SINGULAR TRACES AND THEIR APPLICATIONS TO GEOMETRY

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Introduction

Born as a mathematical curiosity when Dixmier showed their existence on $\mathcal{B}(\mathcal{H})$, singular traces turned out to play a central role in the integration on noncommutative manifolds in Connes' setting [7].

Indeed Connes observed that the (logarithmic) Dixmier trace of a pseudo-differential operator of negative order coincides (up to a constant) with the Wodzicki residue [40] of such an operator, and may be used to “ redefine” the integral of functions on a compact spin manifold.

This finally led Connes to the proposal of a noncommutative (compact) manifold as a triple $(\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{A}$ plays the role of the algebra of functions, $\mathcal{H}$ that of a Clifford bundle, and D of the Dirac operator. A non-commutative dimension d can be associated with this triple according to the Weyl asymptotics relation, namely to the order of growth of the eigenvalues of D. This can be restated by saying that d is characterized by the logarithmic divergence of the trace of D^{-d}.

Then the noncommutative dimension appears as the analogue of the Hausdorff dimension, namely as the unique number such that the corresponding d-integration, the singular trace on the ideal generated by D^{-d}, is non-trivial.

In this paper we review how this idea can be further pursued, making use of different generalizations of singular traces to the von Neumann and C*-algebra settings.

Indeed on the one hand the family of singularly traceable operators has been enlarged in order to contain also trace class elements, and on the other hand a new family of singular traces appears, detecting the “rate of divergence” of some unbounded measurable operators affiliated to a continuous semifinite von Neumann algebra. This family may be defined on C*-algebras with a trace too, with the aid of noncommutative Riemann integration.

The first phenomenon produced a criterion for singular traceability in terms of the infinitesimal order of an operator, irrespective of the trace class membership (cf. Proposition 2.5). Therefore such an order has a dimensional interpretation, because a singular trace (a noncommutative integration) is associated with it.

We then propose an interpretation of the second as associated with the asymptotic dimension of a manifold. This dimension is a large-scale analogue of the Kolmogorov dimension and may be attached to any metric space. On a suitable class of open manifolds, it may be computed in terms of the spectral behaviour of some geometrical operator, e.g. as the "order of infinite" of the inverse of the Laplace-Beltrami operator. In this case also, the asymptotic dimension is a noncommutative dimension, namely a non-trivial singular trace is defined on the bimodule generated by the corresponding power of the Laplacian.

In the particular case of universal coverings, the asymptotic dimension coincides with one of the classical L^2 invariants of the manifold, namely with the 0-th Novikov-Shubin invariant.

Finally we notice the following property of the Novikov-Shubin invariants α_p: there exists a nontrivial singular trace on the bimodule generated by $\Delta_p^{-\alpha_p/2}$. This furnishes a dimensional interpretation of the α_p, as the corresponding (non-commutative) integration is non trivial.

1. Singular traces on the compact operators of a Hilbert space.

In this section we present the theory of singular traces on $\mathcal{B}(\mathcal{H})$ as it was developed by Dixmier [13], who first showed their existence, and then in [37] and [2].

A singular trace on $\mathcal{B}(\mathcal{H})$ is a tracial weight vanishing on the finite rank projections. Any tracial weight is finite on an ideal contained in $\mathcal{K}(\mathcal{H})$ and may be decomposed as a sum of a singular trace and a multiple of the normal trace. Therefore the study of (non-normal) traces on $\mathcal{B}(\mathcal{H})$ is the same of the study of singular traces.

Moreover, making use of unitary invariance, a singular trace should depend only on the eigenvalue asymptotics, namely, if a and b are positive compact operators on $\mathcal{H}$ and $\mu_n(a) = \mu_n(b) + o(\mu_n(b))$, μ_n denoting the n-th eigenvalue, then $\tau(a) = \tau(b)$ for any singular trace τ.

The main problem about singular traces is therefore to detect which asymptotics may be "resummed" by a suitable singular trace, that is to say, which operators are singularly traceable.

In order to state the most general result in this respect we need some notations.

Let a be a compact operator. Then we denote by $\{\mu_n(a)\}$ the sequence of the eigenvalues of $|a|$, arranged in non-increasing order and counted with multiplicity. We consider also the (integral) sequence $\{S_n(a)\}$ defined as follows:

$$S_n(a) := \begin{cases} \sum_{k=1}^{n} \mu_k(a) & a \notin \mathcal{L}^1 \\ \sum_{k=n+1}^{\infty} \mu_k(a) & a \in \mathcal{L}^1, \end{cases}$$

where $\mathcal{L}^1$ denotes the ideal of trace-class operators. A compact operator is called *singularly traceable* if there exists a singular trace which is finite non-zero on $|a|$. We observe that the domain of such singular trace should necessarily contain the ideal $\mathcal{I}(a)$ generated by a. A compact operator is called *eccentric* if

$$\frac{S_{2n_k}(a)}{S_{n_k}(a)} \to 1 \tag{1.1}$$

for a suitable subsequence n_k. Then the following theorem holds.

1.1 Theorem. *A positive compact operator a is singularly traceable iff it is eccentric. In this case there exists a sequence n_k such that condition (1.1) is satisfied and, for any singular state ω on ℓ^∞, the positive functional*

$$\tau_\omega(b) = \begin{cases} \omega\left(\left\{\frac{S_{n_k}(b)}{S_{n_k}(a)}\right\}\right) & b \in \mathcal{I}(a)_+ \\ +\infty & b \notin \mathcal{I}(a),\ b > 0, \end{cases}$$

is a singular trace whose domain is the ideal $\mathcal{I}(a)$ generated by a.

The first result on singular traceability is due to Dixmier, who showed in [13] that $\frac{S_{2n}(a)}{S_n(a)} \to 1$ is a sufficient condition for singular traceability when $a \notin \mathcal{L}^1$. Then Varga proved that the eccentricity condition is necessary and sufficient when $a \notin \mathcal{L}^1$ [37]. Finally it was observed in [1] that singular traces may be non-trivial on trace-class operators, while the theorem in the previous form is contained in [2]

Now we briefly recall what can be the general form of a singular trace. The form presented above is due to Varga for the non $\mathcal{L}^1$ case and was generalized in [1,2]. It is easy to see that a function on positive operators described in terms of a positive functional on the eigenvalue sequences gives a trace if and only if it is additive. It was shown in [23] that all traces (on a factor) may be described in terms of dilation invariant functionals on the sequence $\{\mu_n(a)\}$, and that dilation invariance implies additivity when the functional on the eigenvalues is *monotone* (cf. Section 2), hence traces may be produced in this way. The question of the existence of singular traces determined by non-monotone functionals is still open. The family of singular traces determined by monotone functionals naturally splits in two parts.

