SINGULAR TRACES ON SEMIFINITE VON NEUMANN ALGEBRAS

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Abstract. Singular traces are constructed on a general semifinite von Neumann algebra, thus generalizing the result of Dixmier [6]. Moreover our technique produces singular traces on type II_1 factors. Such traces, though vanishing on all bounded operators, are non trivial on the *-algebra of affiliated unbounded operators.

On a semifinite factor, we show that all traces are given by a dilation invariant functional on the cone of positive decreasing functions on $[0, \infty)$, and we prove that the existence of a singular trace which is non trivial on a given operator is equivalent to an eccentricity condition on the singular values function, a result which generalizes the theorem given in [1] for $B(H)$.

Introduction

The solution of the conjecture on the uniqueness of the trace on $B(H)$ given by Dixmier in 1966, in which he showed the existence of non normal traces, had important consequences in the applications, as it is shown by the extensive use of Dixmier traces in Alain Connes' non commutative geometry [4], and opened some lines of research in the general setting, i.e. the description, or classification, of non normal traces. The present work belongs to the second context, generalizing previous constructions to semifinite von Neumann algebras, finding new types of singular traces, and also clarifying the role of some general properties.

In his remarkable paper, Dixmier [6] observed that traces, being unitarily invariant positive functionals, should be given by a positive function of the singular values. Therefore the problem was to exhibit functions which give rise to linear functionals. Dixmier's construction may be easily described: choose a normalizing sequence with a suitable asymptotic behavior, divide the partial sums of the singular values by the given normalizing sequence, apply to the resulting sequence a two-dilation invariant state on ℓ^∞. This procedure gives a linear functional on positive operators, which is therefore a trace. The normalizing sequence may be identified with the partial sums of the singular values of a reference operator, on which the trace takes value 1. Dixmier traces are *singular*, i.e. they vanish on finite rank operators. In fact they vanish on all trace class operators.

In a paper by Varga [15], the asymptotic condition on the normalizing sequence is replaced by a weaker condition, the *irregularity* of the reference operator a, and such condition is proven to be equivalent to the *traceability* of a, i.e. the existence of a trace which takes value 1 on a. Irregularity turns out to be a weak form of dilation invariance, therefore the ideal generated by an irregular operator inherits some form of invariance. This allows Varga to do without the dilation invariance of the state, but his functionals are traces only when restricted to a smaller domain. The traces constructed by Varga vanish on trace class operators, too.

Then, in [1], it was noted that the irregularity condition, which is trivial on trace class operators, may be replaced by a stricter condition, which we call *eccentricity*. If Dixmier (or Varga) procedure is suitably refined, eccentric sequences, not only reproduce Dixmier traces, but also give rise to a new class of singular traces, whose domain is contained in $L^1(H)$. Moreover, it is proven in [1] that eccentricity condition is equivalent to *singular traceability*, i.e. the existence of a singular trace which takes value 1 on the given operator.

In this paper we extend the theory of singular traces to semifinite von Neumann algebras. As a first result, such an extension produces a new phenomenon, that is the existence of singular traces on bimodules of measurable operators affiliated to a given von Neumann algebra which depend only on the behavior of the "large eigenvalues", therefore vanishing on all bounded operators. Such traces are present when the algebra has a non trivial continuous part, therefore we produce, in particular, singular traces on type II_1 factors. More precisely, the domain of such traces is a bimodule of measurable operators affiliated to the given factor, and they necessarily vanish on the whole algebra. As a further result, the key role of some properties appears in its full generality in this context.

The main technical point is the extensive use of the notion of the (trace-) decreasing rearrangement of an operator [7], which substitutes the sequence of the singular values in the general setting. In fact, on a semifinite factor, both a bimodule and a trace are rearrangement invariant objects, therefore all relevant properties concerning these objects should admit a description in terms of the decreasing rearrangement. More precisely, unitarily invariant spaces (such as ideals or bimodules) may be expressed in terms of the corresponding spaces of decreasing rearrangements, and unitarily invariant functionals (such as traces) may be expressed in terms of functionals on the decreasing rearrangements.

As already pointed out, the non trivial property to prove when using rearrangements is linearity. Following Dixmier, it is natural to think that dilation invariance is the counterpart of linearity on the space $\mathcal{D}$ of decreasing rearrangements. We show that this is precisely the case for bimodules on semifinite factors, i.e. bimodules are in $1-1$ correspondence with dilation invariant faces in $\mathcal{D}$. As far as traces are concerned, we are not able to produce such a complete description. Indeed we prove that dilation invariance is a necessary condition, and it becomes sufficient when a further positivity property (*monotonicity*) is assumed. Further results on this topic are contained in [2], where it is proven that monotonicity is not a necessary property.

As we mentioned before, the counterpart of singular traceability in terms of rearrangements is eccentricity, i.e. the two conditions are equivalent on a semifinite factor, but the explicit construction of traces on bimodules generated by eccentric operators works on a general semifinite von Neumann algebra.

This paper is divided into "commutative" sections, in which the properties of the space of rearrangements are analyzed, and "non commutative" ones, in which "commutative" results are applied to operator algebras.

In section 1 the main inequalities concerning the behavior of rearrangements, or better their integrals, w.r.t. the additive structure of operators, are proved.

In section 2 the relation between linearity and dilation invariance is proven in a commutative setting, the general form of a dilation invariant functional is described and a sort of commutative Calkin theorem is proven for dilation invariant faces. Monotone dilation invariant functionals are also described. These results are used in section 3 and 4 to get a complete description of bimodules on semifinite factors, together with a generalized Calkin theorem, and to give a general description of singular traces on measurable operators affiliated to semifinite factors via dilation invariant functionals. Also, it is proven that monotone dilation invariant functionals always give rise to traces on a semifinite von Neumann algebra.

Section 5 is dedicated to the relationships between eccentric functions and dilation invariant functionals, thus bringing in section 6 to the singular traceability theorem and to the explicit construction of singular traces on semifinite von Neumann algebras.

To better illustrate the unifying role played by the notion of decreasing rearrangement, most of the statements are given for a general semifinite factor (or von Neumann algebra), even though analogous results in the type I case are already contained in [1] or [15]. The hypotheses of factoriality and σ-finiteness are often assumed, but they are not needed in the explicit construction of singular traces.

Section 1. Rearrangements on von Neumann algebras.

In this section (M, tr) is a pair consisting of a semifinite von Neumann algebra with a normal semifinite faithful trace. We refer the reader to [14] for the general theory of von Neumann algebras. Let $\tilde{M}$ be the collection of the closed, densely defined operators on H affiliated with M. Then, $\forall x = \int_0^\infty t\ de_x(t) \in \tilde{M}_+$, $E \subset \mathbf{R} \to \nu_x(E) := tr(e_x(E))$ is a Borel measure on $\mathbf{R}$, and $tr(x) := \int_0^\infty t\ d\nu_x(t)$ is a faithful extension of tr to $\tilde{M}_+$. Define

$$\overline{M} := \{x \in \tilde{M} : tr(e_{|x|}(t, \infty)) < \infty \text{ for some } t > 0\}.$$

Then $\overline{M}$, equipped with strong sense operations [11] and with the topology of convergence in measure ([13], [9]), becomes a topological $*$-algebra, called the algebra of tr-measurable operators.

1.1 Remark. If $M := L^\infty(X, m)$ and $tr(f) := \int f dm$, then $\tilde{M}$ is the *-algebra of m-measurable functions that are finite m-a.e., and $\overline{M}$ is the *-subalgebra of $\tilde{M}$ consisting of functions that are bounded except on a set of finite m-measure.

1.2 Definition. [7] *Let $a \in \overline{M}$, and define, for all $t \geq 0$,*

(i) $\lambda_a(t) := tr(e_{|a|}(t, \infty))$, *the distribution function of a w.r.t. tr,*

(ii) $\mu_a(t) := \inf\{s \geq 0 : \lambda_a(s) \leq t\}$, $t > 0$, *the decreasing rearrangement of a w.r.t. tr,*

(iii) $\sigma_a(t) := \int_0^t \mu_a(r)dr$,

(iv) $\varsigma_a(t) := \int_t^\infty \mu_a(r)dr$.

1.3 Remarks. (i) If $M = B(H)$ and tr is the usual trace, then $\mu_a = \sum_{n=0}^\infty s_n \chi_{[n,n+1)}$, where $\{s_n\}$ is the sequence of singular values of the operator a, arranged in decreasing order and counted with multiplicity. [12]

(ii) If $M := L^\infty(X, m)$ and $tr(f) := \int f dm$, then, $\forall f \in \overline{M}$, $\mu_f \equiv f^*$, the classical decreasing rearrangement of f [3]. Observe that, in this setting, one defines the decreasing rearrangement also for $f \in \tilde{M}$, and for such f the following are equivalent (i) $f \in \overline{M}$, (ii) $\exists s_0 > 0$ s.t. $\lambda_f(s_0) < \infty$, (iii) $\lim_{s\to\infty} \lambda_f(s) = 0$, (iv) $f^*(t) < \infty$, $\forall t > 0$.

1.4 Proposition. [7] *Let $a, b, c \in \overline{M}$, $e \in Proj(M)$ then*

(i) *The function μ_a is non-increasing and right-continuous. Moreover* $\lim_{t\downarrow 0} \mu_a(t) = \|a\| \in [0, \infty]$,

(ii) $\mu_a = \mu_{|a|} = \mu_{a^*}$ *and* $\mu_{\alpha a} = |\alpha|\mu_a$ *for* $\alpha \in \mathbf{C}$,

(iii) $\mu_{a+b}(s + t) \leq \mu_a(s) + \mu_b(t)$, $s, t > 0$,

(iv) $\mu_{abc} \leq \|a\|\|c\|\mu_b$,

(v) $\mu_{ae}(t) = 0$, *for* $t \geq tr(e)$,

(vi) $tr(|a|) = \int_0^\infty \mu_a(t)dt$,

(vii) $\mu_a(t) = \inf\{\|ap\| : p \in Proj(M),\ tr(p^\perp) \leq t\}$.

1.5 Definition. *The pair (S,e), given by a subset S of the interval $[0,tr(1))$ and an increasing map e from S to the projections in M, is called a (partial) trace family when $tr(e(t))=t$, $t\in S$.*

We observe that the set of trace families is not empty since $S:=\{0\}$, $e(0)=0$ is a trace family. This set admits a natural order relation given by $(S_1,e_1)\prec(S_2,e_2)$ if $S_1\subset S_2$ and $e_2|_{S_1}\equiv e_1$. In this case (S_2,e_2) is also called an extension of (S_1,e_1). A trace family is called *global* if $S\equiv[0,tr(1))$, and is specified, in this case, only by the map e. A global trace family is called *complete* if $\sup e(t)=1$.

1.6 Proposition. *Let M have no minimal projection. Then each trace family (S_0,e_0) admits a global extension. In particular, there exists a global trace family, and any such family is strongly continuous. If M is σ-finite, there exist global, complete trace families.*

Proof. It is easy to see that we may apply Zorn lemma to the ordered set $\{(S,e):(S,e)\succeq(S_0,e_0)\}$, therefore we get a maximal element (S,e). Since the continuous extension of e to the closure $\overline{S}$ of S in $[0,tr(1))$ is a trace family, we get, by maximality, $S=\overline{S}$. Finally let (α,β) be a maximal open interval in S^c. Since (S,e) is maximal, $e(\beta)-e(\alpha)$ is a minimal projection, i.e. $\alpha=\beta$ and (S,e) is global.
Now we set

$$e(t_0^{\pm}):=\lim_{t\to t_0^{\pm}}e(t)$$

and, since e is increasing and tr is positive, we get $tr(e(t_0^-))=tr(e(t_0^+))$. Because tr is faithful we get strong continuity of e.
If M is σ-finite, there exists an increasing sequence of projections with finite trace $\{e_n\}$ such that $\sup e_n=1$. Then set $S:=\{tr(e_n):n\in\mathbf{N}\}$, $e(tr(e_n)):=e_n$. Each global extension of such an (S,e) is complete.

□

1.7 Proposition. *Let M have no minimal projection, and set $\mathcal{M}:=L^\infty(0,\infty)$. There exists a (non unital) normal isomorphism $i:\mathcal{M}\to M$ such that*

$$tr(i(f))=\int f dm\ ,\qquad f\in\overline{\mathcal{M}}_+.$$

If M is σ-finite, i may be chosen unital.

