An inequality related to uncertainty principle in von Neumann algebras

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Abstract

Recently Kosaki proved in [8] an inequality for matrices that can be seen as a kind of new uncertainty principle. Independently, the same result was proved by Yanagi *et al.* in [13]. The new bound is given in terms of Wigner-Yanase-Dyson informations. Kosaki himself asked if this inequality can be proved in the setting of von Neumann algebras. In this paper we provide a positive answer to that question and moreover we show how the inequality can be generalized to an arbitrary operator monotone function.

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1 Introduction

If A, B are selfadjoint matrices and ρ is a density matrix, define

$$\begin{aligned} \mathrm{Cov}_\rho(A,B) &:= \mathrm{Re}\{\mathrm{Tr}(\rho AB) - \mathrm{Tr}(\rho A)\cdot \mathrm{Tr}(\rho B)\} \\ \mathrm{Var}_\rho(A) &:= \mathrm{Cov}_\rho(A,A). \end{aligned}$$

The uncertainty principle reads as

$$\mathrm{Var}_\rho(A)\mathrm{Var}_\rho(B) \geq \frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2.$$

This inequality can be refined as

$$\mathrm{Var}_\rho(A)\mathrm{Var}_\rho(B) - \mathrm{Cov}_\rho(A,B)^2 \geq \frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2,$$

(see [5, 12]). Recently a different uncertainty principle has been found [11, 9, 10, 8, 13]. For $\beta \in (0,1)$ define β-correlation and β-information as

$$\begin{aligned} \mathrm{Corr}_{\rho,\beta}(A,B) &:= \mathrm{Re}\{\mathrm{Tr}(\rho AB) - \mathrm{Tr}(\rho^\beta A \rho^{1-\beta} B)\} \\ I_{\rho,\beta}(A) &:= \mathrm{Corr}_{\rho,\beta}(A,A) = \mathrm{Tr}(\rho A^2) - \mathrm{Tr}(\rho^\beta A \rho^{1-\beta} A), \end{aligned}$$

where the latter coincides with the Wigner-Yanase-Dyson information. It has been proved that

$$\mathrm{Var}_\rho(A)\,\mathrm{Var}_\rho(B) - \mathrm{Cov}_\rho(A,B)^2 \geq I_{\rho,\beta}(A) I_{\rho,\beta}(B) - \mathrm{Corr}_{\rho,\beta}(A,B)^2. \tag{1.1}$$

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The quantities involved in the previous inequality make a perfect sense in a von Neumann algebra setting (see for example [7]). In ref. [8] Kosaki asked if the inequality (1.1) is true in this more general setting.

In this paper we provide a positive answer to Kosaki question and moreover we show that, once the inequality is formulated in the context of operator monotone functions, the result can be greatly generalized.

2 Preliminaries

Denote by $M_{n,sa}$ the space of complex self-adjoint $n \times n$ matrices, and recall that a function $f : (0, \infty) \to \mathbb{R}$ is said *operator monotone* if, for any $n \in \mathbb{N}$, any $A, B \in M_{n,sa}$ such that $0 \le A \le B$, the inequalities $0 \le f(A) \le f(B)$ hold. Then, $f : (0, \infty) \to \mathbb{R}$ is operator monotone *iff* for any $A, B \in \mathcal{B}(\mathcal{H})$ such that $0 \le A \le B$, it holds $f(A) \le f(B)$. An operator monotone function is said *symmetric* if $f(x) := xf(x^{-1})$ and *normalized* if $f(1) = 1$. We denote by $\mathfrak{F}$ the class of positive, symmetric, normalized, operator monotone functions.

Examples of operator monotone functions are the so-called Wigner-Yanase-Dyson functions

$$f_\beta(x) := \beta(1-\beta)\frac{(x-1)^2}{(x^\beta - 1)(x^{1-\beta} - 1)}, \qquad \beta \in (0,1).$$

Returning to a general $f \in \mathfrak{F}$, we associate to it a function $\tilde{f} \in \mathfrak{F}$ [2] defined by

$$\tilde{f}(x) := \frac{1}{2}\Big((x+1) - (x-1)^2\frac{f(0)}{f(x)}\Big), \; x > 0.$$

