Uncertainty principle for Wigner-Yanase-Dyson information in semifinite von Neumann algebras

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Abstract

In [9] Kosaki proved an uncertainty principle for matrices, related to Wigner-Yanase-Dyson information, and asked if a similar inequality could be proved in the von Neumann algebra setting. In this paper we prove such an uncertainty principle in the semifinite case.

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1 Introduction

Let $M_n := M_n(\mathbb{C})$ (resp. $M_{n,sa} := M_n(\mathbb{C})_{sa}$) be the set of all $n \times n$ complex matrices (resp. all $n \times n$ self-adjoint matrices). Let $\mathcal{D}_n^1$ be the set of strictly positive density matrices namely

$$\mathcal{D}_n^1 = \{ \rho \in M_n : \text{Tr}\rho = 1, \rho > 0 \}.$$ 

Definition 1.1. For $A, B \in M_{n,sa}$ and $\rho \in \mathcal{D}_n^1$, define covariance and variance as

$$\text{Cov}_\rho(A, B) := \text{Tr}(\rho AB) - \text{Tr}(\rho A) \cdot \text{Tr}(\rho B)$$
$$\text{Var}_\rho(A) := \text{Tr}(\rho A^2) - \text{Tr}(\rho A)^2.$$ 

Then the well known Schrödinger and Heisenberg uncertainty principles are given in the following

Theorem 1.2. [8, 14]

For $A, B \in M_{n,sa}$ and $\rho \in \mathcal{D}_n^1$ one has

$$\text{Var}_\rho(A)\text{Var}_\rho(B) - |\text{Re Cov}_\rho(A, B)|^2 \geq \frac{1}{4}|\text{Tr}(\rho[A, B])|^2,$$

that implies

$$\text{Var}_\rho(A)\text{Var}_\rho(B) \geq \frac{1}{4}|\text{Tr}(\rho[A, B])|^2.$$ 

Recently a different uncertainty principle has been found [12, 10, 11, 9, 15].

Definition 1.3. For $A, B \in M_{n,sa}$, $\beta \in (0, 1)$, and $\rho \in \mathcal{D}_n^1$ define $\beta$-correlation and $\beta$-information as

$$\text{Corr}_{\rho,\beta}(A, B) := \text{Tr}(\rho AB) - \text{Tr}(\rho^\beta A\rho^{1-\beta}B)$$
$$I_{\rho,\beta}(A) := \text{Corr}_{\rho,\beta}(A, A) \equiv \text{Tr}(\rho A^2) - \text{Tr}(\rho^\beta A\rho^{1-\beta}A).$$ 

The latter coincides with the Wigner-Yanase-Dyson information.
Theorem 1.4.

\[
\text{Var}_\rho(A) \text{Var}_\beta(B) \geq |\text{Re Cov}_\rho(A, B)|^2 \geq I_{\rho, \beta}(A)I_{\rho, \beta}(B) - |\text{Re Cov}_{\rho, \beta}(A, B)|^2.
\]

Kosaki [9] asked if the previous inequality, which makes perfect sense in a von Neumann algebra setting, could indeed be proved. In the sequel, we provide such a proof in the semifinite case.

In closing, we mention that different generalizations of Theorem 1.4 have been recently obtained by the authors [2, 3, 4, 5, 6, 7].

2 Auxiliary lemmas

In all this Section we let \((M, \tau)\) be a semifinite von Neumann algebra with a n.s.f. trace, and denote by \(\text{Proj}(M)\) the set of orthogonal projections in \(M\), and by \(\overline{M}\) the topological *-algebra of \(\tau\)-measurable operators. We fix \(\rho, \sigma\), with spectral decompositions \(\rho = \int_{-\infty}^{+\infty} \lambda \, d\rho(\lambda)\), and \(\sigma = \int_{-\infty}^{+\infty} \lambda \, d\sigma(\lambda)\).

Finally, we denote by \(A\) the algebra generated by the sets \(\Omega_1 \times \Omega_2\), for \(\Omega_1, \Omega_2\) Borel subsets of \(R\), and observe that \(\sigma(A)\), the \(\sigma\)-algebra generated by \(A\), coincides with the Borel subsets of \(\mathbb{R}^2\).