Proof. Let e be a global trace family in M. Then the map

$$i(f):=\int f(t)de(t)$$

gives the desired isomorphism. If M is σ-finite, we can choose a complete e, therefore the isomorphism is unital.

$\square$

1.8 Lemma. *Let M have no minimal projection, $a \in \overline{M}_+$. Then for all $r \in (0, tr(1))$, $\exists p_r \in Proj(M)$ s.t. $tr(p_r) = r$, $[a, p_r] = 0$,*

$$\mu_{ap_r} = \mu_a \chi_{[0,r)},$$

and

$$\mu_{ap_r^\perp}(t) = \mu_a(t + r), \ t > 0.$$

Proof. Let $a = \int_0^\infty t de(t)$ be the spectral decomposition of a and set $S := \{tr(e[t, \infty)) : t > 0\} \setminus \{\infty\}$, and, for all $s \in S$, $f(s) := e[t, \infty)$, where $e[t, \infty)$ is the unique projection s.t. $tr(e[t, \infty)) = s$. Then (S, f) is a trace family and, by proposition 1.6, admits a global extension that we denote again by (S, f). Let us set $p_r := f(r)$, so that $tr(p_r) = r$, and, as there exists $t > 0$ s.t. $e(t, \infty) \leq p_r \leq e[t, \infty)$, because a global trace family is increasing, we easily get $[a, p_r] = 0$. Besides $\lambda_{ap_r}(s) = tr(e(s, \infty)p_r) = tr(e(s, \infty)) \wedge r = \lambda_a(s) \wedge r$, and $\lambda_{ap_r^\perp}(s) = \lambda_a(s) - \lambda_{ap_r}(s) = (\lambda_a(s) - r) \vee 0$, so that

$$\begin{aligned}
\mu_{ap_r}(s) &= \inf\{v \geq 0 : \lambda_{ap_r}(v) \leq s\} \\
&= \inf\{v \geq 0 : \lambda_a(v) \wedge r \leq s\} \\
&= \mu_a(s)\chi_{[0,r)}(s),
\end{aligned}$$

and

$$\begin{aligned}
\mu_{ap_r^\perp}(s) &= \inf\{v \geq 0 : \lambda_{ap_r^\perp}(v) \leq s\} \\
&= \inf\{v \geq 0 : \lambda_a(v) - r \leq s\} \\
&= \mu_a(r + s).
\end{aligned}$$

$\square$

1.9 Lemma. *Let M have no minimal projection, $a \in \overline{M}$. Then we have*

(i) $\|a\| = \sup_{0 < tr(q) < \infty} \frac{tr(|a|q)}{tr(q)}$, *where $q \in Proj(M)$,*

(ii) for all $t \in [0, tr(1))$,

$$\mu_a(t) = \inf_{tr(p) \leq t} \sup_{q \leq p^\perp} \frac{tr(|a|q)}{tr(q)},$$

where $p, q \in Proj(M)$, $0 < tr(q) < \infty$,

(iii) for all $r \in [0, tr(1))$, $s \in (0, tr(1) - r]$,

$$\int_r^{r+s} \mu_a(t)dt = \inf_{tr(p) \leq r} \sup_{q \leq p^\perp, tr(q) \leq s} tr(|a|q),$$

where $p, q \in Proj(M)$,

(iv) *in particular* $\sigma_a(t) = \sup_{tr(p)\le t} tr(|a|p)$ *and* $\varsigma_a(t) = \inf_{tr(p)\le t} tr(|a|p^\perp)$.

Proof. As $\mu_a = \mu_{|a|}$ we can assume that $a > 0$. Let $a = \int_0^\infty s\, de_a(s)$ be the spectral decomposition.
(i) Suppose first that a is unbounded, so that $\|a\| = \infty$. As a is tr-measurable, there exists $s_0 > 0$ s.t. $tr(e_a(s,\infty)) < \infty$, for all $s > s_0$. Then we have $tr(ae_a(s,\infty)) \ge s\, tr(e_a(s,\infty))$, so that $\sup_{0<tr(q)<\infty} \frac{tr(aq)}{tr(q)} \ge s$, for all $s > s_0$, and the thesis follows.
Therefore we can suppose that $a \in M$. As $tr(aq) \le \|a\| tr(q)$, for all $q \in Proj(M)$, there follows $\sup \frac{tr(aq)}{tr(q)} \le \|a\|$.
For the opposite inequality let, for all $r \in (0, tr(1))$, $p_r \in Proj(M)$ be as in lemma 1.8. Then $tr(ap_r) = \int_0^{tr(1)} \mu_{ap_r}(s)ds = \int_0^r \mu_a(s)ds$. For all $\varepsilon > 0$, there is $r_\varepsilon > 0$ s.t. $\mu_a(s) \ge \|a\| - \varepsilon$, $\forall s \in [0, r_\varepsilon]$, because of right-continuity of μ_a. Then we get $tr(ap_{r_\varepsilon}) = \int_0^{r_\varepsilon} \mu_a(s)ds \ge r_\varepsilon(\|a\| - \varepsilon)$, so that $\frac{tr(ap_{r_\varepsilon})}{tr(p_{r_\varepsilon})} \ge \|a\| - \varepsilon$, and the thesis follows from the arbitrariness of ε.
(ii) Observe that $\mu_a(t) = \inf_{tr(p)\le t} \|ap^\perp\|$ so the thesis follows from (i).
(iii) Let us first suppose that $s < \infty$. Then for all $p \in Proj(M)$, $tr(p) \le r$ one has

$$\begin{aligned}\int_r^{r+s} \mu_a(t)dt \le \mu_a(r)s \le s\|ap^\perp\| = s \sup_q \frac{tr(aq)}{tr(q)} \\ = \sup_q tr(aq) \frac{s}{tr(q)} \le \sup_q tr(aq),\end{aligned}$$

where the sup is taken over all $q \in Proj(M)$, $0 < tr(q) \le s$, $q \le p^\perp$, so that

$$\int_r^{r+s} \mu_a(t)dt \le \inf_{tr(p)\le r} \sup_{tr(q)\le s, q\le p^\perp} tr(aq). \tag{1.1}$$

Observe that from the above one gets

$$\int_r^\infty \mu_a(t)dt \le \inf_{tr(p)\le r} \sup_{q\le p^\perp} tr(aq),$$

that is (1.1) for $s = \infty$.
For the opposite inequality, let $p_r \in Proj(M)$ be as in lemma 1.8; then we get, for all $q \in Proj(M)$, $tr(q) \le s$, $q \le p_r^\perp$,

$$\begin{aligned}tr(aq) &= \int_0^{tr(1)} \mu_{aq}(t)dt = \int_0^s \mu_{aq}(t)dt \\ &= \int_0^s \mu_{aqp_r^\perp}(t)dt \le \int_0^s \mu_{ap_r^\perp}(t)dt = \int_r^{r+s} \mu_a(t)dt,\end{aligned}$$

from which the thesis follows.
(iv) The result for σ_a is immediate from (iii) (see [7] for a direct proof). As far as ς_a is concerned, from (iii) we get

$$\begin{aligned}\varsigma_a(t) &= \int_t^{tr(1)} \mu_a(r)dr \\ &= \inf_{tr(p)\le t} \sup_{tr(q)\le tr(1)-t, q\le p^\perp} tr(aq) \\ &\le \inf_{tr(p)\le t} tr(ap^\perp),\end{aligned}$$

but for $p = p_t$ of lemma 1.8 we get $\int_t^{tr(1)} \mu_a(r)dr = tr(ap_t^\perp)$, and the thesis follows. □

1.10 Proposition. *Let M be a semifinite von Neumann algebra, tr a n.s.f. trace on M, $a, b \in \overline{M}_+$. Then, for all $0 \le t \le tr(1)$,*
(i) $\sigma_{a+b}(t) \le \sigma_a(t) + \sigma_b(t) \le \sigma_{a+b}(2t)$,
(ii) $\varsigma_{a+b}(t) \ge \varsigma_a(t) + \varsigma_b(t) \ge \varsigma_{a+b}(2t)$.

Proof. The inequalities are trivial for $t = 0$, so we suppose $t > 0$, and set, for the sake of brevity, $\mathcal{P}_r := \{p \in Proj(M) : tr(p) \le r\}$, for $r > 0$.
(i) Let us first suppose that M has no minimal projection. Then the first inequality is in [7]. For the second we get from lemma 1.9(iv),

$$\begin{aligned}\sigma_{a+b}(s+t) &= \sup_{e\in\mathcal{P}_{s+t}} tr((a+b)e) \\ &= \sup_{p\in\mathcal{P}_s, q\in\mathcal{P}_t} tr(a(p\vee q)) + tr(b(p\vee q)) \\ &\ge \sup_{p\in\mathcal{P}_s, q\in\mathcal{P}_t} tr(ap) + tr(bq) \\ &= \sigma_a(s) + \sigma_b(t).\end{aligned}$$

Choosing $s = t$ we get the thesis.
If now M has minimal projections, embed M in $M \otimes L^\infty(0,1)$ equipped with the trace $tr \otimes \int$. Then, by [7, p. 286], we get $\sigma_{a+b}(2t) = \tilde{\sigma}_{a+b}(2t) \ge \tilde{\sigma}_a(t) + \tilde{\sigma}_b(t) = \sigma_a(t) + \sigma_b(t)$, where $\tilde{\sigma}_a(t) = \int_0^t \tilde{\mu}_a(s)ds$ and $\tilde{\mu}_a$ is the rearrangement of a w.r.t. $tr \otimes \int$.
(ii) Consider first the case when M has no minimal projections. Then, from lemma 1.9(iv), we get

$$\begin{aligned}\varsigma_{a+b}(t) &= \inf_{e\in\mathcal{P}_t} tr((a+b)e^\perp) \\ &\ge \inf_{p\in\mathcal{P}_t} tr(ap^\perp) + \inf_{q\in\mathcal{P}_t} tr(bq^\perp) \\ &= \varsigma_a(t) + \varsigma_b(t)\end{aligned}$$

and, for all $s, t \geq 0$, we get

$$\begin{aligned}
\varsigma_{a+b}(s+t) &= \inf_{e \in \mathcal{P}_{s+t}} tr((a+b)e^{\perp}) \\
&= \inf_{p \in \mathcal{P}_s, q \in \mathcal{P}_t} tr((a+b)(p \vee q)^{\perp}) \\
&\leq \inf_{p \in \mathcal{P}_s, q \in \mathcal{P}_t} tr(ap^{\perp}) + tr(bq^{\perp}) \\
&= \varsigma_a(s) + \varsigma_b(t).
\end{aligned}$$

Choosing $s = t$ we get the result.
If now M has minimal projections, we can proceed as in (i), if we observe that $\tilde{\varsigma}_a(t) = \varsigma_a(t)$, where $\tilde{\varsigma}_a(t) = \int_t^{tr(1)} \tilde{\mu}_a(s)ds$.

□

Section 2. Dilation invariance on $\mathbf{R}_+$.

Let us denote by $\mathcal{D}$ the convex cone of positive measurable functions on $\mathbf{R}_+ \equiv (0, \infty)$ which are finite, non increasing and right continuous. If Ω is a closed subset of $\mathbf{R}_+$, $\mathcal{D}(\Omega)$ will denote the subset of $\mathcal{D}$ whose elements are constant on the connected components of $\mathbf{R}_+ \setminus \Omega$ and are zero after $\sup \Omega$. We note that functions in $\mathcal{D}(\Omega)$ are determined by their restriction to Ω. As a consequence, we shall identify $\mathcal{D}(\mathbf{N})$ with the set of positive non increasing sequences.

We shall denote by $\mathcal{D}_b$, resp. $\mathcal{D}_\infty$, resp. $\mathcal{D}_0$ the set of functions in $\mathcal{D}$ which are bounded, resp. infinitesimal, resp. with compact support in $[0, \infty)$, and by $\mathcal{D}_{b,\infty}$, $\mathcal{D}_{b,0}$ the corresponding intersections. These sets are indeed *faces*, i.e. hereditary subcones, of $\mathcal{D}$. $\mathcal{M}$ denotes the von Neumann algebra $L^\infty(\mathbf{R}_+)$, therefore $\overline{\mathcal{M}}$ denotes, according to section 1, the space of (equivalence classes of) measurable functions on $\mathbf{R}_+$ which are finite almost everywhere and bounded on the complement of a set of finite measure.