For example

$$\tilde{f}_\beta(x) = \frac{1}{2}(x^\beta + x^{1-\beta}).$$

Definition 2.1. For $A, B \in M_{n,sa}$, $f \in \mathfrak{F}$, and ρ a faithful density matrix, define f-correlation and f-information as

$$\begin{aligned} \operatorname{Corr}^f_\rho(A,B) &:= \operatorname{Re}\{\operatorname{Tr}(\rho AB) - \operatorname{Tr}(R_\rho \tilde{f}(L_\rho R_\rho^{-1})(A)\cdot B)\}, \\ I^f_\rho(A) &:= \operatorname{Corr}^f_\rho(A,A). \end{aligned}$$

Recall that f-information is also known as metric adjusted skew information (see [4]). The following generalization of inequality (1.1) is proved in [2].

Theorem 2.2.

$$\operatorname{Var}_\rho(A)\operatorname{Var}_\rho(B) - \operatorname{Cov}_\rho(A,B)^2 \ge I^f_\rho(A) I^f_\rho(B) - \operatorname{Corr}^f_\rho(A,B)^2.$$

In the next Section we prove that the above inequality holds true in a general von Neumann algebra, thus answering, in particular, the question raised by Kosaki in [8], and recalled above. A different generalization of Theorem 2.2 has been proved in [3].

3 The main result

Let $\mathcal{M}$ be a von Neumann algebra, and ω a normal faithful state on $\mathcal{M}$, and denote by $\mathcal{H}_\omega$ and ξ_ω the GNS Hilbert space and vector, and by S_ω, J_ω and Δ_ω the modular operators associated to ω.

The proof of the main result is divided in a series of Lemmas. In order to deal with unbounded operators, we introduce some sesquilinear forms on $\mathcal{H}_\omega$, and take [6] as our standard reference.

Definition 3.1. Let $f \in \mathfrak{F}$, and define the following sequilinear forms

$$\begin{aligned} \mathcal{E}(\xi, \eta) &:= \langle \Delta_\omega^{1/2}\xi, \Delta_\omega^{1/2}\eta \rangle, \\ \mathcal{E}_1(\xi, \eta) &:= \mathcal{E}(\xi, \eta) + \langle \xi, \eta \rangle, \\ \mathcal{F}^f(\xi, \eta) &:= \langle \tilde{f}(\Delta_\omega)^{1/2}\xi, \tilde{f}(\Delta_\omega)^{1/2}\eta \rangle, \\ \mathcal{G}^f(\xi, \eta) &:= \frac{1}{2}\mathcal{E}_1(\xi, \eta) - \mathcal{F}^f(\xi, \eta). \end{aligned}$$

It follows from [6], Example VI.1.13, that $\mathcal{E}$, $\mathcal{E}_1$, $\mathcal{F}^f$ are closed, positive and symmetric sesquilinear forms.

Lemma 3.2. *Let $\xi, \eta \in \mathcal{D}(\Delta_\omega^{1/2})$, and $\{\xi_n\}$, $\{\eta_n\} \subset \mathcal{D}(\Delta_\omega)$ be such that $\xi_n \to \xi$, $\mathcal{E}(\xi_n - \xi, \xi_n - \xi) \to 0$, $n \to \infty$, and analogously for η_n and η. Then*

$$\begin{aligned} \mathcal{E}(\xi, \eta) &= \lim_{n\to\infty} \mathcal{E}(\xi_n, \eta_n) = \lim_{n\to\infty} \langle \xi_n, \Delta_\omega \eta_n \rangle, \\ \mathcal{F}^f(\xi, \eta) &= \lim_{n\to\infty} \mathcal{F}^f(\xi_n, \eta_n) = \lim_{n\to\infty} \langle \xi_n, \tilde{f}(\Delta_\omega)\eta_n \rangle. \end{aligned}$$

Proof. It follows from [6] Theorem VI.2.1 that $\mathcal{D}(\Delta_\omega)$ is a core for $\mathcal{D}(\mathcal{E}) \equiv \mathcal{D}(\Delta_\omega^{1/2})$, so that, from [6] Theorem VI.1.21, for any $\xi \in \mathcal{D}(\Delta_\omega^{1/2})$ there is $\{\xi_n\} \subset \mathcal{D}(\Delta_\omega)$ such that $\xi_n \to \xi$, and $\mathcal{E}(\xi_n - \xi, \xi_n - \xi) \to 0$, $n \to \infty$. Then $\mathcal{E}(\xi_n - \xi_m, \xi_n - \xi_m) \to 0$, $m, n \to \infty$. Now observe that $0 \le \tilde{f}(x) \le \frac{1}{2}(x+1)$, for $x > 0$ [2], so that