1.2 Proposition. *Let τ be a singular trace described by a monotone functional φ on the eigenvalue sequences of positive operators. The functional φ is increasing iff the domain of τ contains $\mathcal{L}^1$, indeed τ vanishes on $\mathcal{L}^1$. The functional φ is decreasing, iff the domain of τ is contained in $\mathcal{L}^1$. If τ is non-trivial, the above inclusions are strict.*

Proof. If φ is increasing and c is a positive element with $tr(c) = 1$, we may find a finite rank operator b s.t. $\sum_{k=1}^n \mu_k(c) \le \sum_{k=1}^n \mu_k(b)$ for any $n \in \mathbb{N}$, therefore $\tau(c) = \varphi(\mu_n(c)) \le \varphi(\mu_n(b)) = \tau(b) = 0$ where the inequality holds because φ is increasing (cf. Definition 2.1) and the last equality follows by the singularity of τ. Conversely if φ is decreasing, it is either 0 or ∞ identically outside $\mathcal{L}^1$, but the value 0 is ruled out by singularity. When τ is non-trivial and decreasing, and c is a positive operator s.t. $\tau(c) = 1$, we may find an increasing sequence $\alpha_n \to \infty$ s.t. $\mu_n(c) \cdot \alpha_n$ is still summable, hence an operator $c' \in \mathcal{L}^1$ s.t. $\frac{\mu_n(c')}{\mu_n(c)} = \alpha_n \to \infty$. By positivity and singularity of τ we get $\tau(c') \ge \alpha_n \tau(c)$ for any n. The increasing case is obvious.

A principal ideal corresponding to the domain of a trace vanishing on $\mathcal{L}^1$ described in Theorem 1.1 may be embedded in a maximal symmetrically normed ideal (cf. [19]). It was shown by Varga that singular traces may always be defined on these maximal ideals, and the traces described by Dixmier when $S_{2n}(a)/S_n(a) \to 1$ were indeed defined on such ideals.

The form of the traces on maximal ideals is very similar to the form of the traces in Theorem 1.1. In fact while the eccentricity of a provides enough dilation invariance to ensure additivity of the functional τ on the principal ideal generated by a, the dilation invariance of ω is required in order to get additivity of τ on the maximal ideal generated by a.

It was shown by Varga that the principal ideal $\mathcal{I}(a)$ generated by an eccentric operator a (not in $\mathcal{L}^1$) is not norm closed, and its norm closure is strictly smaller than the corresponding maximal normed ideal.

As for traces inside $\mathcal{L}^1$, symmetrically normed ideals are ruled out, because the minimal symmetrically normed ideal containing a trace class operator is $\mathcal{L}^1$, and the only trace whose domain is $\mathcal{L}^1$ is the normal one (up to a scalar), cf. Proposition 1.2. When a is in $\mathcal{L}^1$ (and eccentric), it is not even true in general that the set $\{b \in \mathcal{K}(\mathcal{H}) : S_n(b)/S_n(a) \in \ell^\infty\}$ is an ideal. When $S_{2n}(a)/S_n(a) \to 1$, it was observed in [2] that this property is true, but the ideal is not symmetrically normed.

2. Singular traces on semifinite von Neumann algebras.

In order to extend the theory developed in the previous section to general semifinite von Neumann algebras, we need a substitute for the sequence of singular values of a compact operator, which is provided by the notion of non-increasing rearrangement of an operator.

Since the parameter of this rearrangement varies continuously, divergences in zero appear when the operator is unbounded. This phenomenon produces a new class of traces (II_1 singular traces, or singular traces at 0), which are defined on $\mathcal{M}-\mathcal{M}$-bimodules of measurable operators, $\mathcal{M}$ a semifinite von Neumann algebra.

Indeed in this section all traces are described as unitary invariant positive functionals on $\overline{\mathcal{M}}$ but, while singular traces at ∞ (like those in Section 1) may be defined on $\mathcal{M}$, and possibly extended to $\overline{\mathcal{M}}$, singular traces at 0 vanish on all bounded operators, hence make sense on $\overline{\mathcal{M}}$ only.

As we shall see below, singular traceability and eccentricity are equivalent when $\mathcal{M}$ is a factor, as in the type I case. In general, eccentricity is only a sufficient condition to produce a singular trace on the generated bimodule.

Let $(\mathcal{M}, tr)$ be a pair consisting of a semifinite von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ with a normal semifinite faithful trace. Let $\widetilde{\mathcal{M}}$ be the collection of the closed, densely defined operators on $\mathcal{H}$ affiliated with $\mathcal{M}$ and define $\overline{\mathcal{M}} := \{x \in \widetilde{\mathcal{M}} : tr(e_{|x|}(t,\infty)) < \infty$ for some $t > 0\}$. Then $\overline{\mathcal{M}}$, equipped with strong sense operations [35] and with the topology of convergence in measure [36,29], becomes a topological $*$-algebra, called the algebra of tr-measurable operators.

For example, if $\mathcal{M} := L^\infty(X,m)$ and $tr(f) := \int f dm$, then $\widetilde{\mathcal{M}}$ is the *-algebra of m-measurable functions that are finite a.e., and $\overline{\mathcal{M}}$ is the *-subalgebra of $\widetilde{\mathcal{M}}$ consisting of functions that are bounded except on a set of finite m-measure, whereas if $\mathcal{M} = \mathcal{B}(\mathcal{H})$, then $\widetilde{\mathcal{M}}$ consists of all closed densely defined operators on $\mathcal{H}$, while $\overline{\mathcal{M}} = \mathcal{M}$.