Let us consider the action $\lambda \to f^\lambda$ of the multiplicative group $\mathbf{R}_+$ on $\mathcal{D}$ given by:

$$f^\lambda(t) = \lambda f(\lambda t) , \qquad \lambda, t \in \mathbf{R}_+ \tag{2.1}$$

2.1 Definition. *A face $\mathcal{F}$ in $\mathcal{D}$ is called dilation invariant if $f \in \mathcal{F} \Rightarrow f^\lambda \in \mathcal{F}$, $\lambda \in \mathbf{R}_+$.*
A positive linear functional φ on $\mathcal{D}$ is called dilation invariant if $\varphi(f) = \varphi(f^\lambda)$, $f \in \mathcal{D}$, $\lambda \in \mathbf{R}_+$.

Let us note that $\mathcal{D}_0$, $\mathcal{D}_\infty$, etc. are dilation invariant faces in $\mathcal{D}$.

We recall that a positive linear functional on a convex cone C is a positive homogeneous, additive function with values in $[0, +\infty]$. If C is the positive cone of an ordered vector space V, we say φ is a positive linear functional on V if it is a positive linear functional on C. We denote by the same symbol the linear extension of φ to the linear span of $\{v \in C : \varphi(v) < +\infty\}$. Now we show that there is a relation between dilation invariance on the space $\mathcal{D}$ and linearity on $\overline{\mathcal{M}}$.

2.2 Proposition.
(a) *Let $\mathcal{F}$ be a subset of $\mathcal{D}$ such that*

$$\mathcal{F}_* := \{f \in \overline{\mathcal{M}}_+ : f^* \in \mathcal{F}\}$$

is a face in $\overline{\mathcal{M}}_+$. Then $\mathcal{F}$ is a dilation invariant face in $\mathcal{D}$.
(b) *Let φ be a positive functional on $\mathcal{D}$ such that the functional φ_* given by*

$$\varphi_*(f) := \varphi(f^*)\ , \quad f \in \overline{\mathcal{M}}_+$$

is a positive linear functional on $\overline{\mathcal{M}}$. Then its domain $\mathcal{F} := \{f \in \mathcal{D} : \varphi(f) < \infty\}$ is a dilation invariant face in $\mathcal{D}$ and φ is a positive linear, dilation invariant functional on $\mathcal{D}$.

Proof. Let $0 < \lambda \leq 1$, f a $\lambda^{\mathbf{Z}}$-valued function in $\mathcal{D}$. Then f may be written as

$$\varphi \equiv \sum_{m\in\mathbf{Z}} \lambda^m \chi_{[\alpha_m,\alpha_{m+1})}$$

where α_m is a non decreasing sequence of positive numbers. Now we fix $n \in \mathbf{N}$, and consider the functions

$$\left.\begin{aligned} f^{(i)} &:= \sum_{m\in\mathbf{Z}} \lambda^m \chi_{[\beta_{m,i-1},\beta_{m,i})}\ , \quad \beta_{m,i} := n\alpha_m + i(\alpha_{m+1}-\alpha_m) \\ f_{(i)} &:= \sum_{m\in\mathbf{Z}} \lambda^m \chi_{[\gamma_{m,i-1},\gamma_{m,i})}\ , \quad \gamma_{m,i} := \alpha_m + \frac{i}{n}(\alpha_{m+1}-\alpha_m) \end{aligned}\right. \qquad i = 1\dots n$$

and note that
(i) $(f^{(i)})^* \equiv f$, $1 \leq i \leq n$, which implies $\frac{1}{n}\sum_i (f^{(i)})^* \equiv f$,
(ii) $\frac{1}{n}\sum_i f^{(i)} \equiv f^{1/n}$,
(iii) $\sum_i f_{(i)} \equiv f$ and in particular $f_{(i)} \leq f$,
(iv) $(f_{(i)})^* \equiv \frac{1}{n} f^n$, $1 \leq i \leq n$, which implies $\sum_i (f_{(i)})^* \equiv f^n$.
By (i) and (ii) we get $f \in \mathcal{F} \Rightarrow f^{(i)} \in \mathcal{F}_*$, $i = 1, \dots, n \Rightarrow f^{1/n} \in \mathcal{F}$, and $\varphi(f) = 1/n \sum_i \varphi_*(f^{(i)}) = \varphi(f^{1/n})$. Analogously, by (iii) and (iv) we get $f \in \mathcal{F} \Rightarrow f_{(i)} \in \mathcal{F}_*$, $i = 1, \dots, n \Rightarrow f^n \in \mathcal{F}$ and $\varphi(f) = \sum_i \varphi_*(f_{(i)}) = \varphi(f^n)$.
As a consequence, for all $\lambda^{\mathbf{Z}}$-valued function $f \in \mathcal{D}$, $\lambda > 0$, and all rational numbers r, we get $\varphi(f) = \varphi(f^r)$ and $f \in \mathcal{F} \iff f^r \in \mathcal{F}$.

Now we define the ε-approximation f_ε of a function $f \in \mathcal{D}$. For a fixed positive λ, let us consider the function $\imath_\lambda$ on $[0, +\infty)$ defined as follows: $\imath_\lambda(x)$ is the greatest integer power of λ which is strictly lower then x when $x > 0$, and $\imath_\lambda(0) := 0$. Then we observe that

$$\imath_\lambda(x) \leq x \leq \lambda \imath_\lambda(x) \ . \tag{2.2}$$

Therefore $f_\varepsilon := \imath_{1+\varepsilon}(f)$ is an $(1+\varepsilon)^{\mathbf{Z}}$-valued function in $\mathcal{D}$ and, by (2.2), $f \in \mathcal{F}$ *iff* $f_\varepsilon \in \mathcal{F}$, and $\varphi(f_\varepsilon) \leq \varphi(f) \leq (1+\varepsilon)\varphi(f_\varepsilon)$. These relations imply that $\mathcal{F}$ and φ are invariant under rational dilations. Finally, if α is irrational and r, s are rational numbers verifying $r < \alpha < s$, we have

$$\frac{r}{s} f^s < f^\alpha < \frac{s}{r} f^r \ , \qquad \forall f \in \mathcal{D}. \tag{2.3}$$

Therefore, by the face property of $\mathcal{F}$, resp. the positivity of φ, we obtain the dilation invariance for $\mathcal{F}$, resp. φ.

□

2.3 Remark. Let $\mathcal{D}_\#$ be a the dilation invariant subface of $\mathcal{D}$, e.g. $\mathcal{D}_b$, $\mathcal{D}_\infty$, $\mathcal{D}_0$ and their intersections $\mathcal{D}_{b,\infty}$, $\mathcal{D}_{b,0}$. It is clear that if $\mathcal{F}$ is a face in $\mathcal{D}_\#$, it is also a face in $\mathcal{D}$, and if it is dilation invariant in $\mathcal{D}_\#$ it is *a fortiori* a dilation invariant face in $\mathcal{D}$. In the same way, if φ is a positive linear functional on $\mathcal{D}_\#$, it uniquely extends to a positive linear functional on $\mathcal{D}$ with the same domain, and if φ is dilation invariant, also its extension is. Therefore the results on faces and functionals on $\mathcal{D}$ apply to faces and functionals on $\mathcal{D}_\#$.

2.4 Proposition. *Each non zero dilation invariant face $\mathcal{F}$ in $\mathcal{D}$ contains $\mathcal{D}_{b,0}$. Moreover, either $\mathcal{F} \subset \mathcal{D}_\infty$ or $\mathcal{F} \supset \mathcal{D}_b$.*

Proof. If $\mathcal{F} \ni f \neq 0$ then, by face property, $\mathcal{F}$ contains $\chi_{[0,\varepsilon]}$ for a suitably small ε, and therefore, by dilation invariance, $\frac{\varepsilon}{\alpha}\chi_{[0,\alpha]} \in \mathcal{F}$ for each positive α. Again by face property $\mathcal{F}$ contains each bounded function with compact support.
Now let $\mathcal{F}$ contain $f \notin \mathcal{D}_\infty$. Then $\inf_{\mathbf{R}_+} f(x) = \alpha > 0$, which implies, by face property, $\mathcal{F} \supset \mathcal{D}_b$.

□

2.5 Lemma. *Let φ be a positive linear dilation invariant bounded functional on $\mathcal{D}_{b,0}$. Then φ coincides with the Lebesgue integral φ_L on $\mathbf{R}_+$ up to a positive constant.*

Proof. Let us denote by $\tilde{\varphi}$ the linear extension of the restriction of φ to the continuous non increasing functions with compact support in $[0, \infty)$. Since each real valued continuous function with compact support in $[0, \infty)$ may be uniquely decomposed into a difference of two continuous non increasing functions with compact support in $[0, \infty)$, by Riesz theorem, $\tilde{\varphi}$ is given by the integral w.r.t. a Borel measure ν on $[0, \infty)$. Now we observe that the Dirac measure on $\{0\}$ is not dilation invariant according to definition 2.1, therefore $\tilde{\varphi}$ is determined by its behavior on the continuous functions with compact support in $\mathbf{R}_+$. Then we note that dilation invariance for $\tilde{\varphi}$ is equivalent to the fact that $dh(t) := \frac{d\nu(t)}{t}$ is the Haar measure on $\mathbf{R}_+$ (with the multiplicative structure), indeed

$$\int_{\mathbf{R}_+} f(t)dh(t) = \int_{\mathbf{R}_+} \frac{f(t)}{t} d\nu(t) = \int_{\mathbf{R}_+} \lambda \frac{f(\lambda t)}{\lambda t} d\nu(t) = \int_{\mathbf{R}_+} f(\lambda t)dh(t).$$

As a consequence ν is the Lebesgue measure (up to a positive constant k). Since for each function $f \in \mathcal{D}_{b,0}$ and for each $\varepsilon > 0$ we may find two continuous functions $f_1, f_2 \in \mathcal{D}_{b,0}$ such that $f_1 \leq f \leq f_2$ and $\int f_2 \leq \varepsilon + \int f_1$, $\varphi \equiv k\varphi_L$ by positivity.

□

2.6 Proposition. *Let φ be a non trivial dilation-invariant positive linear functional on $\mathcal{D}$. Then φ is uniquely decomposed in*

$$\varphi = \varphi_0 + \varphi_\infty + k\varphi_L, \tag{2.4}$$

where φ_0, φ_∞, φ_L are dilation-invariant positive linear functionals on $\mathcal{D}$, φ_0 is identically zero on $\mathcal{D}_b$, φ_∞ is identically zero on $\mathcal{D}_0$, $k \geq 0$, and φ_L is the Lebesgue integral on $\mathbf{R}_+$.

Proof. By proposition 2.2 the domain of φ is a dilation invariant face, and it contains $\mathcal{D}_{b,0}$ by lemma 2.4, since φ is non trivial. Then, by lemma 2.5, φ restricted to $\mathcal{D}_{b,0}$ coincides with $k\varphi_L$ for a suitable constant k. By the inner regularity of the Lebesgue measure and the positivity of φ we get, for each $f \in \mathcal{D}$,

$$k\varphi_L(f) = \sup_{\substack{g \in \mathcal{D}_{b,0} \\ g \leq f}} k\varphi_L(g) = \sup_{\substack{g \in \mathcal{D}_{b,0} \\ g \leq f}} \varphi(g) \leq \varphi(f)$$

i.e. $\varphi - k\varphi_L$ is a dilation invariant positive linear functional on $\mathcal{D}$ which vanishes on $\mathcal{D}_{b,0}$.

Now let us denote by φ_∞ the functional on $\mathcal{D}$ which coincides with $\varphi - k\varphi_L$ on $\mathcal{D}_b$ and vanishes on all functions with compact support. It is easy to see that φ_∞ enjoys all the requested properties, moreover, by definition, $\varphi_\infty \leq \varphi - k\varphi_L$. Then we set $\varphi_0 := \varphi - \alpha\varphi_L - \varphi_\infty$ and the proof is complete.