Lemma 2.1. Let \(a, b \in \mathcal{M} \cap \mathcal{L}^2(M, \tau)\). Let \(\mu_{ab}(\Omega_1 \times \Omega_2) := \tau(e_\rho(\Omega_1)a^*e_\sigma(\Omega_2)b)\), for \(\Omega_1, \Omega_2\) Borel subsets of \(\mathbb{R}\). Then \(\mu_{ab}\) extends uniquely to a bounded Borel measure on \(\mathbb{R}^2\).

Proof. For \(\Omega \subset \mathbb{R}\) Borel subset, \(x \in \mathcal{L}^2(M, \tau)\), let \(P(\Omega)x := e_\rho(\Omega)x, Q(\Omega)x := xe_\sigma(\Omega)\). Then, \(P, Q\) are commuting Borel spectral measures on \(\mathcal{L}^2(M, \tau)\), and their product \(P \otimes Q(\Omega_1 \times \Omega_2) := P(\Omega_1)Q(\Omega_2)\) extends uniquely to a Borel spectral measure on \(\mathbb{R}^2\) ([1], Chapter 5). Observe that \(\mu_{ab}(\Omega_1 \times \Omega_2) = \tau(P \otimes Q(\Omega_1 \times \Omega_2)(a^*) \cdot b)\), and, if \(\{A_n\}\) is a sequence of disjoint Borel sets, then \(P \otimes Q(\bigcup A_n)(a^*) = \sum_n P \otimes Q(A_n)(a^*)\) converges in \(\mathcal{L}^2(M, \tau)\), so that \(\tau(P \otimes Q(\bigcup A_n)(a^*) \cdot b)\) is well defined. So \(\mu_{ab} = \tau(P \otimes Q(\cdot)(a^*) \cdot b)\) is the desired extension.

Observe now that \(\mu_{ab}\) is a bounded Borel (complex) measure on \(A\). Indeed, with \(A \in \mathcal{A}\),

\[
|\mu_{ab}(A)|^2 \leq (\tau(P \otimes Q(A)(a^*) \cdot b))^2 = \|P \otimes Q(A)(a^*)\|^2 \leq \|a\|^2 \|b\|^2.
\]

Therefore, by [13] Corollary 4.6, there is a unique extension of \(\mu_{ab}\) to a bounded (complex) measure on \(\sigma(A)\), the \(\sigma\)-algebra generated by \(A\), i.e. the Borel subsets of \(\mathbb{R}^2\).

Lemma 2.2. Let \(a, b \in \mathcal{M} \cap \mathcal{L}^2(M, \tau)\). Then

(i) \(\mu_{ab} = \frac{1}{4} \sum_{k=1}^4 (-i)^k \mu_{a+b, a+b}\),

(ii) if \(\sigma = \rho\), \(\mu_{aa}\) is a real positive measure,

(iii) if \(a, b\) are self-adjoint, \(\text{Re} \, \mu_{ab} = \text{Re} \, \mu_{ba}\).

Proof. (i) is standard.

(ii) Let \(\Omega_1, \Omega_2\) be Borel sets in \(\mathbb{R}\), and set \(e_j := e_\rho(\Omega_j)\), \(j = 1, 2\). Then \(\mu_{aa}(\Omega_1 \times \Omega_2) = \tau(e_1 a^*e_2 a) = \tau((e_2a e_1^*) e_2 a e_1) \geq 0\), and the thesis follows by uniqueness of the extension from \(A\) to \(\sigma(A)\).

(iii) Let \(\Omega_1, \Omega_2\) be Borel sets in \(\mathbb{R}\), and set \(e_1 := e_\rho(\Omega_1)\), \(e_2 := e_\rho(\Omega_2)\). Then \(\text{Re} \, \mu_{ab}(\Omega_1 \times \Omega_2) = \text{Re} \, \tau(e_1 e_2 b) = \text{Re} \, \tau(e_2 a e_1) = \text{Re} \, \tau(e_1 b e_2 a) = \text{Re} \, \mu_{ba}(\Omega_1 \times \Omega_2)\).