Let $a \in \overline{\mathcal{M}}$, and define [17] the distribution function of a (w.r.t. tr) as $\lambda_a(t) := tr(e_{|a|}(t,\infty))$, $t \geq 0$, and the non-increasing rearrangement of a as

$\mu_a(t) := \inf\{s \geq 0 : \lambda_a(s) \leq t\}$, $t > 0$. μ_a is a non-increasing and right-continuous function. Moreover $\lim_{t \downarrow 0} \mu_a(t) = \|a\| \in [0, \infty]$.

For example, if $\mathcal{M} := L^\infty(X, m)$ and $tr(f) := \int f dm$, then for any $f \in \overline{\mathcal{M}}$, $\mu_f \equiv f^*$, the classical non-increasing rearrangement of f. If instead $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and tr is the usual trace, then $\mu_a = \sum_{n=0}^\infty \mu_n(a) \chi_{[n,n+1)}$.

If $\mathcal{M}$ contains no minimal projections, μ_a is given by a $min - max$ formula: for all $t \in [0, tr(1))$,

$$\mu_a(t) = \inf_{tr(p) \leq t} \sup_{q \leq p^\perp} \frac{tr(|a|q)}{tr(q)},$$

where p and q are tr-finite, non-zero projections in $\mathcal{M}$.

Non-increasing rearrangements are very useful in describing traces and their domains, *i.e.* bimodules of measurable operators. More precisely, on a semifinite factor, unitarily invariant spaces (such as ideals or bimodules) may be expressed in terms of the corresponding spaces of non-increasing rearrangements, and unitarily invariant functionals (such as traces) may be expressed in terms of functionals on the non-increasing rearrangements, but observe that linearity of the obtained function is not at all obvious. Following Dixmier, it is natural to think that dilation invariance is the counterpart of linearity on the space $\mathcal{D}$ of decreasing rearrangements. Indeed dilation invariance is a necessary condition, and it becomes sufficient when a further positivity property (*monotonicity*) is assumed. The following results in this section are taken from [23].

Introduce the notation $\mathcal{D}$ for the convex cone of positive measurable functions on $\mathbb{R}_+ \equiv (0, \infty)$ which are finite, non increasing and right continuous, and consider the action $\lambda \to f^\lambda$ of the multiplicative group $\mathbb{R}_+$ on $\mathcal{D}$ given by $f^\lambda(t) = \lambda f(\lambda t)$. A face $\mathcal{F}$ in $\mathcal{D}$ is called *dilation invariant* if $f \in \mathcal{F} \Rightarrow f^\lambda \in \mathcal{F}$, $\lambda \in \mathbb{R}_+$. A positive linear functional φ on $\mathcal{D}$ is called *dilation invariant* if $\varphi(f) = \varphi(f^\lambda)$, $f \in \mathcal{D}$, $\lambda \in \mathbb{R}_+$. Then measurable bimodules on a semifinite σ-finite factor are described by dilation invariant faces in $\mathcal{D}$, and a generalised Calkin theorem holds:

(i) each non-zero measurable bimodule X on $\mathcal{M}$ contains the two-sided ideal of bounded, finite-rank elements. In particular, if $\mathcal{M}$ is finite, $X \supseteq \mathcal{M}$

(ii) each measurable bimodule X satisfies either $X \subseteq \{a \in \overline{\mathcal{M}} : \mu_a \to 0\}$ or $X \supseteq \mathcal{M}$.

Some known results may be seen as consequences of this theorem. In particular, each type II_1 factor is algebraically simple and each two-sided ideal in a type I_∞ factor is contained in the compact operators.

As regards traces on a general semifinite von Neumann algebra, a positive linear functional on $\overline{\mathcal{M}}$ is said a *trace* if, for each unitary element $u \in \mathcal{M}$, $\tau(a) = \tau(uau^*)$, $a \in \overline{\mathcal{M}}_+$. The bimodule X given by the linear span of the set $X_+ := \{a \in \overline{\mathcal{M}} : \tau(a) < \infty\}$ is called the domain of τ.

There is a relation between dilation invariance on the space $\mathcal{D}$ and linearity on $\overline{\mathcal{M}}$.

2.1 Definition. *A functional φ on a subset X of $\mathcal{D}$ is increasing if for $f, g \in X$,*

$$\int_0^t f(s)ds \leq \int_0^t g(s)ds, \quad \forall t \in (0, \infty) \Rightarrow \varphi(f) \leq \varphi(g),$$

and decreasing if for $f, g \in X$,

$$\int_t^\infty f(s)ds \leq \int_t^\infty g(s)ds, \quad \forall t \in (0,\infty) \Rightarrow \varphi(f) \leq \varphi(g).$$

A functional φ on $\mathcal{D}$ is monotone if $\varphi|_{\mathcal{D}_0}$ and $\varphi|_{\mathcal{D}_b}$ are either increasing or decreasing, where $\mathcal{D}_0 := \{f \in \mathcal{D} : \text{supp } f \text{ is compact}\}$, $\mathcal{D}_b := \{f \in \mathcal{D} : f \text{ is bounded}\}$.

2.2 Theorem. *Let $\mathcal{M}$ be a semifinite von Neumann algebra and φ a monotone positive linear functional on $\mathcal{D}$. If φ is dilation invariant the functional τ given by*

$$\tau(a) := \varphi(\mu_a), \qquad a \in \overline{\mathcal{M}}_+$$

is a trace on $\overline{\mathcal{M}}$, and the converse implication holds when $\mathcal{M}$ is a factor.

To give examples of singular traces, we construct dilation invariant monotone positive linear functionals on $\mathcal{D}$ by making use of eccentric operators. Let us introduce a useful shorthand notation:

$$S_a^\infty(t) := \begin{cases} \int_1^t \mu_a(s)ds & \mu_a \notin L^1(1,\infty) \\ \int_t^\infty \mu_a(s)ds & \mu_a \in L^1(1,\infty), \end{cases}$$

and

$$S_a^0(t) := \begin{cases} \int_0^t \mu_a(s)ds & \mu_a \in L^1(0,1) \\ \int_t^1 \mu_a(s)ds & \mu_a \notin L^1(0,1). \end{cases}$$

2.3 Definition. *Let $(\mathcal{M}, tr)$ be a semifinite algebra with a normal semifinite faithful trace. An element $a \in \overline{\mathcal{M}}$ is called 0-eccentric if 1 is a limit point of $\{\frac{S_a^0(2t)}{S_a^0(t)}\}$ when $t \to 0$, and ∞-eccentric if 1 is a limit point of $\{\frac{S_a^\infty(2t)}{S_a^\infty(t)}\}$, when $t \to \infty$. The element a is called eccentric if it is 0 or ∞ eccentric.*

We remark that the notion of eccentricity in a von Neumann algebra depends on the chosen trace.