□

Now we consider a special class of positive linear functionals on $\mathcal{D}$. Such functionals, which we call monotone, obey a stronger form of positivity. Let us consider the following relations between functions in $\mathcal{D}$, $0 \le \alpha < \beta \le +\infty$:

$$
\begin{aligned}
f \underset{\alpha\uparrow\beta}{\prec} g &\iff \int_\alpha^t f(s)ds \le \int_\alpha^t g(s)ds, \quad \forall t \in (\alpha,\beta), \\
f \underset{\alpha\downarrow\beta}{\prec} g &\iff \int_t^\beta f(s)ds \le \int_t^\beta g(s)ds, \quad \forall t \in (\alpha,\beta).
\end{aligned}
\tag{2.5}
$$

They are pre-order relations (i.e. not necessarily antisymmetric) and are weaker than the usual one. Now let φ be a positive linear functional on $\mathcal{D}$, $0 \le \alpha < \beta \le +\infty$. φ is called (α,β)-*increasing* if it satisfies

$$
f \underset{\alpha\uparrow\beta}{\prec} g \Rightarrow \varphi(f) \le \varphi(g). \tag{2.6}
$$

φ is called (α,β)-*decreasing* if it satisfies

$$
f \underset{\alpha\downarrow\beta}{\prec} g \Rightarrow \varphi(f) \le \varphi(g). \tag{2.7}
$$

2.7 Lemma. *Let φ be a $(0,\infty)$-increasing (resp. -decreasing) dilation invariant positive linear functional on $\mathcal{D}$. Then φ_0 is $(0,1)$-increasing (resp. -decreasing) and φ_∞ is $(1,\infty)$-increasing (resp. -decreasing), where φ_0, φ_∞ refer to the decomposition (2.4).*

Proof. First suppose φ is $(0,\infty)$-increasing. We prove that φ_∞ is $(1,\infty)$-increasing by contradiction. Indeed, let $\mathcal{I}_\varphi$ be the domain of φ, and let f, $g \in \mathcal{I}_\varphi$ be s.t. $f \underset{0\uparrow\infty}{\prec} g$ but $\varphi_\infty(f) - \varphi_\infty(g) =: \varepsilon > 0$. If the constant k in formula (2.4) is not zero, which implies $\mathcal{I}_\varphi \subseteq L^1(\mathbf{R}_+)$, we set $\delta := \frac{\varepsilon}{2k}$ and find $t_0 \in \mathbf{R}_+$ s.t. $\int_{t_0}^\infty f$ and $\int_{t_0}^\infty g$ are less than $\frac{\delta}{4}$. Then let us define

$$
\tilde f(t) := \begin{cases} f(t_0) \vee g(t_0) + \frac{\delta}{t_0} & 0 \le t < \frac{t_0}{2} \\ f(t_0) \vee g(t_0) & \frac{t_0}{2} \le t < t_0 \\ f(t) & t_0 \le t \end{cases}
$$

and

$$
\tilde g(t) := \begin{cases} f(t_0) \vee g(t_0) + \frac{\delta}{t_0} & 0 \le t < t_0 \\ g(t) & t_0 \le t \end{cases}.
$$

It is easy to see that $\varphi_\infty(f) = \varphi_\infty(\tilde f)$, $\varphi_\infty(g) = \varphi_\infty(\tilde g)$, $\tilde f \underset{0\uparrow\infty}{\prec} \tilde g$, $\varphi_0(\tilde f) = \varphi_0(\tilde g) = 0$, $\int \tilde g - \tilde f \le \frac{\delta}{2}$. Then, we get

$$
\varphi(\tilde g) - \varphi(\tilde f) = k\varphi_L(\tilde g - \tilde f) + \varphi_\infty(g) - \varphi_\infty(f) < 0
$$

that is, φ is not $(0,\infty)$-increasing. If $k=0$, i.e. the Lebesgue integral does not appear in the decomposition of φ, we set $t_0 = 1$ and $\delta = 0$ in the preceding construction. As before we get $\varphi_\infty(f) = \varphi_\infty(\tilde f)$, $\varphi_\infty(g) = \varphi_\infty(\tilde g)$, $\tilde f \underset{0\infty}{\prec} \tilde g$, $\varphi_0(\tilde f) = \varphi_0(\tilde g) = 0$, and, trivially,

$$\varphi(\tilde g) - \varphi(\tilde f) = \varphi_\infty(g) - \varphi_\infty(f) < 0.$$

Analogously, we prove that φ_0 is increasing. The decreasing case is proved in the same way.

$\square$

2.8 Lemma. *Let φ be a dilation invariant, positive linear functional, $\alpha > 0$, $\beta < \infty$. Then*

(a) *The following are equivalent*

(i) *φ is (α,∞)-increasing (resp. decreasing)*

(ii) *φ is $(1,\infty)$-increasing (resp. decreasing)*

(iii) *φ vanishes on $\mathcal{D}_0$ and $\varphi|_{\mathcal{D}_b}$ is $(0,\infty)$-increasing (resp. decreasing).*

In this case, φ is determined by its restriction to $\mathcal{D}_b$ and $\varphi|_{\mathcal{D}_b}$ vanishes on (resp. its domain is contained in) $L^1 \cap \mathcal{D}$.

(b) *The following are equivalent*

(i) *φ is $(0,\beta)$-increasing (resp. decreasing)*

(ii) *φ is $(0,1)$-increasing (resp. decreasing)*

(iii) *φ vanishes on $\mathcal{D}_b$ and $\varphi|_{\mathcal{D}_0}$ is $(0,\infty)$-increasing (resp. decreasing).*

In this case, φ is determined by its restriction to $\mathcal{D}_0$ and the domain of $\varphi|_{\mathcal{D}_0}$ is contained in (resp. $\varphi|_{\mathcal{D}_0}$ vanishes on) $L^1 \cap \mathcal{D}$.

(c) *The Lebesgue integral is $(0,\infty)$-increasing <u>and</u> $(0,\infty)$-decreasing.*

Proof. (a), increasing case. First observe that, if φ is (α,∞)-increasing, then φ vanishes on $\mathcal{D}_0$. Indeed if $f \in \mathcal{D}$ with support in $[0,\alpha]$, we get $\varphi \underset{\alpha\infty}{\prec} 0$, therefore $\varphi(f) = 0$, and, by dilation invariance, we get the result.

$(ii) \Rightarrow (iii)$ Let us show that $\varphi|_{\mathcal{D}_b}$ is $(0,\infty)$-increasing. Let $f \underset{0\infty}{\prec} g$, $f,g \in \mathcal{D}_b$. By a straightforward computation, it follows that $f \underset{1\infty}{\prec} \tilde g$, where

$$\tilde g := g + \|g - f\|_{L^\infty[0,2]} \chi_{[0,2)}.$$

Then $\varphi(f) \le \varphi(\tilde g) = \varphi(g)$ because φ vanishes on compact support functions, i.e. the thesis.

$(iii) \Rightarrow (ii)$ It follows from lemma 2.7.

$(ii) \iff (i)$ It is analogous to the case $(ii) \Rightarrow (iii)$.

Finally let φ be $(1,\infty)$-increasing. Since any function in $\mathcal{D}$ may be decomposed in a sum of a function in $\mathcal{D}_0$ and a function in $\mathcal{D}_b$ (see e.g. proposition 5.3), φ is determined

by its restriction to $\mathcal{D}_b$. Besides, for all $f \in \mathcal{D} \cap L^1(\mathbf{R}_+)$, we may find a $g \in \mathcal{D}_0$ such that $f \underset{1\infty}{\prec} g$, which implies that φ is zero on integrable functions.
The proof for the decreasing case and the proof of statement (b) are analogous. Statement (c) is trivial.

$\square$

Motivated by the preceding lemmas, we give the following definition:

2.9 Definition. *A dilation invariant positive linear functional φ on $\mathcal{D}$ is called increasing, resp. decreasing, if φ_0 is $(0,1)$-increasing, resp. decreasing, and φ_∞ is $(1,\infty)$-increasing, resp. decreasing. The functional φ is called monotone if it is increasing or decreasing.*

We note that, by lemma 2.8, the only "constant" (i.e. increasing and decreasing) dilation invariant functional is the Lebesgue integral.

As will be shown in section 4, monotonicity plays an important role in constructing traces. Concrete examples of monotone, dilation invariant, positive linear functionals will be given in section 5.

In the rest of the section we discuss the results on dilation invariance for functions with different domains.

On the space $\mathcal{D}(0,1)$, the dilations described in (2.1) make sense only for $\lambda \geq 1$. A face in $\mathcal{D}(0,1)$ resp. functional on $\mathcal{D}(0,1)$ is called *dilation invariant* if it is invariant under these dilations.

2.10 Remark. Each dilation invariant face in $\mathcal{D}(0,1)$ gives rise to a dilation invariant face in $\mathcal{D}_0$ and each dilation invariant face in $\mathcal{D}_0$ determines, by restriction, a dilation invariant face in $\mathcal{D}(0,1)$. In fact, $\mathcal{D}(0,1)$ modulo dilations coincides with $\mathcal{D}_0$ modulo dilations. In the same way, a dilation invariant functional on $\mathcal{D}(0,1)$ uniquely extends to a dilation invariant functional on $\mathcal{D}_0$ and vice versa.
Therefore all the properties on dilation invariant faces, resp. functionals, we proved for $\mathcal{D}$ apply to $\mathcal{D}(0,1)$. In particular proposition 2.4 implies that each non zero face in $\mathcal{D}(0,1)$ contains $\mathcal{D}_b(0,1)$. If φ is a dilation invariant positive linear functional on $\mathcal{D}(0,1)$, the part φ_∞ in the decomposition given by proposition 2.6 does not appear.

2.11 Remark. For $\mathcal{D}(\mathbf{N})$, the dilations described in (2.1) make sense only for $\lambda \in \mathbf{N}$, therefore a face, resp. a functional on $\mathcal{D}(\mathbf{N})$ is called *dilation invariant* if it is invariant under these dilations. Let us define $\mathcal{D}_N$ as the minimal dilation invariant face in $\mathcal{D}$ which contains $\mathcal{D}(\mathbf{N})$. It is easy to see that dilation invariant faces, resp. functionals on $\mathcal{D}(\mathbf{N})$, are in one to one correspondence with the dilation invariant faces, resp. functionals, on $\mathcal{D}_N$. Again the results we proved for $\mathcal{D}$ apply to $\mathcal{D}(\mathbf{N})$. Since $\mathcal{D}_N \subset \mathcal{D}_b$, we have that each non zero face in $\mathcal{D}(\mathbf{N})$ contains $\mathcal{D}_0(\mathbf{N})$ and the part φ_0

in the decomposition given by proposition 2.6 does not appear. Finally we note that D_N is strictly contained in $\mathcal{D}_b$.

Section 3. Bimodules of measurable operators on semifinite factors.

In this section M will be a semifinite factor and tr the normal trace on it (standard normalization for the trace is assumed). We denote by $d(M)$ the set $\{tr(e) : e$ finite projection$\}$. The set $d(M)$ is a measure space w.r.t. the measure dm given by the counting (resp. Lebesgue measure) in the type I (resp. type II) case.

3.1 Proposition. *Let M be a σ-finite, semifinite factor, and $\mathcal{M} := L^\infty(d(M), dm)$. There exists a normal isomorphism $i : \mathcal{M} \to M$ such that*

$$tr(i(f)) = \int f dm \ , \qquad f \in \overline{\mathcal{M}}_+$$

Proof. Type I case. Let $\{e_n\}_{n\in\mathbf{N}}$ be a maximal orthonormal sequence of minimal projections in M. The map

$$i(\{c_n\}) := \sum_n c_n e_n$$

satisfies the requests.
The type II case follows by proposition 1.7.

$\square$

3.2 Corollary. *Let M be a σ-finite, semifinite factor. Then*

$$\{\mu_a : a \in \overline{M}\} \equiv \mathcal{D}(d(M))$$

Proof. It follows by proposition 1.4(i) and the definition of $\mathcal{D}$ that $\mu_a \in \mathcal{D}$, $\forall a \in \overline{M}$. If M is finite, by proposition 1.4(v) we get $\mu_a(t) = 0$ if $t \geq tr(1)$. If M is type I (and the trace of a minimal projection is 1), then $\mu_a(t)$ is constant on the intervals $[n, n+1)$, $n \in \mathbf{N}$. Therefore we have proved that $\mu_a \in \mathcal{D}(d(M))$ when $a \in \overline{M}$. The converse follows by proposition 3.1 because, for each $f \in \mathcal{D}(d(M))$, the element $i(f)$ satisfies $\mu_{i(f)} = f$.

