A dynamical uncertainty principle in von Neumann algebras by operator monotone functions

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Abstract

Suppose that $A_1, ...A_N$ are observables (selfadjoint matrices) and ρ is a state (density matrix). In this case the standard uncertainty principle, proved by Robertson, gives a bound for the quantum generalized variance, namely for $\det\{\mathrm{Cov}_\rho(A_j, A_k)\}$, using the commutators $[A_j, A_k]$; this bound is trivial when N is odd. Recently a different inequality of Robertson-type has been proved by the authors with the help of the theory of operator monotone functions. In this case the bound makes use of the commutators $[\rho, A_j]$ and is non-trivial for any N. In the present paper we generalize this new result to the von Neumann algebra case. Nevertheless the proof appears to simplify all the existing ones.

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1 Introduction

Suppose that $A_1, ..., A_N$ are observables (self-adjoint matrices) and that ρ is a state (density matrix). The uncertainty principle, in its more general form, is due to Robertson [23] and coincides with the following inequality

$$\det\{\mathrm{Cov}_\rho(A_j, A_k)\} \geq \det\left\{-\frac{i}{2}\mathrm{Tr}(\rho[A_j, A_k])\right\}, \tag{1.1}$$

where $\mathrm{Cov}_\rho(A, B) := \frac{1}{2}\mathrm{Tr}(\rho(AB + BA)) - \mathrm{Tr}(\rho A) \cdot \mathrm{Tr}(\rho B)$. The relevance of the above inequality in quantum physics is well-known and we refer the reader to [25, 26, 27, 4, 3, 13].

Since the matrix $\left\{-\frac{i}{2}\mathrm{Tr}(\rho[A_j, A_k])\right\}$ is antisymmetric, actually it is worth to write the inequality as

$$\det\{\mathrm{Cov}_\rho(A_j, A_k)\} \geq \begin{cases} 0, & N = 2m + 1, \\ \det\{-\frac{i}{2}\mathrm{Tr}(\rho[A_j, A_k])\}, & N = 2m. \end{cases} \tag{1.2}$$

This means that the standard uncertainty principle says nothing "quantum" for an odd number of observables. In the case $N = 1$, a reasonable candidate for a lower bound is an expression involving some commutation relation between A and ρ. An inequality of this kind, valid for any N, has been recently proved in [1, 8] (see also [16, 17, 20, 18, 19, 15, 28, 9, 12, 6, 7]). To describe this result we need the theory of operator monotone functions.

We denote by $\mathfrak{F}$ the class of positive, symmetric, normalized, operator monotone functions on $(0, \infty)$. We may associate to any $f \in \mathfrak{F}$ another function $\tilde{f} \in \mathfrak{F}$ by the formula $\tilde{f}(x) := \frac{1}{2}(x+1) - (x-1)^2 \cdot \frac{f(0)}{2f(x)}$.

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If L_ρ and R_ρ denote the left and right multiplication operators (by ρ) define an f-correlation by the formula

$$\mathrm{Corr}_\rho^f(A,B) := \frac{1}{2}\mathrm{Tr}(\rho(AB+BA)) - \mathrm{Tr}(R_\rho \tilde{f}(L_\rho R_\rho^{-1})(A)\cdot B).$$

The following inequality has been proved [8] for any f and for any N:

$$\det\{\mathrm{Cov}_\rho(A_j,A_k)\} \geq \det\{\mathrm{Corr}_\rho^f(A_j,A_k)\}. \tag{1.3}$$

Two are the main differences between inequality (1.3) and the standard uncertainty principle (1.2): (i) the right hand side of (1.3) is not trivial also for an odd number of observables; (ii) using the theory of quantum Fisher information (see [6, 21]) one can see that the right hand side is a function of the commutators $i[\rho, A_j]$, which measure how different the quantum dynamics generated by the observables $A_1, ..., A_N$ are. Therefore we suggest that inequality (1.3) could be named *dynamical uncertainty principle*.