2.4 Theorem. *Let $a \in \overline{\mathcal{M}}$ be eccentric at 0, resp. at ∞. Then there exists a sequence $t_k \to 0$, resp. $t_k \to \infty$, such that for any singular state ω on ℓ^∞ the (monotone) function on positive elements*

$$\tau(b) := \begin{cases} \omega\left(\frac{S_b^\infty(t_k)}{S_a^\infty(t_k)}\right) & b \in X(a)_+ \\ +\infty & b \notin X(a), b > 0, \end{cases}$$

resp.

$$\tau(b) := \begin{cases} \omega\left(\frac{S_b^0(t_k)}{S_a^0(t_k)}\right) & b \in X(a)_+ \\ +\infty & b \notin X(a), b > 0, \end{cases}$$

is a singular trace on $\overline{\mathcal{M}}$ (i.e. it vanishes on tr-finite projections) whose domain is $X(a)$. If $\mathcal{M}$ is a factor, a is eccentric if and only if it is singularly traceable.

Finally we describe a sufficient condition for the singular traceability of a positive operator.

We say that an operator a is an infinitesimal of order α (cf. [11]) if $\mu_a(t) = O(t^{-\alpha})$, $t \to \infty$, and define the *order of infinitesimal* of a ($\mathrm{ord}_\infty(a)$) to be the supremum of such α.

Analogously we say that a is an infinite of order α if $\mu_a(t)^{-1} = O(t^{-\alpha})$, $t \to 0$, and define the *order of infinite* of a ($\mathrm{ord}_0(a)$) to be the supremum of such α.

The order of infinite(simal) may be easily computed as

$$\mathrm{ord}_\infty(a) = \liminf_{t\to\infty} \frac{\log \mu_a(t)}{\log(1/t)}, \quad \mathrm{ord}_0(a) = \liminf_{t\to 0} \frac{\log \mu_a(t)}{\log(1/t)}.$$

Then the following proposition holds ([26]).

2.5 Proposition. *Let a be a positive measurable operator affiliated to $(\mathcal{M}, \tau)$. Then, if $\varepsilon = ord_0(a) \in (0, \infty)$, resp. $\omega = ord_\infty(a) \in (0, \infty)$, then a^ε is singularly traceable at 0, resp. a^ω is singularly traceable at ∞.*

3. Singular traces on C*-algebras.

The purpose of this section is to describe the extension of the theory of singular traces to C*-algebras as it is developed in [25]. As shown in the previous section, singular traces depend, by their very definition, on a given normal trace. Therefore the basic object in this section will be a pair $(\mathcal{A}, \tau)$ consisting of a C*-algebra and a semicontinuous semifinite trace. In fact, by the classical theory of traces on C*-algebras [14], semicontinuity and semifiniteness are necessary and sufficient conditions for τ to give rise (and to be determined by) a normal faithful semifinite trace on the von Neumann algebra generated by $\mathcal{A}$ in the GNS representation relative to τ.

When τ is infinite, the theory of singular traces at ∞ may be developed, and is a simple extension of the corresponding theory on semifinite von Neumann algebras. For any $a \in \mathcal{A}$, we may set $\mu_a = \mu_{\pi_\tau(a)}$, where both the GNS representation π_τ and the non-increasing rearrangement μ are associated with the trace τ.

Then an operator $b \in \mathcal{A}$ is eccentric at ∞ if $\pi_\tau(b)$ is, and in this case the singular traces defined on the ideal generated by $\pi_\tau(b)$ in $\pi_\tau(\mathcal{A})''$ in the previous section may be pulled back on the ideal generated by b in $\mathcal{A}$.

The question is not that simple for singular traces at 0, as are the traces that are needed to give noncommutative measures associated with the asymptotic dimension of open manifolds (see Section 7).

Indeed, as shown in the previous section, singular traces at 0 are defined on bimodules of (unbounded) operators affiliated to a von Neumann algebra, therefore we need a noncommutative integration for C*-algebras, namely a *-bimodule over $\mathcal{A}$ of unbounded operators "affiliated to $\mathcal{A}$" and measurable w.r.t. τ in such a way that this bimodule can be represented in every faithful representation of $\mathcal{A}$, the trace being still well defined on the image. Singular traces will then be defined on suitable sub-bimodules.

It turns out that if this bimodule has to contain enough operators, its bounded part should be strictly larger then the C*-algebra $\mathcal{A}$. Therefore our first goal is the construction of a *-algebra $\mathcal{R}$ containing $\mathcal{A}$ and with some universal measurability property, namely τ has to extend to a trace on $\mathcal{R}$, and any faithful representation π of $\mathcal{A}$ should extend to a representation of $\mathcal{R}$ in such a way that τ is well defined on $\pi(\mathcal{R})$. In the following $\mathcal{A}$ is considered as a subalgebra of its enveloping von Neumann algebra $\mathcal{A}^{**}$, the normal extension of τ to $\mathcal{A}^{**}$ is still denoted by τ, and $\mathcal{R}$ will be chosen as the *enveloping Riemann algebra* of $\mathcal{A}$ [25].

A class A of selfadjoint elements in $\mathcal{A}^{**}$ is bounded if it is bounded in norm and has τ finite support if there exists a τ-finite projection in $\mathcal{A}^{**}$ which contains the supports of all elements of A. Two classes A_+, A_- of selfadjoint elements are called *separated* if $a_- \leq a_+$ for all $a_\pm \in A_\pm$, *τ-contiguous* if $\forall \varepsilon > 0$ $\exists a_\pm \in A_\pm$ such that $\tau(a_+ - a_-) < \varepsilon$. An element x is a *separating element* for two separated classes $A_\pm$ if for any $a_\pm \in A_\pm$ we have $a_- \leq x \leq a_+$.