$\square$

In the following we shall consider the *-algebra $\overline{M}$ of (trace-)measurable operators on a semifinite factor as a bimodule on M. A *measurable bimodule* X on M is a vector subspace of $\overline{M}$ which is a bimodule on M. We observe that if X is contained in M, bimodule property corresponds to two-sided ideal property. We note also that *-invariance for X is implied by the bimodule property. The space of compact operators

$$\overline{K}_M := \{a \in \overline{M} : \mu_a \in \mathcal{D}_\infty\}$$

and the space of finite-rank operators

$$\overline{F}_M := \{a \in \overline{M} : \mu_a \in \mathcal{D}_0\}$$

are measurable bimodules on M.

3.3 Proposition. *Let M be a semifinite (not necessarily σ-finite) factor. A vector subspace X of $\overline{M}$ is a measurable bimodule on M iff its positive part is a unitarily invariant face in $\overline{M}_+$. As a consequence, each unitarily invariant face is invariant under partial isometries.*

Proof. Let X be a measurable bimodule on M, $a^2 \in X_+$, $0 \le b^2 \le a^2$. Then we take the function $\kappa : [0, +\infty) \to [0, +\infty)$ given by

$$\kappa(x) = \begin{cases} \frac{1}{x} & x > 0 \\ \\ 0 & x = 0 \end{cases}$$

and observe that $\kappa(a) \in \overline{M}$ and $\kappa(a)a^2\kappa(a) = e$, where e is the support of a. Hence $\kappa(a)b^2\kappa(a) \le \kappa(a)a^2\kappa(a) \le 1$, which implies $\kappa(a)b$, $b\kappa(a) \in M$. Since the support of b is contained in the support of a, $b^2 = b\kappa(a)a^2\kappa(a)b \in X_+$ by the bimodule property. Then X_+ is a hereditary cone in $\overline{M}_+$, and unitary invariance is obvious.
Now let F be a unitarily invariant face in $\overline{M}_+$. Then the proof follows by classical arguments (see e.g. Theorem 2.5.2 [10]).

□

A subset S of $\overline{M}$ is called rearrangement invariant if $a \in S$ and $\mu_a = \mu_b$ imply $b \in S$. We recall that elements which have the same rearrangement are called equimeasurable.

3.4 Lemma. *Let a_1, a_2 be positive equimeasurable elements of $\overline{K}_M$ with discrete spectrum. Then there exists a (partial) isometry u such that $ua_1u^* = a_2$, $u^*a_2u = a_1$.*

Proof. By hypothesis there exist sequences $\{e_n^i\}$ of finite, mutually orthogonal projections such that $a_i = \sum_n \alpha_n e_n^i$, $i = 1, 2$, $tr(e_n^1) = tr(e_n^2)$, $n \in \mathbf{N}$, therefore we find partial isometries v_n s.t. $v_n e_n^1 v_n^* = e_n^2$. By construction, $u := \sum_n v_n$ is a (partial) isometry that satisfies $ua_1u^* = a_2$ and $u^*a_2u = a_1$.

□

3.5 Remark. Proposition 3.3 implies that in a type I factor two positive compact operators are unitarily equivalent if and only if they have the same rearrangement. This is no longer true in the type II case, in fact there exist maximal abelian subalgebras of a type II_1 factor which are not even conjugate (hence not unitarily conjugate) [5], and it is easy to find generators of such algebras with the same rearrangement. In spite of this, when positivity or facial property is assumed, rearrangement invariance and unitary invariance coincide, as it is shown by propositions 3.6 and 4.2.3.

3.6 Proposition. *Let M be a σ-finite, semifinite factor. A face in $\overline{M}$ is unitarily invariant if and only if it is rearrangement invariant. As a consequence bimodules on M are determined by their image under the rearrangement operation.*

Proof. ($\Rightarrow$) First we use the function $\imath_\lambda$ defined in the proof of proposition 2.2, and define the ε approximation of an element $a \in \overline{M}$ as $\imath_{1+\varepsilon}(a)$, and observe that

$$\imath_\varepsilon(a) \le a \le (1+\varepsilon)\imath_\varepsilon(a).$$

Therefore, due to the face property, we may restrict to elements with discrete spectrum. Now let a, b with discrete spectrum in the face F, $\mu_a = \mu_b$. If $a \in \overline{K}_M$ then, by lemma 3.4 and proposition 3.3, unitary invariance implies rearrangement invariance. If $a \notin \overline{K}_M$, then a majorizes an infinite projection (up to a positive constant), hence, by σ-finiteness, unitary equivalence and face property, $M_+ \subseteq F$. Then, since $a \in \overline{M}$, there exists $t \in \mathbf{R}_+$ s.t $e_a(t, +\infty)$ is finite. As $a\, e_a(t, +\infty)$ and $b\, e_b(t, +\infty)$ are equimeasurable and belong to $\overline{K}_M$, $b\, e_b(t, +\infty) \in F$. On the other hand $b\, e_b[0, t]$ is bounded, therefore is in F, which implies $b = b\, e_b[0, t] + b\, e_b(t, +\infty) \in F$.
($\Leftarrow$) Since the trace is unitarily invariant, the (trace-)rearrangement is unitarily invariant. Therefore unitarily equivalent elements have the same (trace-) rearrangement.

□

As the preceding proposition shows, a measurable bimodule is determined by its image in $\mathcal{D}$ via decreasing rearrangement. Now we show that a subset in $\mathcal{D}$ comes from a bimodule iff it is a dilation invariant face. Partial results for the type I case are already contained in [15].

3.7 Theorem. *Let M be a semifinite factor, $\mathcal{F}$ be a subset of $\mathcal{D}(d(M))$. Then*

$$X := \{a \in \overline{M} : \mu_a \in \mathcal{F}\}$$

is a measurable bimodule if and only if $\mathcal{F}$ is a dilation invariant face.

Proof. ($\Rightarrow$) Let us consider the set

$$\mathcal{F}_* := \{f \in \overline{\mathcal{M}}_+ : f^* \in \mathcal{F}\}.$$

Since $i(f) \in X_+ \iff \mu_{i(f)} \in \mathcal{F}$ and $\mu_{i(f)} = f^*$, we get

$$\mathcal{F}_* \equiv \{f \in \overline{\mathcal{M}}_+ : i(f) \in X_+\},$$

therefore $\mathcal{F}_*$ is a face in $\overline{\mathcal{M}}_+$. The result follows by proposition 2.2.
($\Leftarrow$) We have only to show that X_+ is a unitarily invariant face. Let $a, b \in X_+$, then, from $1.4(ii) - (iii)$,

$$\mu_{\alpha a+\beta b}(t) \leq \alpha\mu_a(t/2) + \beta\mu_b(t/2) \in F \ , \quad \alpha, \beta \in \mathbf{R}_+$$

i.e. X_+ is a cone. Then inequality $\mu_a \leq \mu_b$ if $0 \leq a \leq b$ implies X_+ is a face. Unitary invariance of the rearrangement operation completes the proof.

□

3.8 Corollary. *Let M be a σ-finite, semifinite factor. Then the map*

$$\mathcal{F} \subset \mathcal{D}(d(M)) \to X := \{a \in \overline{M} : \mu_a \in \mathcal{F}\} \tag{3.1}$$

is a one to one correspondence between dilation invariant faces in $\mathcal{D}(d(M))$ and measurable bimodules in $\overline{M}$.

We give the correspondence between faces and bimodules in some particular cases:

$$\begin{array}{ccc}
\mathcal{D} & \leftrightarrow & \overline{M} \\
\mathcal{D}_b & \leftrightarrow & M \\
\mathcal{D}_\infty & \leftrightarrow & \overline{K}_M \\
\mathcal{D}_0 & \leftrightarrow & \overline{F}_M \\
\mathcal{D}_{b,\infty} & \leftrightarrow & K_M \\
\mathcal{D}_{b,0} & \leftrightarrow & F_M
\end{array}$$

and we observe that ($\overline{\ }$) makes no difference for type I factors and $M \equiv K_M \equiv F_M$ for finite factors. Then corollary 3.8 and proposition 2.4 immediately imply the following

3.9 Generalized Calkin Theorem. *Let M be a semifinite factor. Then*
(a) *Each non-zero measurable bimodule X on M contains F_M, the two-sided ideal of bounded, finite-rank elements. In particular, if M is finite, $X \supseteq M$.*
(b) *Let M be an infinite σ-finite factor. Then each measurable bimodule X satisfies either $X \subseteq \overline{K}_M$ or $X \supseteq M$.*

Some known results may be seen as consequences of this theorem. In particular, each type II_1 factor is algebraically simple and each two-sided ideal in a type I_∞ factor is contained in the compact operators. Condition (b) of the preceding theorem may also be seen as a characterization of σ-finiteness.

3.10 Proposition. *Let M be an infinite, semifinite factor. Then the following are equivalent:*

(i) *M is σ-finite*

(ii) *each measurable bimodule X satisfies either $X \subseteq \overline{K}_M$ or $X \supseteq M$*

(iii) *K_M is the unique non trivial norm closed ideal in M*

(iv) *two projections are equivalent $\iff$ they have the same trace.*

Proof. $(i) \Rightarrow (ii)$ Follows by 3.9(b).
$(ii) \Rightarrow (iii)$ From (ii) it follows that each non trivial ideal is contained in K_M. As F_M is norm dense in K_M, by theorem 3.9(a) each non trivial norm closed ideal coincides with K_M.
$(iii) \Rightarrow (i)$ Indeed, if M is not σ-finite, the closed ideal generated by all σ-finite projection is a proper ideal which is not compact.
$(iv) \iff (i)$ Trivial.

□

When M is not σ-finite, dilation invariant faces $\mathcal{F} \subset \mathcal{D}$ give still rise to measurable bimodules on M via map (3.1), but this map is not surjective. More precisely the bimodules generated by such a map are characterized by condition (ii) of the preceding proposition. For instance, the ideal generated by all σ-finite projections cannot be described via a dilation invariant face in $\mathcal{D}$.

Section 4. Traces and dilation invariance.

4.1. Traces on semifinite factors. In the following we denote by τ a (non normal) trace on a semifinite factor M.

4.1.1 Lemma. *Let τ be a trace on a factor M, e an infinite projection. Then $\tau(e)$ is zero or $+\infty$.*
In particular, the domain of a non trivial trace on a σ-finite semifinite factor is contained in K_M.

Proof. Since e is infinite, we may find two orthogonal subprojections of e which are equivalent to e, hence $\tau(e) = 2\tau(e)$.
Now, if M is σ-finite, 1 is equivalent to all infinite projections, therefore if τ is non zero it is infinite on all non-compact operators.

□

4.1.2 Proposition. *Let τ be a trace on a σ-finite, semifinite factor M, $a, b \in M$ be equimeasurable positive elements. Then $\tau(a) = \tau(b)$.*

Proof. Due to the previous lemma, we may suppose $a, b \in K$. Making use of the ε approximation and the positivity of τ we may restrict to positive operators with discrete spectrum, as it was done in the proof of proposition 3.6. Then we apply lemma 3.4 and the proof is concluded.

□

4.1.3 Corollary. *Let (M, tr) be a semifinite σ-finite factor with a normal semifinite faithful trace. Then a positive linear functional τ on M is a trace if and only if it is rearrangement invariant. As a consequence, τ defines, and is determined by, the functional $\varphi_\tau : \mathcal{D}_b(d(M)) \to \mathbf{R}_+$ given by*

$$\varphi_\tau(\mu_a) := \tau(a), \qquad a \in M_+. \tag{4.1.1}$$

Proof. Since tr is unitarily invariant, its rearrangement operation is unitarily invariant, as a consequence rearrangement invariant functionals are unitarily invariant. The converse follows by proposition 4.1.2. Since the space of rearrangements of M is $\mathcal{D}_b(d(M))$ (see corollary 3.2), the theorem follows.

□

4.1.4 Theorem. *Let M be a semifinite factor, φ a functional on $\mathcal{D}_b(d(M))$ such that $\tau_\varphi(a) := \varphi(\mu_a)$, $a \in M_+$ is a trace on M. Then φ is a positive linear dilation invariant functional.*

Proof. Let us define the functional $\varphi_*(f) := \tau_\varphi(i(f))$, $f \in L^\infty(d(M))_+$. By definition, φ_* is linear. Since $\mu_{i(f)} = f^*$, we get

$$\varphi_*(f) = \tau_\varphi(i(f)) = \varphi(\mu_{i(f)}) = \varphi(f^*).$$

Then the thesis follows by theorem 2.2(b) and remark 2.3.