Having in mind the needs of quantum physics, it is natural to seek a generalization of the dynamical uncertainty principle (1.3) that could hold in a setting more general then the matrix case. In the present paper we prove that, with due modifications, inequality (1.3) is true on an arbitrary von Neumann algebra. Despite the general setting, the proof we present here appears simpler than the existing ones (see [8, 1]). Intermediate results have been previously proved by the authors [10, 11]. Notice that a different generalization of inequality (1.3) has recently been proved in [5].

The structure of the paper is the following. In Section 2 we collect some preliminary notions. In Section 3 we explain the relation of inequality (1.3) with quantum dynamics via the theory of quantum Fisher information. In Section 4 we prove the main result.

2 Preliminaries

Denote by M_n the space of complex $n\times n$ matrices. Let us recall that a function $f:(0,\infty)\to\mathbb{R}$ is said *operator monotone* if, for any $n\in\mathbb{N}$, any A, $B\in M_n$ such that $0\leq A\leq B$, the inequalities $0\leq f(A)\leq f(B)$ hold. Note that $f:(0,\infty)\to\mathbb{R}$ is operator monotone *iff* for any $A,B\in\mathcal{B}(\mathcal{H})$ such that $0\leq A\leq B$, it holds $f(A)\leq f(B)$. An operator monotone function is said *symmetric* if $f(x):=xf(x^{-1})$ and *normalized* if $f(1)=1$. We denote by $\mathfrak{F}$ the class of positive, symmetric, normalized, operator monotone functions.

We associate to $f\in\mathfrak{F}$ a function $\tilde{f}\in\mathfrak{F}$ [6] defined by

$$\tilde{f}(x) := \frac{1}{2}\left[(x+1)-(x-1)^2\frac{f(0)}{f(x)}\right], \quad x>0.$$

Definition 2.1. For $A,B\in M_{n,sa}$, $f\in\mathfrak{F}$, and density matrix ρ define the f-correlation and the f-information (also known as metric adjusted skew information, see [12, 6]) as

$$\begin{aligned}\mathrm{Corr}_\rho^f(A,B) &:= \frac{1}{2}\mathrm{Tr}(\rho(AB+BA)) - \mathrm{Tr}(R_\rho\tilde{f}(L_\rho R_\rho^{-1})(A)\cdot B)\\ I_\rho^f(A) &:= \mathrm{Corr}_\rho^f(A,A).\end{aligned}$$

Remark 2.2. The Wigner-Yanase-Dyson skew information is defined as $I_{\rho,\beta}(A):=\mathrm{Tr}(\rho A^2)-\mathrm{Tr}(\rho^\beta A\rho^{1-\beta}A)$. It is easy to see that WYD-information is a particular case of the f-information defined above, which was first shown in [22]. Indeed, if $\beta\in(0,1)$ and $f_\beta(x):=\beta(1-\beta)\frac{(x-1)^2}{(x^\beta-1)(x^{1-\beta}-1)}$, then $\tilde{f}_\beta=1/2(x^\beta+x^{1-\beta})$; this implies that $I_\rho^{f_\beta}(A)=I_{\rho,\beta}(A)$.

Theorem 2.3. [8] *For any $N\in\mathbb{N}$, $A_1,\dots,A_N\in M_{n,sa}$, and any $f\in\mathcal{F}_{op}$ we have*

$$\det\{\mathrm{Cov}_\rho(A_j,A_k)\}_{j,k=1,\dots,N} \geq \det\{\mathrm{Corr}_\rho^f(A_j,A_k)\}_{j,k=1,\dots,N}.$$

In Section 4 we prove that the above inequality holds true in a general von Neumann algebra.