3.1 Definition. *A linear subspace X of $\mathcal{A}^{**}$ is called Dedekind complete (w.r.t. bounded τ-contiguous classes with τ-finite support) if, given two classes in X as before and a separating element in $\mathcal{A}^{**}$ we have $x \in X$. The minimal Dedekind complete C*-subalgebra $\mathcal{R}$ of $\mathcal{A}^{**}$ containing $\mathcal{A}$ is called the enveloping Riemann algebra of $\mathcal{A}$.*

In the commutative case, the algebra $\mathcal{R}$ coincides with the algebra of Riemann integrable functions (see e.g. [34]).

3.2 Theorem. *The Riemann enveloping algebra $\mathcal{R}$ of $\mathcal{A}$ is universally τ-measurable, namely for any faithful representation π of $\mathcal{A}$, there exists $\rho_\pi : \pi(\mathcal{R}) \to \pi_\tau(\mathcal{A})''$ such that the following diagram commutes*

$$\begin{tikzcd} \mathcal{R} \arrow[r, "\pi"] \arrow[dr, "\pi_\tau"'] & \pi(\mathcal{R}) \arrow[d, "\rho_\pi"] \\ & \pi_\tau(\mathcal{A})'' \end{tikzcd}$$

Moreover the closure $\mathcal{R}_0$ of the τ-finite elements of $\mathcal{R}$ coincides with the norm closure of the Dedekind completion of the linear span of the τ-finite projections in $\mathcal{R}$.

In the last part of this section we deal with a concrete C*-algebra $\mathcal{A}$ of operators on a Hilbert space $\mathcal{H}$, equipped with a semicontinuous semifinite trace, and write (by an abuse of notation) $\mathcal{R}$ for the image of $\mathcal{R}$ under (the normal extension to $\mathcal{A}^{**}$ of) the given representation of $\mathcal{A}$. We observe that, by universal τ-measurability, τ is well defined on this algebra. Our aim is the construction of a (τ-a.e.) bimodule $\overline{\mathcal{R}}$ of (unbounded) operators on $\mathcal{H}$ "affiliated to" $\mathcal{R}$, on which the trace still makes sense.

The elements of $\overline{\mathcal{R}}$ are closed unbounded operators on $\mathcal{H}$ affiliated to $\mathcal{A}''$ which possess a strongly dense domain [5] of τ-cofinite projections of $\mathcal{R}$. The morphism ρ_π extends to $\overline{\mathcal{R}}$ with values in the measurable operators affiliated with $\pi_\tau(\mathcal{A})''$, therefore (singular) traces on this von Neumann algebra may be pulled back to $\overline{\mathcal{R}}$.

3.3 Definition. *A strongly dense domain is an increasing sequence $\{e_n\}$ of projections in $\mathcal{R}$ such that $\tau(1 - e_n) \to 0$. We denote by e the supremum of e_n. $\overline{\mathcal{R}}$ denotes the set of closed densely defined operators A affiliated to $\mathcal{A}''$ such that there exists a strongly dense domain $\{e_n\}$ with eAe_n, $eA^*e_n \in \mathcal{R}$.*

If A is a general linear operator on $\mathcal{H}$, its adjoint is a closed densely defined operator from the closure of the range of A to $\mathcal{H}$, with range in the closure of the domain of A. Then the "double" adjoint, which we shall denote with $A^\natural$, is a closed densely defined operator on $\mathcal{H}$. It coincides with the closure of A when A is closable and densely defined. Two operators S, T are said to be equal τ-almost everywhere if there exists a strongly dense domain e_n such that $\mathcal{H}_0 := \cup_n e_n \mathcal{H}$ is contained in the domains of S and T and $eS|_{\mathcal{H}_0} = eT|_{\mathcal{H}_0}$. Strong sense operations are defined as $T \oplus S = (T+S)^\natural$, $a \odot T = (aT)^\natural$. We observe that when τ is faithful and S and T are closed $S = T$ τ-a.e. implies $S = T$ [35]. Let us denote by $\mathcal{M}$ the von Neumann algebra $\pi_\tau(\mathcal{A})''$, and by $\overline{\mathcal{M}}$ the algebra of measurable elements [35]. The following theorem holds:

3.4 Theorem. *The set $\overline{\mathcal{R}}$ is a τ-a.e. *-bimodule over $\mathcal{R}$, namely it is closed under the (strong sense) *-bimodule operations and the module identities hold τ-a.e. The morphism ρ_π extends to a (τ-a.e.) bimodule morphism from $\overline{\mathcal{R}}$ to $\overline{\mathcal{M}}$ whose kernel consists exactly of the τ-a.e. null elements.*

Now we can define singular traces at 0 on $\mathcal{A}$ as positive functionals on $\overline{\mathcal{R}}$ which are invariant under conjugation by unitaries in $\mathcal{R}$ and vanish on τ-finite projections.

We define the non-increasing rearrangement of an element $A \in \overline{\mathcal{R}}$ in terms of ρ_π as

$$\mu_A(t) := \mu_{\rho_\pi(A)}(t)$$

and call A *eccentric at 0* accordingly (see the previous section).

3.5 Theorem. *If $T \in \overline{\mathcal{R}}$ is eccentric at 0 then T is singularly traceable, namely there exists a singular trace defined on the sub-bimodule over $\mathcal{R}$ generated by T. Such a trace may be defined as the pull back of the singular trace defined on $\overline{\mathcal{M}}$. On the positive elements A of the $\mathcal{R}$-bimodule generated by T, singular traces may then be written as*

$$\tau_\omega(A) = \omega\left(\left\{\frac{S^0_A(t_k)}{S^0_T(t_k)}\right\}\right)$$

for a suitable sequence $t_k \to 0$ and any singular state ω on ℓ^∞.

4. Compact manifolds

In this section we briefly recall some aspects of the applications of singular traces to noncommutative geometry. We refer the reader to [7,8,9,11] for basic results and the latest developments in this context.