□

We note that proposition 2.6 together with the preceding theorem implies that each trace on a finite factor is normal.

Corollary 4.1.3, theorem 4.1.4 and remarks 2.10 and 2.11 prove that the map

$$\varphi \longrightarrow \tau_\varphi$$

where $\tau_\varphi(a) = \varphi(\mu_a)$ is a one to one correspondence between a suitable class of dilation invariant positive linear functionals on $\mathcal{D}_b$ and the traces on a σ-finite semifinite factor. Now we show that dilation invariance is a sufficient condition for monotone functionals.

4.1.5 Lemma. *Let φ be a dilation invariant monotone functional on $\mathcal{D}_b$ which vanishes on $\mathcal{D}_0$. Then $\tau_\varphi(a) := \varphi(\mu_a)$ is additive on positive elements.*

Proof. The identity

$$\sigma_a(\lambda t) = \int_0^t \mu_a^\lambda(t)dt,$$

where μ_a^λ is the λ dilation of μ_a, implies that the inequalities in proposition 1.10 may be rephrased as

$$\begin{aligned} \mu_{a+b} \underset{0\uparrow\infty}{\prec} \mu_a + \mu_b \underset{0\uparrow\infty}{\prec} \mu^2_{a+b}, \\ \mu^2_{a+b} \underset{0\downarrow\infty}{\prec} \mu_a + \mu_b \underset{0\downarrow\infty}{\prec} \mu_{a+b}. \end{aligned}$$

By definition 2.9 and lemma 2.7 φ is $(0,\infty)$-decreasing or $(0,\infty)$-increasing, therefore the thesis trivially holds.

□

4.1.6 Theorem. *Let M be a semifinite factor and φ a positive linear monotone functional on $\mathcal{D}_b$. Then $\tau(a) := \varphi(\mu_a)$, $a > 0$, is a trace on M if and only if φ is dilation invariant.*

Proof. The implication ($\Rightarrow$) is a consequence of theorem 4.1.4.
($\Leftarrow$) We observe that the only non trivial property is $\tau(a+b) = \tau(a)+\tau(b)$, $a, b \in M_+$. By lemma 2.7, we only need to prove that the functional τ_∞ induced by the φ_∞ component of decomposition (2.4) is additive, and this follows by lemma 4.1.5.

□

4.2. Traces on measurable bimodules over a semifinite factor. In this subsection we extend previous results to traces on the bimodule $\overline{M}$ of trace-measurable operators on a semifinite factor M.

4.2.1 Definition. *Let M be a semifinite factor. A positive linear functional on $\overline{M}$ is said a trace if, for each unitary element $u \in M$,*

$$\tau(a) = \tau(uau^*), \qquad \forall a \in \overline{M}_+ \ . \tag{4.2.1}$$

Making use of polar decomposition, it is easy to show that equation (4.2.1) is equivalent to

$$\tau(a^*a) = \tau(aa^*) \qquad \forall a \in \overline{M} \ . \tag{4.2.2}$$

The vector space X given by the linear span of the set $X_+ := \{a \in \overline{M} : \tau(a) < \infty\}$ is called the domain of τ.

4.2.2 Proposition. *The domain of a trace on $\overline{M}$ is a measurable bimodule X on M and the linear extension of τ to X is well defined and verifies*

$$\tau(ac) = \tau(ca) \qquad \forall a \in X, \ c \in M. \tag{4.2.3}$$

Moreover, a finite positive linear functional τ on a measurable bimodule X on M which satisfies (4.2.3) gives rise to a trace on $\overline{M}$ according to the above definition.

Proof. Since τ is finite and positive linear on X_+, it extends uniquely to X. Moreover, by the properties of τ, X_+ is a unitarily invariant face in $\overline{M}$, therefore, by

proposition 3.3, X is a bimodule on M. As a consequence, left and right hand side of equation (4.2.3) are defined and, since each $c \in M$ is a finite linear combination of unitary elements and each $a \in \overline{M}$ is a finite linear combination of positive elements, equation (4.2.3) holds.
On the other hand, if τ is a finite positive linear functional on a measurable bimodule X which satisfies equation (4.2.3), equation (4.2.1) holds for each $a \in X_+$. Setting $\tau(a) = +\infty$ when $a \in M_+ \setminus X_+$, we get a positive linear functional on $\overline{M}_+$ which still verifies (4.2.1) because X_+, and therefore $\overline{M}_+ \setminus X_+$, is unitarily invariant.

□

4.2.3 Theorem. *Let M be a semifinite σ-finite factor. Then, a positive linear functional τ on $\overline{M}$ is a trace iff it is rearrangement invariant. As a consequence τ determines, and is determined by, a positive linear functional φ on $\mathcal{D}$ via the equation $\tau(a) = \varphi(\mu_a)$. Moreover, φ is dilation invariant.*

Proof. It is analogous to that of the corresponding statements of the preceding subsection.

□

4.2.4 Lemma. *Let φ be a monotone functional on $\mathcal{D}_0$ which vanishes on $\mathcal{D}_b$. Then $\tau_\varphi(a) := \varphi(\mu_a)$ is additive on positive elements.*

Proof. Since φ is $(0,\infty)$-increasing or $(0,\infty)$-decreasing by lemma 2.8, the thesis follows as in lemma 4.1.5.

□

4.2.5 Theorem. *Let M be a semifinite σ-finite factor and φ a dilation invariant monotone positive linear functional on $\mathcal{D}$. Then the functional τ given by*

$$\tau(a) := \varphi(\mu_a), \qquad a \in \overline{M}_+ \tag{4.2.4}$$

is a trace on $\overline{M}$.

Proof. By definition, τ is a positive, positively homogeneous, rearrangement invariant functional on $\overline{M}_+$. Applying lemmas 2.8, 4.1.5 and 4.2.4 we obtain that both φ_0 and φ_∞ in the decomposition (2.4) give rise to additive functionals, therefore the thesis follows.

□

4.2.6 Remark. Even though all traces on a finite factor are normal (cf. preceding subsection), previous discussion shows that monotone dilation invariant functionals of type φ_0 give rise to traces on $\overline{M}$ when M is a type II_1 factor. Such traces necessarily vanish on M, but are non trivial on suitable elements affiliated to M. The explicit construction of monotone dilation invariant functionals (cf. sections 5 and 6) gives rise to the first examples of singular traces on a type II_1 factor.

4.3. Traces on measurable bimodules over a semifinite von Neumann algebra. In this subsection (M, tr) denotes a semifinite von Neumann algebra with a normal semifinite faithful trace. By rearrangement we shall always mean the *tr*-rearrangement.

4.3.1 Proposition. *Let τ be a rearrangement invariant positive linear functional on $\overline{M}$. Then τ is a trace.*

Proof. It is analogous to that of corollary 4.1.3.

□

We denote by Φ_M the set of positive linear dilation invariant functionals on $\mathcal{D}$ such that $\tau_\varphi(a) := \varphi(\mu_a)$ is a trace on $\overline{M}$. Now we give a result that concerns the decomposition of the traces coming from functionals in Φ_M, thus generalizing proposition 2.3 in [1]. Here, the singular part, i.e the part which vanishes on finite projections, is further decomposed into a trace which is sensitive to the behavior at infinity and a trace which is sensitive to the behavior in zero.

4.3.2 Theorem. *Let (M, tr) be a semifinite von Neumann algebra with n.s.f. trace, and choose $\varphi \in \Phi_M$. Then $\tau \equiv \tau_\varphi$ is uniquely decomposed as $\tau = \tau_0 + \tau_\infty + k\ tr$, where τ_0 is identically zero on M, τ_∞ is identically zero on $\overline{F}_M$, and $k \geq 0$.*

Proof. By construction, the functional φ_0, resp. φ_∞ in the decomposition given in proposition 2.6 gives rise to a trace vanishing on M, resp. on $\overline{F}_M$, while φ_L gives rise to the normal trace.

□

We note that the τ_∞ part necessarily vanishes on finite factors, while the τ_0 part vanishes on type I factors.

If M is a σ-finite semifinite factor all traces come from dilation invariant functionals, therefore the decomposition in theorem 4.3.2 always applies. If in particular M is of type II_∞ we get that traces on $\overline{M}$ may be completely described in terms of traces on M and traces on $\overline{eMe}$, where eMe is the type II_1 factor obtained by restriction via a finite projection e.

4.3.3 Theorem. *Let (M, tr) be a semifinite von Neumann algebra with a n.s.f. trace, φ a monotone dilation invariant positive linear functional on $\mathcal{D}$. Then $\tau_\varphi(a) := \varphi(\mu_a)$ is a trace on $\overline{M}$.*

Proof. It is analogous to that of theorem 4.2.5.

□

In this section we did not give concrete examples of non normal traces: this amounts to show that there exist non constant monotone, positive linear, dilation invariant functionals on $\mathcal{D}$, and will be done in the following sections.

Section 5. Eccentric functions and dilation-invariant functionals.

Throughout the following we use the notation

$$S_f^\infty(t) := \begin{cases} \int_0^t f(s)ds & f \in \mathcal{D}_b \setminus L^1 \\ \int_t^\infty f(s)ds & f \in \mathcal{D}_b \cap L^1, \end{cases}$$

and

$$S_f^0(t) := \begin{cases} \int_0^t f(s)ds & f \in \mathcal{D}_0 \cap L^1 \\ \int_t^\infty f(s)ds & f \in \mathcal{D}_0 \setminus L^1. \end{cases}$$

5.1 Definition. *Let $f \in \mathcal{D}$ and $g \in \mathcal{D}_0$, $h \in \mathcal{D}_b$ be s.t. $f = g + h$. Then f is said 0-eccentric if 1 is a limit point of $\{\frac{S_g^0(2t)}{S_g^0(t)}\}$ when $t \to 0$, and ∞-eccentric if $h \notin \mathcal{D}_0$ and 1 is a limit point of $\{\frac{S_h^\infty(2t)}{S_h^\infty(t)}\}$, when $t \to \infty$.*

5.2 Remark. Every $f \in \mathcal{D}_{b0}$ is not eccentric. Indeed it is not ∞-eccentric, by definition, and, as $\lim_{t\to 0} \frac{S_f^0(2t)}{S_f^0(t)} = 2$, it is not 0-eccentric.

Observe that definition 5.1 is well posed. Indeed we have

5.3 Proposition.

(i) *For all $f \in \mathcal{D}$ there are $g \in \mathcal{D}_0$, $h \in \mathcal{D}_b$ s.t. $f = g + h$*

(ii) *If $f = g + h = g' + h'$, with $g, g' \in \mathcal{D}_0$, $h, h' \in \mathcal{D}_b$, then g is 0-eccentric $\iff$ g' is, and h is ∞-eccentric $\iff$ h' is. As a consequence, eccentricity of f does not depend on the decomposition.*

Proof. (i) Indeed define

$$g(t) := \begin{cases} f(t) - f(1) & 0 < t < 1 \\ 0 & t \geq 1 \end{cases} \qquad h(t) := \begin{cases} f(1) & 0 < t < 1 \\ f(t) & t \geq 1. \end{cases}$$

(ii) Indeed $g' - g = h - h' =: \lambda$ is a bounded right-continuous function with compact support, so that $g \in L^1 \iff g' \in L^1$, $h \in L^1 \iff h' \in L^1$ and there is $s > 0$ s.t. $h(t) = h'(t),\ t \geq s$.

Suppose first that $h, h' \in L^1$, then $S^\infty_{h'}(t) = S^\infty_h(t),\ t \geq s$ and the thesis follows easily. If $h, h' \notin L^1$, then $S^\infty_{h'}(t) = S^\infty_h(t) + c,\ t \geq s$, where $c := -\int_0^s \lambda(r)dr$. Therefore, if h is ∞-eccentric, there is a sequence $t_k \nearrow \infty$ s.t. $\frac{S^\infty_h(2t_k)}{S^\infty_h(t_k)} \to 1$, so that,

$$\frac{S^\infty_{h'}(2t_k)}{S^\infty_{h'}(t_k)} = \frac{\frac{S^\infty_h(2t_k)}{S^\infty_h(t_k)} + \frac{c}{S^\infty_h(t_k)}}{1 + \frac{c}{S^\infty_h(t_k)}} \to 1$$

as $\frac{c}{S^\infty_h(t_k)} \to 0$, that is h' is ∞-eccentric. Reversing the argument we get the desired equivalence for h, h'.