3 Relation with quantum dynamics

Suppose that H is a positive (self-adjoint) operator. The associated Schrödinger equation for density matrices has the form

$$\dot{\rho}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = i[\rho(t), H]. \tag{3.1}$$

The solution of the evolution equation (3.1) is given by

$$\rho_H(t) := e^{-itH}\rho e^{itH}. \tag{3.2}$$

Therefore the commutator $i[\rho, H]$ appears as the tangent vector to the quantum trajectory (3.2) (at the initial point $\rho = \rho_H(0)$) generated by H. Suppose we are considering two different evolutions determined, through the Schrödinger equation, by H and K. If we want to quantify how "different" the trajectories $\rho_H(t), \rho_K(t)$ are, then it would be natural to measure the "area" spanned by the tangent vectors $i[\rho, H], i[\rho, K]$.

This is precisely what the right-hand side of inequality (1.3) measures. What follows explain this point.

First of all let us recall that, to each $f \in \mathfrak{F}$, one associates a scalar product, on the tangent space to the state manifold, that is a quantum version of Fisher information through the formula

$$\langle A, B\rangle_{\rho,f} := \mathrm{Tr}(A \cdot m_f(L_\rho, R_\rho)^{-1}(B)).$$

In this formula $m_f(L_\rho, R_\rho)$ is the operator mean associated to f (see [21] [8] for extended explanations).

Let $(V, g(\cdot,\cdot))$ be a real inner-product vector space. Define

$$\mathrm{Vol}^g(v_1, \ldots, v_N) := \sqrt{\det\{g(v_j, v_k)\}}.$$

If the inner product depends on a further parameter so that $g(\cdot,\cdot) = g_\rho(\cdot,\cdot)$, we write $\mathrm{Vol}^g_\rho(v_1, \ldots, v_N) = \mathrm{Vol}^g(v_1, \ldots, v_N)$. As an instance suppose that $(\Omega, \mathcal{G}, \rho)$ is a probability space and let $(V, g_\rho(\cdot,\cdot)) = (\mathcal{L}^2_{\mathbb{R}}(\Omega, \mathcal{G}, \rho), \mathrm{Cov}_\rho(\cdot,\cdot))$. The number $\mathrm{Vol}^{\mathrm{Cov}}_\rho(A_1, \ldots, A_N)^2$ is known as the *generalized variance* of the random vector $(A_1, \ldots, A_N)$.

Now we move to the noncommutative case. Here $A_1, \ldots A_N$ are self-adjoint matrices, ρ is a (faithful) density matrix and $g(\cdot,\cdot) = \mathrm{Cov}_\rho(\cdot,\cdot)$ has been defined in the Introduction. By Vol^f_ρ we denote the volume associated to the quantum Fisher information $\langle\cdot,\cdot\rangle_{\rho,f}$ given by a normalized symmetric operator monotone function $f \in \mathfrak{F}$. For example,

$$I^f_\rho(A) = \frac{f(0)}{2}\mathrm{Vol}^f_\rho(i[\rho, A]) = \frac{f(0)}{2}\langle i[\rho, A], i[\rho, A]\rangle_{\rho,f}.$$

With the above notation, the Robertson uncertainty principle (inequality (1.1)) takes the following form

$$\mathrm{Vol}^{\mathrm{Cov}}_\rho(A_1, \ldots, A_N) \geq \begin{cases} 0, & N = 2m+1, \\ \det\{-\frac{i}{2}\mathrm{Tr}(\rho[A_j, A_k])\}^{\frac{1}{2}}, & N = 2m. \end{cases}$$

On the other hand one can prove that (see [6]) inequality (1.3) can be written as

$$\mathrm{Vol}^{\mathrm{Cov}}_\rho(A_1, \ldots, A_N) \geq \left(\frac{f(0)}{2}\right)^{\frac{N}{2}} \mathrm{Vol}^f_\rho(i[\rho, A_1], \ldots, i[\rho, A_N]).$$

We see that the bound appearing in the right-hand side of inequality (1.3) measures the volume spanned by the tangent vectors at time $t = 0$ to the trajectories associated to $A_1, \ldots, A_N$, and this explains the terminology we propose.