The first appearance of singular traces in geometry was by means of a result by A. Connes in the late 1980's [6], which was later used to define the dimension of a noncommutative compact manifold in terms of the Weyl asymptotics, namely as the inverse of the order of growth of the eigenvalues of differential operators of order one (the Dirac operator for example). Moreover Connes observed that a noncommutative measure (trace) may be attached to such noncommutative dimension via the Dixmier trace, setting $\tau(a) = \mathrm{Tr}_\omega(a|D|^{-d})$, where a is a "function" on the noncommutative manifold, D is the Dirac operator, d is the noncommutative dimension and Tr_ω is the (logarithmic) Dixmier trace. According to the identification of the Dixmier trace with the Wodzicki residue, such trace gives back the ordinary integration in the case of commutative Riemannian manifolds. To be more precise the following holds

4.1 Theorem. ([7], Proposition 5, p. 307) *Let M be an n-dimensional compact manifold and let T be a pseudodifferential operator of order $-n$ acting on sections of a complex vector bundle E on M. Then*

(i) *the corresponding operator on $\mathcal{H} := L^2(M,E)$ belongs to the ideal $\mathcal{L}^{(1,\infty)}(\mathcal{H})$*

(ii) *the (logarithmic) Dixmier trace $Tr_\omega(T)$ is independent of ω and is equal to the Wodzicki residue $Res(T)$.*

Recall that the Wodzicki residue $Res(T)$ is given by an explicit formula [40] involving the principal symbol $\sigma_{-n}(T)$, which is a homogeneous function of degree $-n$ on the cotangent bundle T^*M of M

$$Res(T) := \frac{1}{n(2\pi)^n}\int_{S^*M} trace_E(\sigma_{-n}(T))$$

where S^*M is the unit-sphere bundle induced by a Riemannian metric on M. The same formula makes sense for pseudodifferential operators of arbitrary order [40], and is independent of any choice of coordinate charts and metrics. Wodzicki residue is the unique trace on the algebra of pseudodifferential operators which extends the Dixmier trace on operators of order $\leq -n$.

The previous result applied to the Dirac operator on a compact spin manifold M of dimension n, says that the inverse of the Dirac operator is an infinitesimal of order n (cf [11] and also Section 2), hence D^{-n} is in the domain of the Dixmier trace.

Then Connes proposed a description of a noncommutative compact manifold of dimension n in terms of a *spectral triple* $(\mathcal{A},\mathcal{H},D)$ where the Dirac operator D has to be an infinitesimal of order n. Of course this does not determine the number n, in particular does not imply that the logarithmic Dixmier trace is non trivial on D^{-n}. Connes proposed a cohomological determination of the dimension, and showed that in this case the Dixmier trace of D^{-n} is not zero [7, Corollary 10, p. 309].

However, in many cases, the dimension may be recovered as the order of infinitesimal of D^{-1} defined in Section 2. Even though this dimension does not guarantee that the logarithmic trace is non-zero on D^{-n}, we remark that a singular trace is nevertheless defined (and non-zero) on the ideal generated by D^{-n} (Proposition 2.5), hence a noncommutative integration may be defined in full generality on the spectral triple.

Finally we mention that the notion of dimension has been refined introducing the dimension spectrum, and refer the reader to [8,9,10,11] for details on this argument.

5. Asymptotic dimension for metric spaces.

The definition of asymptotic dimension is given in the context of metric dimension theory, as a suitable large scale analogue of the metric dimension of Kolmogorov and Tihomirov [27]. The results of this section are from [24]. In the following (X,δ) will denote a metric space, $B_X(x,R)$ the open ball in X with centre x and radius R, and $n_r(\Omega)$ the least number of open balls of radius r which cover $\Omega\subset X$.

5.1 Definition. *Let (X,δ) be a metric space. We call*

$$d_\infty(X) := \lim_{r\to\infty} \limsup_{R\to\infty} \frac{\log n_r(B_X(x,R))}{\log R},$$

the asymptotic dimension of X.

It is easy to show that $d_\infty(X)$ does not depend on x. Moreover it is a dimension, namely it satisfies

(i) If $X\subset Y$ then $d_\infty(X)\leq d_\infty(Y)$.

(ii) If $X_1,X_2\subset X$ then $d_\infty(X_1\cup X_2)=\max\{d_\infty(X_1),d_\infty(X_2)\}$.

(iii) If X and Y are metric spaces, then $d_\infty(X\times Y)\leq d_\infty(X)+d_\infty(Y)$.

Now we give some examples.

(i) $\mathbb{R}^n$ has asymptotic dimension n.

(ii) Set $X := \cup_{n\in\mathbb{Z}}\{(x,y)\in\mathbb{R}^2 : \delta((x,y),(n,0)) < \frac{1}{4}\}$, endowed with the Euclidean metric, then $d_\infty(X)=1$.

(iii) Let X be the unit ball in an infinite dimensional Banach space. Then $d_\infty(X)=0$.

(iv) Let Γ be a finitely generated discrete group. Then $d_\infty(\Gamma)$ is equal to the growth of Γ (cf. below and Section 6).

The asymptotic dimension is easier to compute when there is a uniformly bounded Borel measure μ on X *i.e.* there are functions β_1,β_2, s.t. $0<\beta_1(r)\leq \mu(B(x,r))\leq\beta_2(r)$, for all $x\in X$, $r>0$. In this case

$$d_\infty(X) = \limsup_{R\to\infty}\frac{\log\mu(B(x,R))}{\log R}.$$

In particular, if M is a complete Riemannian manifold of bounded geometry, namely it has positive injectivity radius, sectional curvature bounded from above, and Ricci curvature bounded from below, then the volume form is a uniformly bounded Borel measure. Compact Riemannian manifolds and their universal coverings, leaves of a compact Riemannian manifold are in this class.

If M has bounded geometry, and satisfies Grigor'yan isoperimetric inequality [20], then

$$d_\infty(M) = \limsup_{t\to\infty}\frac{-2\log p_t(x_0,x_0)}{\log t},$$

where p_t is the integral kernel of $\mathrm{e}^{-t\Delta}$ and Δ the Laplace-Beltrami operator on M.

This shows that $d_\infty(M)$ is related to the *dimension at infinity* of the heat semigroup defined by Varopoulos [39] (cf. next section).