$$\begin{aligned} \mathcal{F}^f(\xi_n - \xi_m, \xi_n - \xi_m) &= \langle \tilde{f}(\Delta_\omega)^{1/2}(\xi_n - \xi_m), \tilde{f}(\Delta_\omega)^{1/2}(\xi_n - \xi_m) \rangle \\ &= \langle \xi_n - \xi_m, \tilde{f}(\Delta_\omega)(\xi_n - \xi_m) \rangle \\ &\le \frac{1}{2}\langle \xi_n - \xi_m, \xi_n - \xi_m \rangle + \frac{1}{2}\langle \xi_n - \xi_m, \Delta_\omega(\xi_n - \xi_m) \rangle \\ &= \frac{1}{2}\|\xi_n - \xi_m\| + \frac{1}{2}\mathcal{E}(\xi_n - \xi_m, \xi_n - \xi_m) \to 0, \ m, n \to \infty. \end{aligned}$$

This implies $\xi \in \mathcal{D}(\mathcal{F}^f)$ and $\mathcal{F}^f(\xi_n - \xi, \xi_n - \xi) \to 0$, $n \to \infty$.

Therefore, if $\xi, \eta \in \mathcal{D}(\Delta_\omega^{1/2})$, and $\{\xi_n\}$, $\{\eta_n\} \subset \mathcal{D}(\Delta_\omega)$ approximate ξ, η in the above sense, we obtain, from [6] Theorem VI.1.12, that $\mathcal{F}^f(\xi, \eta) = \lim_{n\to\infty} \mathcal{F}^f(\xi_n, \eta_n)$, and analogously for $\mathcal{E}$. $\square$

Lemma 3.3.
(i) $\mathcal{D}(\mathcal{F}^f) \supset \mathcal{D}(\Delta_\omega^{1/2})$,
(ii) $\mathcal{G}^f$ is a symmetric sesquilinear form on $\mathcal{D}(\mathcal{G}^f) \supset \mathcal{D}(\Delta_\omega^{1/2})$, which is positive on $\mathcal{D}(\Delta_\omega^{1/2})$.

Proof. *(i)* It follows from the proof of the previous Lemma.
(ii) We only need to prove positivity. To begin with, let $\xi \in \mathcal{D}(\Delta_\omega)$. Then, setting $g(x) := \frac{1}{2}(x+1) - \tilde{f}(x) \ge 0$, for all $x > 0$, we have $\mathcal{G}^f(\xi, \xi) = \frac{1}{2}\mathcal{E}_1(\xi, \xi) - \mathcal{F}^f(\xi, \xi) = \frac{1}{2}\langle \xi, \xi \rangle + \frac{1}{2}\langle \xi, \Delta_\omega \xi \rangle - \langle \xi, \tilde{f}(\Delta_\omega)\xi \rangle = \langle \xi, g(\Delta_\omega)\xi \rangle \ge 0$.

Moreover, if $\xi \in \mathcal{D}(\Delta_\omega^{1/2})$, and $\xi_n \in \mathcal{D}(\Delta_\omega)$ is such that $\xi_n \to \xi$, and $\mathcal{E}(\xi_n - \xi, \xi_n - \xi) \to 0$, then, from Lemma 3.2 it follows $\mathcal{G}^f(\xi, \xi) = \lim_{n\to\infty} \mathcal{G}^f(\xi_n, \xi_n) \ge 0$. $\square$

We can now introduce the main objects of study. In the sequel, we denote by $T \widehat{\in} \mathcal{M}$ the fact that T is a closed, densely defined, linear operator on $\mathcal{H}_\omega$, and is affiliated with $\mathcal{M}$.