4 The main result

Let $\mathcal{M}$ be a von Neumann algebra, and ω a normal faithful state on $\mathcal{M}$, and denote by ξ_ω the GNS vector, and by J_ω and Δ_ω the modular conjugation and modular operator associated to ω. See [24] for the general theory of von Neumann algebras.

The proof of the main result is divided in a series of Lemmas. In order to deal with unbounded operators, we introduce some sesquilinear forms on $\mathcal{H}_\omega$, and take [14] as our standard reference.

Definition 4.1. Let $f \in \mathfrak{F}$, and define the following sequilinear forms

$$\begin{aligned}
\mathcal{E}(\xi,\eta) &:= \langle \Delta_\omega^{1/2}\xi, \Delta_\omega^{1/2}\eta\rangle,\\
\mathcal{E}_1(\xi,\eta) &:= \mathcal{E}(\xi,\eta) + \langle \xi,\eta\rangle,\\
\mathcal{F}^f(\xi,\eta) &:= \langle \tilde{f}(\Delta_\omega)^{1/2}\xi, \tilde{f}(\Delta_\omega)^{1/2}\eta\rangle,\\
\mathcal{G}^f(\xi,\eta) &:= \frac{1}{2}\mathcal{E}_1(\xi,\eta) - \mathcal{F}^f(\xi,\eta).
\end{aligned}$$

It follows from [14], Example VI.1.13, that $\mathcal{E}$, $\mathcal{E}_1$, $\mathcal{F}^f$ are closed, positive and symmetric sesquilinear forms.

Lemma 4.2. *Let $\xi,\eta \in \mathcal{D}(\Delta_\omega^{1/2})$, and $\{\xi_n\}$, $\{\eta_n\} \subset \mathcal{D}(\Delta_\omega)$ be such that $\xi_n \to \xi$, $\mathcal{E}(\xi_n - \xi, \xi_n - \xi) \to 0$, $n \to \infty$, and analogously for η_n and η. Then*

$$\begin{aligned}
\mathcal{E}(\xi,\eta) &= \lim_{n\to\infty} \mathcal{E}(\xi_n,\eta_n) = \lim_{n\to\infty}\langle \xi_n, \Delta_\omega \eta_n\rangle,\\
\mathcal{F}^f(\xi,\eta) &= \lim_{n\to\infty} \mathcal{F}^f(\xi_n,\eta_n) = \lim_{n\to\infty}\langle \xi_n, \tilde{f}(\Delta_\omega) \eta_n\rangle.
\end{aligned}$$

Proof. It follows from [14] Theorem VI.2.1 that $\mathcal{D}(\Delta_\omega)$ is a core for $\mathcal{D}(\mathcal{E}) \equiv \mathcal{D}(\Delta_\omega^{1/2})$, so that, from [14] Theorem VI.1.21, for any $\xi \in \mathcal{D}(\Delta_\omega^{1/2})$ there is $\{\xi_n\} \subset \mathcal{D}(\Delta_\omega)$ such that $\xi_n \to \xi$, and $\mathcal{E}(\xi_n - \xi, \xi_n - \xi) \to 0$, $n \to \infty$. Then $\mathcal{E}(\xi_n - \xi_m, \xi_n - \xi_m) \to 0$, $m, n \to \infty$. Now observe that $0 \le \tilde{f}(x) \le \frac{1}{2}(x+1)$, for $x > 0$ [6], so that

$$\begin{aligned}
\mathcal{F}^f(\xi_n - \xi_m, \xi_n - \xi_m) &= \langle \tilde{f}(\Delta_\omega)^{1/2}(\xi_n - \xi_m), \tilde{f}(\Delta_\omega)^{1/2}(\xi_n - \xi_m)\rangle\\
&= \langle \xi_n - \xi_m, \tilde{f}(\Delta_\omega)(\xi_n - \xi_m)\rangle\\
&\le \frac{1}{2}\langle \xi_n - \xi_m, \xi_n - \xi_m\rangle + \frac{1}{2}\langle \xi_n - \xi_m, \Delta_\omega(\xi_n - \xi_m)\rangle\\
&= \frac{1}{2}\|\xi_n - \xi_m\| + \frac{1}{2}\mathcal{E}(\xi_n - \xi_m, \xi_n - \xi_m) \to 0,\ m,n\to\infty.
\end{aligned}$$

This implies $\xi \in \mathcal{D}(\mathcal{F}^f)$ and $\mathcal{F}^f(\xi_n - \xi, \xi_n - \xi) \to 0$, $n \to \infty$.