Let now g be 0-eccentric, so that there is a sequence $t_k \searrow 0$ s.t. $\frac{S^0_g(2t_k)}{S^0_g(t_k)} \to 1$. Suppose first that $g, g' \notin L^1$. Then $S^0_{g'}(t) = S^0_g(t) + \int_t^\infty \lambda(r)dr$, so that

$$\frac{S^0_{g'}(2t_k)}{S^0_{g'}(t_k)} = \frac{\frac{S^0_g(2t_k)}{S^0_g(t_k)} + \frac{\int_{2t_k}^\infty \lambda(r)dr}{S^0_g(t_k)}}{1 + \frac{\int_{t_k}^\infty \lambda(r)dr}{S^0_g(t_k)}} \to 1$$

as $S^0_g(t) \to \infty,\ t \to 0$, and we get the thesis.

Finally, if $g, g' \in L^1$, then $S^0_{g'}(t) = S^0_g(t) + \int_0^t \lambda(r)dr$, so that

$$\frac{S^0_{g'}(2t_k)}{S^0_{g'}(t_k)} = \frac{\frac{S^0_g(2t_k)}{S^0_g(t_k)} + \frac{\int_0^{2t_k} \lambda(r)dr}{S^0_g(t_k)}}{1 + \frac{\int_0^{t_k} \lambda(r)dr}{S^0_g(t_k)}} \to 1$$

as $\frac{\int_0^t |\lambda(r)|dr}{S^0_g(t)} \leq \frac{\sup|\lambda|}{g(t)} \to 0$ if g is 0-eccentric (see remark 5.2) and we get the thesis. $\square$

5.4 Lemma. *Let $f \in \mathcal{D}_b \setminus \mathcal{D}_0$, then the following are equivalent*

(i) *1 is a limit point of* $\{\frac{S^\infty_f(2t)}{S^\infty_f(t)}\}$, *when* $t \to \infty$

(ii) $\exists\{t_k\},\ t_k \nearrow \infty$, *s.t.* $\lim_{k\to\infty} \frac{S^\infty_f(kt_k)}{S^\infty_f(t_k)} = 1$

(iii) *if* $f \notin L^1$, $\inf_{t>0} \frac{S^\infty_f(kt)}{S^\infty_f(t)} < 3,\ \forall k \in \mathbf{N}$;

if $f \in L^1$, $\sup_{t>0} \frac{S^\infty_f(kt)}{S^\infty_f(t)} > 1/3,\ \forall k \in \mathbf{N}$.

Proof. We give the proof only in the case $f \notin L^1$, the other case being analogous. Let us set, for the sake of simplicity, $S \equiv S^\infty_f$ and observe that S is concave and increasing.

$(i) \Rightarrow (ii)$ From the concavity of S it follows $S(2t) \geq \frac{1}{k-1} S(kt) + \frac{k-2}{k-1} S(t)$, so that, with simple manipulations, one has $0 \leq \frac{S(kt)}{S(t)} - 1 \leq (k-1)(\frac{S(2t)}{S(t)} - 1)$. Let now $t_k \nearrow \infty$ be s.t. $0 \leq \frac{S(2t_k)}{S(t_k)} - 1 \leq \frac{1}{k^2}$, then $0 \leq \frac{S(kt_k)}{S(t_k)} - 1 \leq \frac{k-1}{k^2}$, and the thesis follows.

$(ii) \Rightarrow (iii)$ Let $k_0 \in \mathbf{N}$ be s.t. $\frac{S(kt_k)}{S(t_k)} < 3$, for all $k \geq k_0$. Let now $k < k_0$, then $\frac{S(kt_{k_0})}{S(t_{k_0})} \leq \frac{S(k_0 t_{k_0})}{S(t_{k_0})} < 3$. Then $\inf_{t>0} \frac{S_f(kt)}{S_f(t)} < 3$, $\forall k \in \mathbf{N}$.

$(iii) \Rightarrow (i)$ Let us prove that $\inf_{t>0} \frac{S_f^\infty(kt)}{S_f^\infty(t)} = 1$, $\forall k \in \mathbf{N}$, from which the thesis follows easily. Suppose on the contrary that there exists $k \in \mathbf{N}$ s.t. $\inf_{t>0} \frac{S(kt)}{S(t)} =: \alpha > 1$. Then $\exists n \in \mathbf{N}$ s.t.

$$\frac{S(k^n t)}{S(t)} = \frac{S(k^n t)}{S(k^{n-1} t)} \cdots \frac{S(k^2 t)}{S(kt)} \frac{S(kt)}{S(t)} \geq \alpha^n > 3,$$

so that $\inf_{t>0} \frac{S(k^n t)}{S(t)} > 3$.

$\square$

5.5 Lemma. *Let $f \in \mathcal{D}_0$ then the following are equivalent*

(i) *1 is a limit point of* $\{\frac{S_f^0(2t)}{S_f^0(t)}\}$, *when* $t \to 0$

(ii) $\exists \{t_k\}$, $kt_k \searrow 0$, *s.t.* $\lim_{k\to\infty} \frac{S_f^0(kt_k)}{S_f^0(t_k)} = 1$

(iii) *if* $f \in L^1$, $\inf_{t>0} \frac{S_f^0(kt)}{S_f^0(t)} < 3$, $\forall k \in \mathbf{N}$;

if $f \notin L^1$, $\sup_{t>0} \frac{S_f^0(kt)}{S_f^0(t)} > 1/3$, $\forall k \in \mathbf{N}$.

Now we show the deep relationship between eccentricity and dilation-invariance.

5.6 Proposition. *Let $f \in \mathcal{D}$ and φ a dilation-invariant positive linear functional on $\mathcal{D}$, which vanishes on $\mathcal{D}_0$, and s.t. $\varphi(f) = 1$. Then f is ∞-eccentric.*

Proof. Let us first observe that we may assume $f \in \mathcal{D}_b \setminus \mathcal{D}_0$. Set $S := S_f^\infty$ and, for all $k \in \mathbf{N}$,

$$g_k(t) := \begin{cases} f(1) & 0 \leq t < 1 \\ \frac{\int_{\imath_k(t)}^{\imath_k(kt)} f(s)ds}{\imath_k(kt) - \imath_k(t)} & t \geq 1, \end{cases}$$

where $\imath_k$ is defined in section 2 (see equation (2.2)) and verifies $\imath_k(kt) = k\imath_k(t)$. Then $f(\imath_k(kt)) \leq g_k(t) \leq f(\imath_k(t))$, $t \geq 1$, so that $f(kt) \leq g_k(t) \leq f(\frac{t}{k})$, $t \geq 1$, that is $\frac{1}{k} f^k \leq g_k \leq k f^{\frac{1}{k}}$ in $[1, \infty)$. From the hypotheses on φ, we get $\frac{1}{k}\varphi(f) = \frac{1}{k}\varphi(f^k) \leq$

$\varphi(g_k) \leq k\varphi(f^{\frac{1}{k}}) = k\varphi(f)$, so that φ is trivial on f (*i.e.* either $\varphi(f) = 0$ or $\varphi(f) = \infty$) $\iff$ φ is trivial on g_k.
We want to proceed by contradiction. Let us first suppose that $f \in L^1(1, \infty)$. As f is not ∞-eccentric, there exists $k \in \mathbf{N}$ s.t. $\sup_{t>0}\{\frac{S(kt)}{S(t)}\} \leq \frac{1}{3}$. Then $3S(kt) \leq S(t)$ so that, with $t \in [k^n, k^{n+1})$, we get

$$\begin{aligned}
S(\imath_k(t)) - S(\imath_k(\frac{t}{k})) &= S(k^n) - S(k^{n-1}) \\
&= 3S(k^n) - S(k^{n-1}) - 2S(k^n) \leq -2S(k^n) \\
&\leq -2S(k^n) + 2S(k^{n+1}) = 2\{S(\imath_k(kt)) - S(\imath_k(t))\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
g_k(t) &= \frac{S(\imath_k(t)) - S(\imath_k(kt))}{\imath_k(kt) - \imath_k(t)} \\
&\geq 2\frac{S(\imath_k(kt)) - S(\imath_k(k^2t))}{\imath_k(kt) - \imath_k(t)} \\
&= 2g_k(kt)\frac{\imath_k(k^2t) - \imath_k(kt)}{\imath_k(kt) - \imath_k(t)} \\
&= 2kg_k(kt)
\end{aligned}$$

for all $t \geq 1$, that is $g_k \geq 2g_k^k$ in $[1, \infty)$, which implies $\varphi(g_k) \geq 2\varphi(g_k^k) = 2\varphi(g_k)$, *i.e.* φ is trivial on g_k.
Suppose now that $f \notin L^1(1, \infty)$. As f is not ∞-eccentric, there exists $k \in \mathbf{N}$ s.t. $\inf_{t>0}\{\frac{S(kt)}{S(t)}\} \geq 3$. Then $S(kt) \geq 3S(t)$ so that, with $t \in [k^n, k^{n+1})$, we get

$$\begin{aligned}
S(\imath_k(kt)) - S(\imath_k(t)) &= S(k^{n+1}) - S(k^n) \\
&= S(k^{n+1}) - 3S(k^n) + 2S(k^n)) \geq 2S(k^n) \\
&\geq 2S(k^n) - 2S(k^{n-1}) = 2\{S(\imath_k(t)) - S(\imath_k(\frac{t}{k}))\}.
\end{aligned}$$

Therefore, proceeding as above, we get $g_k^k \geq 2g_k$ in $[1, \infty)$, which implies the thesis.

$\square$

5.7 Proposition. *Let $f \in \mathcal{D}$ and φ a dilation-invariant positive linear functional on $\mathcal{D}$, which vanishes on $\mathcal{D}_b$, and s.t. $\varphi(f) = 1$. Then f is 0-eccentric.*

Proof. As this proof is very similar to that of proposition 5.6, we will be sketchy. Let us observe that we can assume that $f \in \mathcal{D}_0$ and *suppf* $\subset [0, 1]$. Set $S := S_f^0$ and, for all $k \in \mathbf{N}$,

$$g_k(t) := \frac{\int_{\eta_k(\frac{t}{k})}^{\eta_k(t)} f(s)ds}{\eta_k(t) - \eta_k(\frac{t}{k})}, \quad t \in (0, 1],$$

where $\eta_k := \imath_{\frac{1}{k}}$, so that $\eta_k(\frac{t}{k}) = \frac{1}{k}\eta_k(t)$, for $t > 0$. Then $\frac{1}{k}f^k \leq g_k \leq kf^{\frac{1}{k}}$ in $(0, \frac{1}{k})$. From the hypotheses on φ, we get φ is trivial on $f \iff \varphi$ is trivial on g_k.
We want to proceed by contradiction. Let us first suppose that $f \notin L^1(0,1)$. As f is not 0-eccentric, there exists $k \in \mathbf{N}$ s.t. $\sup_{t>0}\{\frac{S(kt)}{S(t)}\} \leq \frac{1}{3}$. Then $3S(kt) \leq S(t)$ so that $S(\eta_k(t)) - S(\eta_k(\frac{t}{k})) \leq 2\{S(\eta_k(kt)) - S(\eta_k(t))\}$, $t \in (0, \frac{1}{k})$. Therefore we get $g_k \geq 2g_k^k$ in $(0, \frac{1}{k})$, which implies $\varphi(g_k) \geq 2\varphi(g_k^k) = 2\varphi(g_k)$, *i.e.* φ is trivial on g_k.
Suppose now that $f \in L^1(0,1)$. As f is not 0-eccentric, there exists $k \in \mathbf{N}$ s.t. $\inf_{t>0}\{\frac{S(kt)}{S(t)}\} \geq 3$. Then $S(kt) \geq 3S(t)$ so that $S(\eta_k(kt)) - S(\eta_k(t)) \geq 2\{S(\eta_k(t)) - S(\eta_k(\frac{t}{k}))\}$, $t \in (0, \frac{1}{k})$. Therefore we get $g_k^k \geq 2g_k$ in $(0, \frac{1}{k})$, which implies the thesis.

□

Let us set $\mathcal{F}(f_0) := \{f \in \mathcal{D} : \exists\beta, \ell > 0 \text{ s.t. } f \leq \ell f_0^\beta\}$, the dilation-invariant face generated by $f_0 \in \mathcal{D}$.