A function f is called a rough isometry if $f:X\to Y$ s.t. $a^{-1}\delta_X(x_1,x_2)-b\leq \delta_Y(f(x_1),f(x_2))\leq a\delta_X(x_1,x_2)+b$, for all $x_1,x_2\in X$, and $\bigcup_{x\in X}B_Y(f(x),\varepsilon)= Y$, for some $a\geq 1$, $b,\varepsilon\geq 0$.

In [4] a notion of discretization of a manifold is introduced and it is proved that complete Riemannian manifolds with Ricci curvature bounded from below are roughly isometric to any of their discretizations, endowed with the combinatorial metric.

Since the asymptotic dimension is invariant under rough isometries, it may be computed from any discretization of the manifold.

In particular, if M is the universal covering of a compact manifold X, and $\Gamma := \pi_1(X)$ is the fundamental group of X, each orbit of Γ is a discretisation of M, hence

$$d_\infty(M) = d_\infty(\Gamma) = growth(\Gamma).$$

Our final examples are inspired by a recent work of E. B. Davies' [12]. Let $E \subset M$ be a cylindrical end of a Riemannian manifold M, that is homeomorphic to $(1, \infty) \times A$, where A is a compact Riemannian manifold. Set $\partial E := \{1\} \times A$, $E_r := \{x \in E : \delta(x, \partial E) < r\}$, where δ is the restriction of the metric in M. Assume there are positive constants c, D s.t. $c^{-1}r^D \leq vol(E_r) \leq cr^D$, for all $r \geq 1$. If the volume form on E is a uniformly bounded measure then $d_\infty(E) = D$. A particular case is when the metric is given by $ds^2 = dx^2 + f(x)^2 d\omega^2$, where f is an increasing smooth function. Then the volume form is a uniformly bounded measure, so that $d_\infty(E) = D$.

6. Singular traces on covering spaces

In this section we consider an open manifold M which is the universal covering of a compact manifold X, and denote by Γ the homotopy group of X.

In order to state an index theorem for covering manifolds, M. Atiyah [3] had to measure the size of the kernel and the cokernel of elliptic differential operators on M in spite of their infinite-dimensionality.

He observed that the Γ-invariant operators on $L^2(M)$ belong to a type II_∞ von Neumann algebra, isomorphic to $\mathcal{R}(\Gamma) \otimes \mathcal{B}(L^2(M/\Gamma))$. Since Γ is discrete, its von Neumann algebra is type II_1 with a canonical trace, therefore a canonical trace Tr_Γ is defined on Γ-invariant operators. Then the projections on the kernel and the cokernel of an elliptic Γ-invariant operator D have finite Γ-trace, hence the index of D can be defined as the Fredholm Γ-index of D.

Atiyah also observed that (finite) Betti – von Neumann numbers may be defined as

$$\beta_p := Tr_\Gamma(e_{\{0\}}(\Delta_p)),$$

where we denoted by $e_\Omega(A)$ the spectral projection on the set Ω relative to the operator A. It was proved by Dodziuk [15] that such numbers are homotopy invariants on X.

Since M is not compact, the spectrum of Δ_p is not necessarily discrete, hence the function $Tr_\Gamma(e_{[0,\lambda]}(\Delta_p))$ is not necessarily constant on a (right) neighborhood of 0. In fact Novikov and Shubin [30] conjectured that the order of infinitesimal of $Tr_\Gamma(e_{[0,\lambda]}(\Delta_p)) - \beta_p$, when $\lambda \to 0$, has a geometrical meaning.

The joint efforts of Novikov-Shubin [31] (cf. [16]), Lott [28] and Gromov-Shubin [22] showed that the numbers, in Lott's normalization,

$$\begin{aligned}
\underline{\alpha}_p &= 2 \liminf_{\lambda \to 0} \frac{\log(N_p(\lambda) - \beta_p)}{\log \lambda} \\
&= 2 \liminf_{t \to \infty} \frac{-\log(\vartheta_p(t) - \beta_p)}{\log t} \\
\overline{\alpha}_p &= 2 \limsup_{\lambda \to 0} \frac{\log(N_p(\lambda) - \beta_p)}{\log \lambda} \\
\overline{\alpha}'_p &= 2 \limsup_{t \to \infty} \frac{-\log(\vartheta_p(t) - \beta_p)}{\log t},
\end{aligned}$$

where $N_p(\lambda) = Tr_\Gamma(e_{[0,\lambda]}(\Delta_p))$ and $\vartheta_p(t) = Tr_\Gamma(e^{-t\Delta_p})$, are invariant under homotopies of the base space M/Γ. Since then several papers have been devoted to the study of these (and other L^2) invariants on coverings, and we refer to [18] for further references.

It is well known that if the $\lim_{t\to 0}$ in the definition of the Novikov-Shubin invariants is replaced by the $\lim_{t\to\infty}$, one gets the dimension of the manifold. Then one expects that the global invariants α_p play the role of asymptotic dimensions.

This interpretation is well motivated when $p = 0$.

It has been shown by Gromov [21] that a discrete group is quasi-nilpotent if and only if the number $\gamma(n)$ of distinct words of no more than n letters in the generators grows polinomially. In this case $\gamma(n) \simeq n^d$ for some integer d, which is called the *growth* of Γ. When Γ is the homotopy group of a compact manifold, a result of Varopoulos [38] implies that $N_0(\lambda) \simeq \lambda^{\text{growth}(\Gamma)/2}$ (cf. [28]), hence $\underline{\alpha}_0 = \overline{\alpha}_0 = \overline{\alpha}'_0 = \text{growth}(\Gamma)$.

Moreover Varopoulos, Saloff-Coste and Coulhon [39] say that a semigroup T_t has dimension at infinity equal to n if $\|T_t\|_{1\to\infty} \leq Ct^{-n/2}$, $t \geq 1$. The family of such n's being a left half line, it is determined by its supremum. Then we may set

$$d_\infty(T_t) = \sup\{n > 0 : \|T_t\|_{1\to\infty} \leq Ct^{-n/2},\ t \geq 1\}.$$

Then the following holds.

6.1 Theorem. *[26] Let M be the universal covering of a compact manifold whose homotopy group is denoted by Γ. Then*

$$d_\infty(M) = \mathit{growth}(\Gamma) = d_\infty(e^{-t\Delta}).$$

The previous theorem shows in particular that $\alpha_0(M)$ depends on M up to rough isometries, hance is independent of the particular covering structure we put on it.