Definition 3.4. For any $A, B \widehat{\in} \mathcal{M}_{sa}$, such that $\xi_\omega \in \mathcal{D}(A) \cap \mathcal{D}(B)$, and any $f \in \mathfrak{F}$, we set $A_0 := A - \langle \xi_\omega, A\xi_\omega \rangle$, $B_0 := B - \langle \xi_\omega, B\xi_\omega \rangle$, and define the bilinear forms

$$\begin{aligned} \mathrm{Cov}_\omega(A, B) &:= \mathrm{Re}\langle A_0\xi_\omega, B_0\xi_\omega \rangle, \\ \mathrm{Var}_\omega(A) &:= \mathrm{Cov}_\omega(A, A), \\ \mathrm{Corr}_\omega^f(A, B) &:= \mathrm{Re}\langle A_0\xi_\omega, B_0\xi_\omega \rangle - \mathrm{Re}\langle \tilde{f}(\Delta_\omega)^{1/2}A_0\xi_\omega, \tilde{f}(\Delta_\omega)^{1/2}B_0\xi_\omega \rangle, \\ I_\omega^f(A) &:= \mathrm{Corr}_\omega^f(A, A). \end{aligned}$$

Remark 3.5. Observe that in the matrix case $\omega = \mathrm{Tr}(\rho\cdot)$, for some density matrix ρ, and $\Delta_\omega = L_\rho R_\rho^{-1}$, so that the previous Definition is a true generalization of covariance and f-correlation in the matrix case.

For the reader's convenience, we prove the following folklore result.

Lemma 3.6. $\mathcal{D}(\Delta_\omega^{1/2}) = \{T\xi_\omega : T\widehat{\in}\mathcal{M}, \xi_\omega \in \mathcal{D}(T) \cap \mathcal{D}(T^*)\}$.

Proof. (1) Let us first prove that $\mathcal{D}(\Delta_\omega^{1/2}) \subset \{T\xi_\omega : T\widehat{\in}\mathcal{M}, \xi_\omega \in \mathcal{D}(T)\cap\mathcal{D}(T^*)\}$. Indeed, let $\eta \in \mathcal{D}(\Delta_\omega^{1/2})$, and define the linear operator $T_0 : x'\xi_\omega \in \mathcal{M}'\xi_\omega \mapsto x'\eta \in \mathcal{H}_\omega$, which is densely defined, and affiliated with $\mathcal{M}$. Let us show that is preclosed: indeed, if $x'_n\xi_\omega \to 0$, and $x'_n\eta \to \zeta$, then, for any $y' \in \mathcal{M}'$, we get

$$\begin{aligned}\langle \zeta, y'\xi_\omega\rangle &= \lim_{n\to\infty} \langle x'_n\eta, y'\xi_\omega\rangle = \lim_{n\to\infty} \langle \eta, {x'_n}^* y'\xi_\omega\rangle = \lim_{n\to\infty} \langle \eta, S_\omega^*({y'}^* x'_n\xi_\omega)\rangle \\ &= \lim_{n\to\infty} \langle {y'}^* x'_n\xi_\omega, S_\omega\eta\rangle = \lim_{n\to\infty} \langle x'_n\xi_\omega, y' S_\omega\eta\rangle = 0,\end{aligned}$$

which shows that T_0 is preclosed. Let $T_\eta := \overline{T_0}$. Then, $T_\eta\widehat{\in}\mathcal{M}$, and $T_\eta\xi_\omega = \eta$. It remains to be proved that $\xi_\omega \in \mathcal{D}(T_\eta^*)$. Since $S_\omega\eta \in \mathcal{D}(\Delta_\omega^{1/2})$, we can also consider $T_{S_\omega\eta}$. Let us show that $T_{S_\omega\eta} \subset T_\eta^*$. Indeed, for any $x', y' \in \mathcal{M}'$, we have

$$\langle T_{S_\omega\eta} x'\xi_\omega, y'\xi_\omega\rangle = \langle x' S_\omega\eta, y'\xi_\omega\rangle = \langle S_\omega\eta, {x'}^* y'\xi_\omega\rangle = \langle {y'}^* x'\xi_\omega, \eta\rangle = \langle x'\xi_\omega, y'\eta\rangle = \langle x'\xi_\omega, T_\eta y'\xi_\omega\rangle.$$