5.8 Proposition. *Let $f_0 \in \mathcal{D}_b$ be ∞-eccentric, $\{t_k\}$ the sequence of lemma 5.4, ω a singular state on ℓ^∞ (i.e. vanishing on c_0). Then*

$$\varphi(f) := \begin{cases} \omega\left(\left\{\frac{S_f^\infty(kt_k)}{S_{f_0}^\infty(kt_k)}\right\}\right) & f \in \mathcal{F}(f_0) \\ \\ +\infty & f \in \mathcal{D} \setminus \mathcal{F}(f_0), \end{cases}$$

is a dilation-invariant monotone positive linear functional on $\mathcal{D}$.

Proof. It is easy to see that, if $f_0 \in L^1$, then $[\mathcal{F}(f_0) \subset L^1$ and$]$ φ is $(0,\infty)$-decreasing, whereas, if $f_0 \notin L^1$, then φ [vanishes on $\mathcal{D}_b \cap L^1$ and] is $(0,\infty)$-increasing. So we must only prove invariance. Let us observe that $S_f^\infty(\alpha t) = S_{f^\alpha}^\infty(t)$ so that

$$\begin{aligned}
\varphi(f^\alpha) &= \omega\left(\left\{\frac{S_{f^\alpha}^\infty(kt_k)}{S_{f_0}^\infty(kt_k)}\right\}\right) \\
&= \omega\left(\left\{\frac{S_f^\infty(\alpha kt_k)}{S_{f_0}^\infty(kt_k)}\right\}\right) \\
&= \omega\left(\left\{\frac{S_f^\infty(kt_k)}{S_{f_0}^\infty(kt_k)}\right\}\right) + \omega\left(\left\{\frac{S_f^\infty(\alpha kt_k) - S_f^\infty(kt_k)}{S_{f_0}^\infty(kt_k)}\right\}\right) \\
&= \varphi(f) + \omega\left(\left\{\frac{S_f^\infty(\alpha kt_k) - S_f^\infty(kt_k)}{S_{f_0}^\infty(kt_k)}\right\}\right).
\end{aligned}$$

We want to prove that $\omega\left(\left\{\frac{S_f^\infty(\alpha kt_k) - S_f^\infty(kt_k)}{S_{f_0}^\infty(kt_k)}\right\}\right) = 0$, if $f \in \mathcal{F}(f_0)$.

Let $f \in \mathcal{F}(f_0)$ so that there exist $\beta, \ell > 0$ s.t. $f \leq \ell f_0^\beta$. Let us set $\gamma := \min\{1, \beta, \frac{1}{\alpha}\}$, then, for $k \geq \max\{\frac{1}{\gamma}, \frac{1}{\alpha\gamma}\}$, we get, if $\alpha > 1$,

$$
\begin{aligned}
|S_f^\infty(\alpha k t_k) - S_f^\infty(k t_k)| &= \int_{kt_k}^{\alpha k t_k} f(s)ds \\
&\leq \int_{\frac{1}{\gamma}t_k}^{\frac{1}{\gamma}kt_k} f(s)ds \\
&\leq \ell' \int_{\frac{t_k}{\gamma}}^{k\frac{t_k}{\gamma}} \gamma f_0(\gamma s)ds \\
&= \ell' \int_{t_k}^{kt_k} f_0(r)dr \\
&= \ell' |S_{f_0}^\infty(kt_k) - S_{f_0}^\infty(t_k)|,
\end{aligned}
$$

where $\ell' := \frac{\beta}{\gamma}\ell$. Analogously if $\alpha < 1$. So that we get

$$
\left| \frac{S_f^\infty(\alpha k t_k) - S_f^\infty(k t_k)}{S_{f_0}^\infty(kt_k)} \right| \leq \ell' \left| 1 - \frac{S_{f_0}^\infty(t_k)}{S_{f_0}^\infty(kt_k)} \right| \to 0,
$$

that is $\left\{ \frac{S_f^\infty(\alpha k t_k) - S_f^\infty(k t_k)}{S_{f_0}^\infty(kt_k)} \right\} \in c_0$ and the thesis follows. $\square$

5.9 Proposition. *Let $f_0 \in \mathcal{D}_0$ be 0-eccentric, $\{t_k\}$ the sequence of lemma 5.4, ω a singular state on ℓ^∞. Then*

$$
\varphi(f) := \begin{cases} \omega\left(\left\{\frac{S_f^0(kt_k)}{S_{f_0}^0(kt_k)}\right\}\right) & f \in \mathcal{F}(f_0) \\ +\infty & f \in \mathcal{D} \setminus \mathcal{F}(f_0), \end{cases}
$$

is a dilation-invariant monotone positive linear functional on $\mathcal{D}$.

Proof. It is easy to see that, if $f_0 \in L^1$, then [$\mathcal{F}(f_0) \subset L^1$ and] φ is $(0, \infty)$-increasing, whereas, if $f_0 \notin L^1$, then φ [vanishes on $\mathcal{D}_0 \cap L^1$ and] is $(0, \infty)$-decreasing. The proof of the invariance of φ is analogous to proposition 5.8. $\square$

Section 6. Examples of singular traces and traceability conditions.

In this section we explicitly construct singular traces on semifinite algebras, and relate the traceability with the eccentricity of an operator.

6.1 Definition. *Let (M, tr) be a semifinite algebra with a normal semifinite faithful trace. An element $a \in \overline{M}$ is called 0-eccentric [resp. ∞-eccentric] if its decreasing tr-rearrangement μ_a is a 0-eccentric [resp. ∞-eccentric] function. The element a is called eccentric if it is 0 or ∞ eccentric.*

We remark that the notion of eccentricity in a von Neumann algebra depends on the chosen trace. Our notion is more general than the usual notion of eccentric operator in $B(H)$ (cf. [8]). Now we give the description of singular traces announced in section 4.

Let M be a type II_∞ factor. Then definition 6.1, together with lemma 5.4 imply that $a \in M_+$ is ∞-eccentric if there is a sequence $\{t_k\}$ such that $t_k \nearrow +\infty$ for which

$$\lim_{k \in \mathbf{N}} \frac{\sigma_a(kt_k)}{\sigma_a(t_k)} = 1 \quad \text{if } a \notin L^1(M, tr)$$

$$\lim_{k \in \mathbf{N}} \frac{\varsigma_a(kt_k)}{\varsigma_a(t_k)} = 1 \quad \text{if } a \in L^1(M, tr)$$

Now let $a \in M_+$ be ∞-eccentric, $\{t_k\}$ as before, $X(a)$ the ideal generated by a and ω a singular state on $\ell^\infty(\mathbf{N})$. If $a \notin L^1(M, tr)$, we consider the positive functional

$$\tau_{(i)}(b) := \begin{cases} \omega\left(\dfrac{\sigma_b(kt_k)}{\sigma_a(kt_k)}\right) & b \in X(a)_+ \\ +\infty & b \in M_+ \setminus X(a) \end{cases} \tag{6.1}$$

If $a \in L^1(M, tr)$, we consider the positive functional

$$\tau_{(d)}(b) := \begin{cases} \omega\left(\dfrac{\varsigma_b(kt_k)}{\varsigma_a(kt_k)}\right) & b \in X(a)_+ \\ +\infty & b \in M_+ \setminus X(a) \end{cases} \tag{6.2}$$

6.2 Proposition. *The functional $\tau_{(i)}$ [resp. $\tau_{(d)}$] described in formula (6.1) [resp. (6.2)] is a singular trace on M whose domain is $X(a)$. In particular, $\tau(a) = 1$.*

Proof. We note that the functional $\tau_{(i)}$ [resp. $\tau_{(d)}$] is written in terms of the positive linear functional described in proposition 5.8, which is monotone increasing [resp. decreasing] and dilation invariant. Then the thesis follows by theorem 4.1.6.

□

Now let M be a type II_1 factor. It is easy to see that a positive $a \in \overline{M}$ is 0-eccentric if there is a sequence $\{t_k\}$ such that $kt_k \searrow 0$ for which

$$\lim_{k\in\mathbf{N}} \frac{\sigma_a(kt_k)}{\sigma_a(t_k)} = 1 \quad \text{if } a \in L^1(M,tr)$$

$$\lim_{k\in\mathbf{N}} \frac{\varsigma_a(kt_k)}{\varsigma_a(t_k)} = 1 \quad \text{if } a \notin L^1(M,tr)$$

If $a \in L^1(M,tr)$ [resp. $a \notin L^1(M,tr)$] we consider the positive functional $\tau_{(i)}$ [resp. $\tau_{(d)}$] given by (6.1) [resp (6.2)]. Then we have

6.3 Proposition. *Let $a \in \overline{M}$ be an eccentric operator. Then, if $a \in L^1(M,tr)$ [resp. $a \notin L^1(M,tr)$] the functional $\tau_{(i)}$ [resp. $\tau_{(d)}$] is a singular trace on $\overline{M}$ whose domain is the bimodule $X(a)$ generated by a. In particular, $\tau(a) = 1$.*

Proof. It is analogous to that of proposition 6.2.

□

The previous constructions parallel the construction of singular traces on type I factors, as it has been done in [15], [1]. Indeed, the possibility of constructing traces whose domain is generated by eccentric operators does not depend on the fact that M is a factor.

6.4 Theorem. *Let (M,tr) be a semifinite von Neumann algebra with a semifinite normal faithful trace, a an eccentric element of $\overline{M}_+$. Then there exists a trace τ whose domain is the measurable bimodule generated by a and such that $\tau(a) = 1$. Such a trace is singular, i.e. it vanishes on projections which are finite w.r.t. tr.*

Proof. It follows from propositions 5.8-5.9 and theorem 4.3.3.

□

In particular, previous theorem states that an eccentric element $a \in \overline{M}$ is *singularly traceable*, i.e. there exists a singular trace which takes value 1 on this element. These conditions are equivalent on a factor.

6.5 Theorem. *Let M be a semifinite factor, $a \in \overline{M}_+$. Then a is eccentric if and only if it is singularly traceable.*

Proof. The implication $\Rightarrow$ follows by Theorem 6.4. Now suppose that M is σ-finite. If a is singularly traceable, by theorem 4.2.3 there exists a positive linear, dilation invariant functional on $\mathcal{D}$, associated with the singular trace, which takes value 1 on μ_a. Then, by propositions 5.6 and 5.7 and definition 6.1, a is eccentric. If M is not σ-finite, and τ is a singular trace such that $\tau(a) = 1$, we observe that decomposition

$\tau = \tau_0 + \tau_\infty$ still holds. If $\tau_0(a) \neq 0$, since τ_0 is determined by its restriction to $\overline{F}_M$, the preceding argument applies and a is 0-eccentric. If $\tau_0(a) = 0$ we may restrict to the case $a \in M$. The theorem is proved if we show that τ is determined by its restriction to a σ-finite (non unital) subalgebra, i.e. if we show that if τ takes a finite non zero value on some operator, then it is infinite on all infinite projections. Suppose on the contrary that τ vanishes on some infinite projection but $\tau(a) = 1$ for some positive $a \in M$. We note that in this case τ vanishes on all σ-finite projections. First, via discretization, we may restrict to the case that a has finite spectrum, then, since a countable sum of finite projections is σ-finite, to the case that all eigenprojections are infinite. Finally, decomposing each eigenprojection in a sum of two projections equivalent to the original one, we easily get $\tau(a) = 2\tau(a)$, which completes the proof.

$\square$

It is possible to prove that if (M, tr) is a semifinite von Neumann algebra with a normal semifinite faithful trace, there exists an element $a \in \overline{M}_+$ s.t. a is eccentric w.r.t. tr. Moreover, if M is not finite of type I, it is possible to find an a which is algebraically eccentric, i.e. is eccentric w.r.t. all the semifinite normal faithful traces. The traces associated with such an element are algebraically singular, i.e. they vanish on all projections which are finite in the sense of Murray von Neumann.

Then we recall that eccentricity gives some information on the decreasing rate of $\mu(t)$, cf. [1].

Finally we observe that the traces we exhibited are strictly non trivial, i.e. they take finite non-zero values on some elements. We may consider also quasi-trivial traces, i.e. traces with range $\{0, +\infty\}$. Such traces are determined by their domain, which in this case coincides with their kernel. Indeed they are in one-to-one correspondence with the measurable bimodules on M.

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