Therefore, if $\xi,\eta \in \mathcal{D}(\Delta_\omega^{1/2})$, and $\{\xi_n\}$, $\{\eta_n\} \subset \mathcal{D}(\Delta_\omega)$ approximate ξ,η in the above sense, we obtain, from [14] Theorem VI.1.12, that $\mathcal{F}^f(\xi,\eta) = \lim_{n\to\infty}\mathcal{F}^f(\xi_n,\eta_n)$, and analogously for $\mathcal{E}$. □

Lemma 4.3.

(i) $\mathcal{D}(\mathcal{F}^f) \supset \mathcal{D}(\Delta_\omega^{1/2})$,

(ii) $\mathcal{G}^f$ *is a symmetric sesquilinear form on* $\mathcal{D}(\mathcal{G}^f) \supset \mathcal{D}(\Delta_\omega^{1/2})$, *which is positive on* $\mathcal{D}(\Delta_\omega^{1/2})$.

Proof. *(i)* It follows from the proof of the previous Lemma.
(ii) We only need to prove positivity. To begin with, let $\xi \in \mathcal{D}(\Delta_\omega)$. Then, setting $g(x) := \frac{1}{2}(x+1) - \tilde{f}(x) \ge 0$, for all $x > 0$, we have $\mathcal{G}^f(\xi,\xi) = \frac{1}{2}\mathcal{E}_1(\xi,\xi) - \mathcal{F}^f(\xi,\xi) = \frac{1}{2}\langle\xi,\xi\rangle + \frac{1}{2}\langle \xi, \Delta_\omega\xi\rangle - \langle \xi, \tilde{f}(\Delta_\omega)\xi\rangle = \langle \xi, g(\Delta_\omega)\xi\rangle \ge 0$.

Moreover, if $\xi \in \mathcal{D}(\Delta_\omega^{1/2})$, and $\xi_n \in \mathcal{D}(\Delta_\omega)$ is such that $\xi_n \to \xi$, and $\mathcal{E}(\xi_n - \xi, \xi_n - \xi) \to 0$, then, from Lemma 4.2 it follows $\mathcal{G}^f(\xi,\xi) = \lim_{n\to\infty}\mathcal{G}^f(\xi_n,\xi_n) \ge 0$. □

We can now introduce the main objects of study.

Definition 4.4. For any $A, B \widehat{\in} \mathcal{M}_{sa}$, such that $\xi_\omega \in \mathcal{D}(A) \cap \mathcal{D}(B)$, and any $f \in \mathfrak{F}$, we set $A_0 := A - \langle \xi_\omega, A\xi_\omega \rangle$, $B_0 := B - \langle \xi_\omega, B\xi_\omega \rangle$, and define the bilinear forms

$$\begin{aligned} \mathrm{Cov}_\omega(A,B) &:= \mathrm{Re}\langle A_0\xi_\omega, B_0\xi_\omega \rangle, \\ \mathrm{Var}_\omega(A) &:= \mathrm{Cov}_\omega(A,A), \\ \mathrm{Corr}^f_\omega(A,B) &:= \mathrm{Re}\langle A_0\xi_\omega, B_0\xi_\omega \rangle - \mathrm{Re}\langle \tilde{f}(\Delta_\omega)^{1/2} A_0\xi_\omega, \tilde{f}(\Delta_\omega)^{1/2} B_0\xi_\omega \rangle, \\ I^f_\omega(A) &:= \mathrm{Corr}^f_\omega(A,A). \end{aligned}$$

Remark 4.5. Observe that, in case $\mathcal{M} = M_n$, then $\omega = \mathrm{Tr}(\rho\cdot)$, for some $\rho \in \mathcal{D}^1_n$, and $\Delta_\omega = L_\rho R_\rho^{-1}$, so that the previous Definition is a true generalization of covariance and f-correlation in the matrix case.