Then, $\xi_\omega \in \mathcal{D}(T_{S_\omega\eta}) \subset \mathcal{D}(T_\eta^*)$, which shows that $\mathcal{D}(\Delta_\omega^{1/2}) \subset \{T\xi_\omega : T\widehat{\in}\mathcal{M}, \xi_\omega \in \mathcal{D}(T) \cap \mathcal{D}(T^*)\}$.
(2) Let us now prove that $\mathcal{D}(\Delta_\omega^{1/2}) \supset \{T\xi_\omega : T\widehat{\in}\mathcal{M}, \xi_\omega \in \mathcal{D}(T) \cap \mathcal{D}(T^*)\}$. Indeed, if $T\widehat{\in}\mathcal{M}$ is such that $\xi_\omega \in \mathcal{D}(T) \cap \mathcal{D}(T^*)$, we can consider its polar decomposition $T = v|T|$, and let $e_n := \chi_{[0,n]}(|T|)$, $T_n := v|T|e_n$, for any $n \in \mathbb{N}$. Since $\xi_\omega \in \mathcal{D}(T)$, we have $T_n\xi_\omega = ve_n|T|\xi_\omega \to T\xi_\omega$. Moreover, since $\xi_\omega \in \mathcal{D}(T^*)$, we have $T_n^*\xi_\omega = |T|e_n v^*\xi_\omega = e_n T^*\xi_\omega \to T^*\xi_\omega$. Since S_ω is a closed operator, it follows that $T\xi_\omega \in \mathcal{D}(S_\omega) = \mathcal{D}(\Delta_\omega^{1/2})$ [and $S_\omega T\xi_\omega = T^*\xi_\omega$], which is what we wanted to prove. □

Lemma 3.7. *For any $A, B\widehat{\in}\mathcal{M}_{sa}$, such that $\xi_\omega \in \mathcal{D}(A) \cap \mathcal{D}(B)$, and any $f \in \mathfrak{F}$, we have*
(i) $\mathrm{Cov}_\omega(A, B) = \frac{1}{2}\mathrm{Re}\,\mathcal{E}_1(A_0\xi_\omega, B_0\xi_\omega)$ *is a positive bilinear form,*
(ii) $\mathrm{Corr}^f_\omega(A, B) = \mathrm{Re}\,\mathcal{G}^f(A_0\xi_\omega, B_0\xi_\omega)$ *is a positive bilinear form.*

Proof. (i) Observe that

$$\begin{aligned}\langle B_0\xi_\omega, A_0\xi_\omega\rangle &= \langle B_0^*\xi_\omega, A_0^*\xi_\omega\rangle = \langle J_\omega\Delta_\omega^{1/2}B_0\xi_\omega, J_\omega\Delta_\omega^{1/2}A_0\xi_\omega\rangle \\ &= \langle \Delta_\omega^{1/2}A_0\xi_\omega, \Delta_\omega^{1/2}B_0\xi_\omega\rangle = \mathcal{E}(A_0\xi_\omega, B_0\xi_\omega).\end{aligned}$$

The thesis follows from this and the fact that $\mathcal{D}(\Delta_\omega^{1/2}) = \{T\xi_\omega : T\widehat{\in}\mathcal{M}, \xi_\omega \in \mathcal{D}(T) \cap \mathcal{D}(T^*)\}$.
(ii) It follows from (i) and Lemma 3.3 (ii). □

Lemma 3.8. *Let $\xi, \eta \in \mathcal{H}_\omega$, $\Delta_\omega = \int_0^\infty t\, de(t)$, and define, for Ω a Borel subset of $[0,\infty)$, $\mu_{\xi\eta}(\Omega) := \mathrm{Re}\langle\xi, e(\Omega)\eta\rangle$, and*

$$\mu := \mu_{\xi\xi} \otimes \mu_{\eta\eta} + \mu_{\eta\eta} \otimes \mu_{\xi\xi} - 2\mu_{\xi\eta} \otimes \mu_{\xi\eta}.$$

Then, μ is a bounded positive Borel measure on $[0,\infty)^2$.

Proof. Let Ω_1, Ω_2 be Borel subsets of $[0,\infty)$, and set $e_j := e(\Omega_j)$, $j = 1, 2$. Observe that $|\mathrm{Re}\langle\xi, e_1\eta\rangle \cdot \mathrm{Re}\langle\xi, e_2\eta\rangle| \le \|e_1\xi\| \cdot \|e_1\eta\| \cdot \|e_2\xi\| \cdot \|e_2\eta\|$, so that

$$\mu(\Omega_1 \times \Omega_2) \ge \|e_1\xi\|^2 \cdot \|e_2\eta\|^2 + \|e_2\xi\|^2 \cdot \|e_1\eta\|^2 - 2\|e_1\xi\| \cdot \|e_1\eta\| \cdot \|e_2\xi\| \cdot \|e_2\eta\| \ge 0.$$