We conclude this section giving an argument in favour of the dimensional interpretation of the α_p, $p > 0$.

In Alain Connes' noncommutative geometry, the dimension is the exponent to be given to a geometric pseudodifferential operator of order -1 in order to obtain a singularly traceable operator. Such a property is enjoyed by the $\overline{\alpha}_p$'s, as the following corollary of Theorem 2.5 shows.

6.2 Theorem. *Let M be the universal covering of a compact manifold X whose homotopy group is denoted by Γ. Then if $\overline{\alpha}_p$ is finite, $\Delta^{-\frac{\overline{\alpha}_p}{2}}$ is singularly traceable, namely a singular trace is defined on the bimodule generated by $\Delta^{-\frac{\overline{\alpha}_p}{2}}$ on the von Neumann algebra of Γ-invariant operators on the L^2 sections of $\Lambda^p T^* M$.*

7. Novikov-Shubin invariant for open manifolds.

In this section we define the 0-th Novikov-Shubin invariant for open manifolds with bounded geometry and regular polynomial growth, showing that this number carries a singular trace. Indeed it coincides with the asymptotic dimension on a suitable subclass of manifolds. The results in this section are extracted from [24].

The trace which is to replace Atyiah's trace is essentially the one constructed by J. Roe in [33]. We need some regularizations in order to get a semicontinuous semifinite trace on a C*-algebra and therefore apply the results described in Section 3.

A natural C*-algebra of operators $\mathcal{A}$ on a manifold of bounded geometry M is that of almost local operators, obtained as the norm closure of the *-algebra of finite propagation speed operators, *i.e.* operators $A \in \mathcal{B}(L^2(M))$ for which there is a constant $u(A) > 0$ s.t. for any compact subset K of M, any $\varphi \in L^2(M)$, supp $\varphi \subset K$, we have supp $A\varphi \subset \{x \in M : \delta(x, K) \leq u(A)\}$.

$\mathcal{A}$ contains any $\mathcal{C}_0$-functional calculus of the Laplace-Beltrami operator.

As concerns a trace on this C*-algebra, its construction is more involved. First we need to consider a more restricted class of manifolds that we call of regular polynomial growth, that is

$$\lim_{r\to\infty} \frac{V(x, r+R)}{V(x,r)} = 1$$

for all $x \in M$, $R > 0$.

Recall that an operator T on $L^2(M)$ is called locally trace class if, for any compact set $K \subset M$, $E_K T E_K$ is trace class, where E_K denotes the projection given by the characteristic function of K. It is known that the functional $\mu_T(K) := Tr(E_K T E_K)$ extends to a Radon measure on M.

Then consider the set $\mathcal{J}_{0+}$ of positive locally trace class operators T, such that $\limsup_{r\to\infty} \frac{\mu_T(B(x,r))}{V(x,r)}$ is finite and independent of $x \in M$, which turns out to be a hereditary (positive) cone in $\mathcal{B}(L^2(M))$.

Choose a translationally invariant state ω on $L^\infty([0,\infty))$, and consider the functional φ_0 on $\mathcal{B}(L^2(M))_+$ given by

$$\varphi_0(A) := \begin{cases} \omega\left(\frac{\mu_A(B(x,r))}{V(x,r)}\right) & A \in \mathcal{J}_{0+} \\ +\infty & A \in \mathcal{B}(L^2(M))_+ \setminus \mathcal{J}_{0+} \end{cases}$$

Then φ_0 is a weight on $\mathcal{B}(L^2(M))$ and does not depend on $x \in M$.

The functional φ_0 was considered by J. Roe in [33]. Indeed regular polynomial growth implies that $\{B(x, kr)\}_{k\in\mathbb{N}}$ is a regular exhaustion according to [33]. The further hypothesis that ω is translationally invariant plays a crucial role in our construction.

Applying results in [32] and classical results about traces on C*-algebras, we may produce a trace τ on $\mathcal{A}$ which is a semicontinuous semifinite regularisation of φ_0. It turns out that τ coincides with φ_0 on operators with a suitably regular kernel, such as the heat semigroup $e^{-t\Delta}$.

The semicontinuous regularisation φ of $\varphi_0|_{\mathcal{A}}$ is given by

$$\varphi(A) := \sup\{\psi(A) : \psi \in \mathcal{A}^*_+,\ \psi \leq \varphi_0\} \equiv \sup_{\psi\in\mathcal{F}(\varphi_0)} \psi(A),$$

where $\mathcal{F}(\varphi_0) := \{\psi \in \mathcal{A}^*_+ : \exists\ \varepsilon > 0,\ (1+\varepsilon)\psi < \varphi_0\}$. φ is a trace on $\mathcal{A}$.

Finally τ is the semifinite extension of φ [14], obtained as the pull back of the normal semifinite faithful trace on the weak closure of $\mathcal{A}$ in the GNS representation induced by φ.

We can now define the 0-th Novikov-Shubin invariant of M as

7.1 Definition. *Let M be an open manifold with bounded geometry and regular polynomial growth. Then the (0-th) Novikov-Shubin invariant of M is defined as*

$$\alpha_0(M) = 2 \limsup_{t\to 0} \frac{\log(N(t))}{\log t} = 2 \limsup_{t\to\infty} \frac{\log(\vartheta(t))}{\log \frac{1}{t}},$$

where $N(t)$ and $\vartheta(t)$ are defined as in Section 6, the Γ-trace being replaced by τ.

As illustrated in Section 3, this produces a singular trace on $\mathcal{A}$.

7.2 Theorem. *Let M be an open manifold with bounded geometry and regular polynomial growth. Then there exists a singular trace on (the unbounded operators affiliated to) $\mathcal{A}$ which is finite on the *-bimodule over $\mathcal{A}$ generated by $\Delta^{-\alpha_0(M)/2}$.*

Finally, if M satisfies Grigor'yan isoperimetric inequality [20], Theorem 6.1 may be generalized as follows.

7.3 Theorem. *Let M be an open manifold with bounded geometry, regular polynomial growth and satisfying Grigor'yan isoperimetric inequality. Then*

$$d_\infty(M) = \alpha_0(M).$$

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