Lemma 4.6. *For any $A, B \widehat{\in} \mathcal{M}_{sa}$, such that $\xi_\omega \in \mathcal{D}(A) \cap \mathcal{D}(B)$, and any $f \in \mathfrak{F}$, we have*
(i) $\mathrm{Cov}_\omega(A,B) = \frac{1}{2} \mathrm{Re}\, \mathcal{E}_1(A_0\xi_\omega, B_0\xi_\omega)$ is a positive bilinear form,
(ii) $\mathrm{Corr}^f_\omega(A,B) = \mathrm{Re}\, \mathcal{G}^f(A_0\xi_\omega, B_0\xi_\omega)$ is a positive bilinear form.

Proof. (i) Observe that

$$\begin{aligned} \langle B_0\xi_\omega, A_0\xi_\omega \rangle &= \langle B_0^*\xi_\omega, A_0^*\xi_\omega \rangle = \langle J_\omega \Delta_\omega^{1/2} B_0\xi_\omega, J_\omega \Delta_\omega^{1/2} A_0\xi_\omega \rangle \\ &= \langle \Delta_\omega^{1/2} A_0\xi_\omega, \Delta_\omega^{1/2} B_0\xi_\omega \rangle = \mathcal{E}(A_0\xi_\omega, B_0\xi_\omega). \end{aligned}$$

The thesis follows from this and the fact that $\mathcal{D}(\Delta_\omega^{1/2}) = \{ T\xi_\omega : T \widehat{\in} \mathcal{M}, \xi_\omega \in \mathcal{D}(T) \cap \mathcal{D}(T^*) \}$.
(ii) It follows from (i) and Lemma 4.3 (ii). ◻

Remark 4.7. Observe that, for any $A, B \widehat{\in} \mathcal{M}_{sa}$, such that $\xi_\omega \in \mathcal{D}(A) \cap \mathcal{D}(B)$, and any $f \in \mathfrak{F}$, we have $\mathcal{F}^f(B_0\xi_\omega, A_0\xi_\omega) = \mathcal{F}^f(A_0\xi_\omega, B_0\xi_\omega)$. Indeed,

$$\begin{aligned} \mathcal{F}^f(B_0\xi_\omega, A_0\xi_\omega) &= \langle \tilde{f}(\Delta_\omega)^{1/2} B_0\xi_\omega, \tilde{f}(\Delta_\omega)^{1/2} A_0\xi_\omega \rangle = \langle \tilde{f}(\Delta_\omega)^{1/2} J_\omega \Delta_\omega^{1/2} B_0\xi_\omega, \tilde{f}(\Delta_\omega)^{1/2} J_\omega \Delta_\omega^{1/2} A_0\xi_\omega \rangle \\ &= \langle J_\omega \tilde{f}(\Delta_\omega)^{1/2} J_\omega \Delta_\omega^{1/2} A_0\xi_\omega, J_\omega \tilde{f}(\Delta_\omega)^{1/2} J_\omega \Delta_\omega^{1/2} B_0\xi_\omega \rangle \\ &= \langle \tilde{f}(\Delta_\omega^{-1})^{1/2} \Delta_\omega^{1/2} A_0\xi_\omega, \tilde{f}(\Delta_\omega^{-1})^{1/2} \Delta_\omega^{1/2} B_0\xi_\omega \rangle \\ &= \langle \tilde{f}(\Delta_\omega)^{1/2} A_0\xi_\omega, \tilde{f}(\Delta_\omega)^{1/2} B_0\xi_\omega \rangle = \mathcal{F}^f(A_0\xi_\omega, B_0\xi_\omega), \end{aligned}$$

where in the last but one equality we used $\tilde{f}(x^{-1}) = x^{-1}\tilde{f}(x)$, for $x > 0$.