The thesis follows by standard measure theoretic arguments. □

Theorem 3.9. *For any $A, B\widehat{\in}\mathcal{M}_{sa}$, such that $\xi_\omega \in \mathcal{D}(A) \cap \mathcal{D}(B)$, and any $f \in \mathfrak{F}$, we have*

$$\mathrm{Var}_\omega(A)\,\mathrm{Var}_\omega(B) - \mathrm{Cov}_\omega(A, B)^2 \ge I^f_\omega(A) I^f_\omega(B) - \mathrm{Corr}^f_\omega(A, B)^2.$$

Proof. Set

$$
\begin{aligned}
G(A,B) &:= \operatorname{Var}_\omega(A)\operatorname{Var}_\omega(B) - \operatorname{Cov}_\omega(A,B)^2 - I^f_\omega(A)I^f_\omega(B) + \operatorname{Corr}^f_\omega(A,B)^2 \\
&\overset{(a)}{=} \frac{1}{2}\mathcal{E}_1(A_0\xi_\omega, A_0\xi_\omega)\cdot\frac{1}{2}\mathcal{E}_1(B_0\xi_\omega, B_0\xi_\omega) - \Big(\frac{1}{2}\operatorname{Re}\mathcal{E}_1(A_0\xi_\omega, B_0\xi_\omega)\Big)^2 \\
&\quad - \Big(\frac{1}{2}\mathcal{E}_1(A_0\xi_\omega, A_0\xi_\omega) - \mathcal{F}^f(A_0\xi_\omega, A_0\xi_\omega)\Big)\Big(\frac{1}{2}\mathcal{E}_1(B_0\xi_\omega, B_0\xi_\omega) - \mathcal{F}^f(B_0\xi_\omega, B_0\xi_\omega)\Big) \\
&\quad + \Big(\frac{1}{2}\operatorname{Re}\mathcal{E}_1(A_0\xi_\omega, B_0\xi_\omega) - \operatorname{Re}\mathcal{F}^f(A_0\xi_\omega, B_0\xi_\omega)\Big)^2 \\
&= \frac{1}{2}\mathcal{E}_1(A_0\xi_\omega, A_0\xi_\omega)\cdot\mathcal{F}^f(B_0\xi_\omega, B_0\xi_\omega) + \frac{1}{2}\mathcal{F}^f(A_0\xi_\omega, A_0\xi_\omega)\cdot\mathcal{E}_1(B_0\xi_\omega, B_0\xi_\omega) \\
&\quad - \mathcal{F}^f(A_0\xi_\omega, A_0\xi_\omega)\cdot\mathcal{F}^f(B_0\xi_\omega, B_0\xi_\omega) - \operatorname{Re}\mathcal{E}_1(A_0\xi_\omega, B_0\xi_\omega)\cdot\operatorname{Re}\mathcal{F}^f(A_0\xi_\omega, B_0\xi_\omega) \\
&\quad + \big(\operatorname{Re}\mathcal{F}^f(A_0\xi_\omega, B_0\xi_\omega)\big)^2,
\end{aligned}
$$

where in (a) we have used Lemma 3.7. Let us now introduce the function, for $\xi,\eta \in \mathcal{D}(\Delta_\omega^{1/2})$,

$$
H(\xi,\eta) := \frac{1}{2}\mathcal{E}_1(\xi,\xi)\cdot\mathcal{F}^f(\eta,\eta) + \frac{1}{2}\mathcal{F}^f(\xi,\xi)\cdot\mathcal{E}_1(\eta,\eta) - \mathcal{F}^f(\xi,\xi)\cdot\mathcal{F}^f(\eta,\eta) - \operatorname{Re}\mathcal{E}_1(\xi,\eta)\cdot\operatorname{Re}\mathcal{F}^f(\xi,\eta) + \big(\operatorname{Re}\mathcal{F}^f(\xi,\eta)\big)^2,
$$

and recall that $\mathcal{D}(\Delta_\omega^{1/2}) = \{T\xi_\omega : T\widehat{\in}\mathcal{M}, \xi_\omega \in \mathcal{D}(T)\cap\mathcal{D}(T^*)\}$, so that, if A, B are as in the statement of the Theorem, we obtain $G(A,B) = H(A_0\xi_\omega, B_0\xi_\omega)$, and to prove the theorem it suffices to show that $H(\xi,\eta) \geq 0$, for all $\xi,\eta \in \mathcal{D}(\Delta_\omega^{1/2})$. Observe that, for $\xi,\eta \in \mathcal{D}(\Delta_\omega)$, we get

