Lemmas 3.11, 3.12, we get

$$\begin{aligned} \|T_t\|_{1\to\infty} &= \sup_{x\in X} \|k(t,x,x)\| \\ &\bowtie \sup_{x\in X} Tr(|k(t,x,x)|) = \sup_{x\in X} Tr(k(t,x,x)). \end{aligned}$$

Therefore

$$\begin{aligned} d_\infty(T) &= \liminf_{t\to\infty} \frac{-2\log \|T_t\|_{1\to\infty}}{\log t} \\ &= \liminf_{t\to\infty} \frac{-2\log(\sup_{x\in X} Tr(k(t,x,x)))}{\log t}. \end{aligned}$$

□

Using these results we can show the relation between the asymptotic dimension of the heat kernel semigroup and the Novikov-Shubin numbers.

Corollary 3.14. *Let M be a complete connected Riemannian n-manifold, and G an infinite discrete group of isometries of M, acting freely and properly discontinuously, and with $X := M/G$ a compact manifold. Then $d_\infty(\mathrm{e}^{-t\Delta_k}) = \underline{\alpha}_k(M,G)$.*

Proof. In the following we use the notation $f \bowtie g$, as in the proof of Theorem 3.13. Let us denote by $H_k(t,x,y)$ the kernel of the integral operator $\mathrm{e}^{-t\Delta_k}$, and observe that

$$\sup_{x\in M} Tr(H_k(t,x,x)) = \sup_{x\in\mathcal{F}} Tr(H_k(t,x,x)) \bowtie \inf_{x\in\mathcal{F}} Tr(H_k(t,x,x)),$$

where the last relation follows from the fact that $\overline{\mathcal{F}}$ is compact and $Tr(H_k(t,x,x)) > 0$. Therefore

$$\sup_{x\in M} Tr(H_k(t,x,x)) \bowtie \int_{\mathcal{F}} Tr(H_k(t,x,x))dvol(x) = \vartheta_k(t) - b_k.$$

Then, using Theorem 3.13, we get

$$\begin{aligned} d_\infty(T) &= \liminf_{t\to\infty} \frac{-2\log \|\mathrm{e}^{-t\Delta_k}\|_{1\to\infty}}{\log t} \\ &= \liminf_{t\to\infty} \frac{-2\log(\vartheta_k(t) - b_k)}{\log t} = \underline{\alpha}_k(M,G). \end{aligned}$$

□

3.4 Comparison between the algebras associated to a covering manifold and a general open manifold

In this subsection we study the relation between the von Neumann algebra of G-invariant operators considered here, and the C*-algebra of almost local operators considered in [13], namely the norm closure of the finite propagation operators, and the traces on these algebras.

Proposition 3.15. *Let us denote by $\mathcal{A}_k$ the C*-algebra of almost local operators on k-forms. Then $\mathcal{A}_k \cap \mathcal{M}_k$ is weakly dense in $\mathcal{M}_k$.*

Proof. First we choose a fundamental domain $\mathcal{F}$ in M and denote by e_g the projection given by the multiplication operator by the characteristic function of $g\mathcal{F}$, $g \in G$. Then denote by G_n the ball of radius n in G, namely the set of elements which can be written as words of length $\leq n$ in terms of a prescribed set of generators for G.
For any selfadjoint operator a acting on $L^2\Lambda^k(M)$ and any $n \in \mathbb{N}$ set

$$a_n := \sum_{g^{-1}h\in G_n} e_g a e_h,$$

and note that a_n has finite propagation, hence it belongs to $\mathcal{A}_k$.
Observe then that if a is bounded, a_n is bounded too. Indeed

$$\begin{aligned} (x, a_n x) &= \sum_{g^{-1}h\in G_n} (x, e_g a e_h x) \leq \|a\| \sum_{g^{-1}h\in G_n} (e_g x, e_h x) \\ &\leq \|a\| \sum_{g^{-1}h\in G_n} \frac{\|e_g x\|^2 + \|e_h x\|^2}{2} = \|a\| \|x\| \, \#(G_n). \end{aligned}$$

Also, if a is periodic, a_n is periodic too. Indeed, for any $\gamma \in G$,

$$L_\gamma a_n = \sum_{g^{-1}h \in G_n} L_\gamma e_g a e_h = \sum_{g^{-1}h \in G_n} e_{\gamma g} a e_{\gamma h} L_\gamma = a_n L_\gamma .$$

Since a_n converges weakly to a, the thesis follows. □

Consider now the case that G is amenable, and the corresponding (regular) exhaustion $\mathcal{K}$ on M (see [18]). Denoting with $Tr_{\mathcal{K}}$ the trace on $\mathcal{A}_k$ introduced in [13], the following holds

Corollary 3.16. *Tr_G and $Tr_{\mathcal{K}}$ coincide on $\mathcal{A}_k \cap \mathcal{M}_k$, hence Tr_G is uniquely determined by $Tr_{\mathcal{K}}$.*

Proof. In this case $Tr_{\mathcal{K}}$ is given, for $T \in \mathcal{A}_k \cap \mathcal{M}_k$, by

$$Tr_{\mathcal{K}}(T) \equiv \frac{Tr(e_1 T e_1)}{vol(\mathcal{F})} = Tr_G(T),$$

where $1 \in G$ is the identity element, and we have chosen $vol(\mathcal{F}) = 1$. □

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