Lemma 4.8. *Let $X, Y \in M_n$ be positive self-adjoint matrices. Then* $\det(X+Y) \geq \det X + \det Y$.

Proof. It is a particular case of [2], Theorem VI.7.1. ◻

Theorem 4.9. *For any $N \in \mathbb{N}$, $A_1, \ldots, A_N \widehat{\in} \mathcal{M}_{sa}$, such that $\xi_\omega \in \mathcal{D}(A_j)$, $j = 1, \ldots, N$, and any $f \in \mathfrak{F}$ we have*

$$\begin{aligned} \det\{\mathrm{Cov}_\omega(A_j, A_k)\}_{j,k=1,\ldots,N} \geq{} & \det\{\mathrm{Corr}^f_\omega(A_j, A_k)\}_{j,k=1,\ldots,N} \\ & + \det\{\mathrm{Cov}_\omega(A_j, A_k) - \mathrm{Corr}^f_\omega(A_j, A_k)\}_{j,k=1,\ldots,N}. \end{aligned}$$

In particular,

$$\det\{\mathrm{Cov}_\omega(A_j, A_k)\}_{j,k=1,\ldots,N} \geq \det\{\mathrm{Corr}^f_\omega(A_j, A_k)\}_{j,k=1,\ldots,N}.$$

Proof. Set $\mathcal{X} = \{ A \widehat{\in} \mathcal{M}_{sa} : \xi_\omega \in \mathcal{D}(A) \}$. Then the map $(A,B) \in \mathcal{X} \times \mathcal{X} \to \mathrm{Cov}_\omega(A,B) - \mathrm{Corr}^f_\omega(A,B) \in \mathbb{R}$ is a positive bilinear form. Therefore, we can apply Lemma 4.8 to the Gram matrices $\{\mathrm{Corr}^f_\omega(A_j, A_k)\}_{j,k=1,\ldots,N}$ and $\{\mathrm{Cov}_\omega(A_j, A_k) - \mathrm{Corr}^f_\omega(A_j, A_k)\}_{j,k=1,\ldots,N}$ to show the claim. ◻

Moreover, using the same techniques, one can prove the following monotonicity property.

Proposition 4.10. *For any $N \in \mathbb{N}$, $A_1, \ldots, A_N \widehat{\in} \mathcal{M}_{sa}$, such that $\xi_\omega \in \mathcal{D}(A_j)$, $j = 1, \ldots, N$, and any $f, g \in \mathfrak{F}$ we have*

$$\frac{f(0)}{f(x)} \geq \frac{g(0)}{g(x)}, \quad x > 0 \iff \tilde{f} \leq \tilde{g} \implies$$

$$\det\{\mathrm{Corr}^f_\omega(A_j, A_k)\}_{j,k=1,\ldots,N} \geq \det\{\mathrm{Corr}^g_\omega(A_j, A_k)\}_{j,k=1,\ldots,N} + \det\{\mathrm{Corr}^f_\omega(A_j, A_k) - \mathrm{Corr}^g_\omega(A_j, A_k)\}_{j,k=1,\ldots,N}.$$

Proof. The equivalence $\frac{f(0)}{f(x)} \geq \frac{g(0)}{g(x)}, \quad x > 0 \iff \tilde{f} \leq \tilde{g}$ is proved in [6], Proposition 5.7. It is easy to see that $\tilde{f} \leq \tilde{g}$ implies that $(A, B) \in \mathcal{X} \times \mathcal{X} \to \mathrm{Corr}^f_\omega(A, B) - \mathrm{Corr}^g_\omega(A, B)$ is a positive bilinear form, so the proof follows from Lemma 4.8. □

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