$$
\begin{aligned}
H(\xi,\eta) &= \frac{1}{2}\langle\xi,(1+\Delta_\omega)\xi\rangle\cdot\langle\eta,\tilde{f}(\Delta_\omega)\eta\rangle + \frac{1}{2}\langle\eta,(1+\Delta_\omega)\eta\rangle\cdot\langle\xi,\tilde{f}(\Delta_\omega)\xi\rangle \\
&\quad - \langle\xi,\tilde{f}(\Delta_\omega)\xi\rangle\cdot\langle\eta,\tilde{f}(\Delta_\omega)\eta\rangle - \operatorname{Re}\langle\xi,(1+\Delta_\omega)\eta\rangle\cdot\operatorname{Re}\langle\xi,\tilde{f}(\Delta_\omega)\eta\rangle + \big(\operatorname{Re}\langle\xi,\tilde{f}(\Delta_\omega)\eta\rangle\big)^2 \\
&\overset{(b)}{=} \frac{1}{2}\int_0^\infty (s+1)\,d\mu_{\xi\xi}(s)\int_0^\infty \tilde{f}(t)\,d\mu_{\eta\eta}(t) + \frac{1}{2}\int_0^\infty \tilde{f}(s)\,d\mu_{\xi\xi}(s)\int_0^\infty (t+1)\,d\mu_{\eta\eta}(t) \\
&\quad - \int_0^\infty \tilde{f}(s)\,d\mu_{\xi\xi}(s)\int_0^\infty \tilde{f}(t)\,d\mu_{\eta\eta}(t) - \frac{1}{2}\int_0^\infty (s+1)\,d\mu_{\xi\eta}(s)\int_0^\infty \tilde{f}(t)\,d\mu_{\xi\eta}(t) \\
&\quad - \frac{1}{2}\int_0^\infty \tilde{f}(s)\,d\mu_{\xi\eta}(s)\int_0^\infty (t+1)\,d\mu_{\xi\eta}(t) - \int_0^\infty \tilde{f}(s)\,d\mu_{\xi\eta}(s)\int_0^\infty \tilde{f}(t)\,d\mu_{\xi\eta}(t) \\
&\overset{(c)}{=} \frac{1}{2}\int_{[0,\infty)^2}\big((s+1)\tilde{f}(t) + (t+1)\tilde{f}(s) - 2\tilde{f}(s)\tilde{f}(t)\big)\,d\mu_{\xi\xi}\otimes\mu_{\eta\eta}(s,t) \\
&\quad - \frac{1}{2}\int_{[0,\infty)^2}\big((s+1)\tilde{f}(t) + (t+1)\tilde{f}(s) - 2\tilde{f}(s)\tilde{f}(t)\big)\,d\mu_{\xi\eta}\otimes\mu_{\xi\eta}(s,t) \\
&\overset{(d)}{=} \frac{1}{4}\iint_{[0,\infty)^2}\big((s+1)\tilde{f}(t) + (t+1)\tilde{f}(s) - 2\tilde{f}(s)\tilde{f}(t)\big)\,d\mu(s,t),
\end{aligned}
$$

where we used in (b) notation as in Lemma 3.8, in (c) Fubini-Tonelli Theorem, and in (d) the symmetries of the first integrand and notation as in Lemma 3.8.
Since μ is a positive measure, and

$$
(s+1)\tilde{f}(t) + (t+1)\tilde{f}(s) - 2\tilde{f}(s)\tilde{f}(t) = \big(s+1-\tilde{f}(s)\big)\tilde{f}(t) + \big(t+1-\tilde{f}(t)\big)\tilde{f}(s) \geq 0,
$$

we obtain $H(\xi,\eta) \geq 0$, for any $\xi,\eta \in \mathcal{D}(\Delta_\omega)$.

It follows from Lemma 3.2 that, for any $\xi,\eta \in \mathcal{D}(\Delta_\omega^{1/2})$, we have $H(\xi,\eta) = \lim_{n\to\infty} H(\xi_n,\eta_n) \geq 0$, which ends the proof. □

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