Lemma 2.3. Let \(a, b \in \mathcal{M} \cap \mathcal{L}^2(M, \tau)\). Let \(g, h: \mathbb{R} \rightarrow \mathbb{C}\) be bounded Borel functions. Then

\[
\tau(g(\rho)a^*h(\sigma)b) = \int \int g(x)h(y) \, d\mu_{ab}(x, y).
\]

Proof. We use notation as in the proof of Lemma 2.1. Let \(s = \sum_{i=1}^k s_i \chi_{A_i}\), \(t = \sum_{j=1}^k t_j \chi_{B_j}\) be simple Borel functions. Then

\[
\tau(s(\rho)a^*t(\sigma)b) = \sum_{i=1}^h \sum_{j=1}^k s_i t_j \tau(\chi_{A_i}(\rho)a^*\chi_{B_j}(\sigma)b) = \sum_{i=1}^h \sum_{j=1}^k s_i t_j \tau(P \otimes Q(A_i \times B_j)(a^*) \cdot b)
\]

\[
= \sum_{i=1}^h \sum_{j=1}^k s_i t_j \int \int \chi_{A_i \times B_j} \, d\mu_{ab} = \int \int s(x) t(y) \, d\mu_{ab}(x, y).
\]
Let now $g, h$ be bounded Borel functions, and $\{s_n\}, \{t_n\}$ sequences of simple Borel functions such that $s_m \to g, t_n \to h$ and $|s_m| \leq |g|, |t_n| \leq |h|$. Denote $r_n(x, y) := s_n(x) t_n(y), k(x, y) := g(x) h(y)$. Then, by ([1], Theorem V.3.2), $s_n(\rho) a^* t_n(\sigma) = P \otimes Q(r_n)(a^*) = P \otimes Q(k)(a^*) = g(\rho) a^* h(\sigma)$ in $L^2(\mathcal{M}, \tau)$, so that $\tau(s_n(\rho) a^* t_n(\sigma)) \to \tau(g(\rho) a^* h(\sigma))$. Moreover, $\int r_n \, d\mu_{ab} \to \int k \, d\mu_{ab}$, because $\mu_{ab}$ is a bounded measure. The thesis follows.

**Lemma 2.4.** Let $a, b \in \mathcal{M} \cap L^2(\mathcal{M}, \tau), \rho \in L^1(\mathcal{M}, \tau)_+, \beta \in (0, 1)$. Then

$$
\tau(\rho^\beta a^* \rho^{1-\beta} b) = \iint_{(0, \infty)^2} x^\beta y^{1-\beta} \, d\mu_{ab}(x, y).
$$

**Proof.** Let $n \in \mathbb{N}$, and set

$$f_n(x) := \begin{cases} x, & 0 \leq x \leq n \\ 0, & \text{else} \end{cases} \quad f(x) := \begin{cases} x, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Then

$$\tau(f_n(\rho) a^* f_n(\rho) \rho^{1-\beta} b) = \int_{\mathbb{R}^2} f_n(x)^\beta f_n(y)^{1-\beta} \, d\mu_{ab}(x, y).$$

Observe now that $f_n(\rho)^\beta \to f(\rho)^\beta = \rho^\beta$ in $L^{1/(\beta)}(\mathcal{M}, \tau)$, so that $f_n(\rho)^\beta a^* f_n(\rho) \rho^{1-\beta} b \to \rho^\beta a^* \rho^{1-\beta} b$ in $L^1(\mathcal{M}, \tau)$, which implies

$$\tau(f_n(\rho) a^* f_n(\rho) \rho^{1-\beta} b) \to \tau(\rho^\beta a^* \rho^{1-\beta} b).$$

Moreover, in case $\sigma = \rho, \mu_{aa}$ is a positive measure, so that, by monotone convergence,

$$\int_{\mathbb{R}^2} f_n(x)^\beta f_n(y)^{1-\beta} \, d\mu_{aa}(x, y) \to \int_{(0, \infty)^2} x^\beta y^{1-\beta} \, d\mu_{aa}(x, y).$$

Therefore, the thesis holds for $a = b$. By polarization (Lemma 2.2 (i)) the result is true in general.

**Lemma 2.5.** Let $a, b \in \mathcal{M} \cap L^2(\mathcal{M}, \tau)$. Then,

$$\mu := \mu_{aa} \otimes \mu_{bb} + \mu_{bb} \otimes \mu_{aa} - 2 \text{Re} \mu_{ab} \otimes \text{Re} \mu_{ab}$$

is a real positive Borel measure on $\mathbb{R}^4$.

**Proof.** Indeed, if $\Omega_1, \ldots, \Omega_4 \subset \mathbb{R}$ are measurable subsets, and $E_j := e_{\sigma_j}(\Omega_j) \in \text{Proj}(\mathcal{M}), j = 1, 3, E_j := e_{\sigma}(\Omega_j) \in \text{Proj}(\mathcal{M}), j = 2, 4$, then

$$\mu(\Omega_1 \times \cdots \times \Omega_4) = \tau(E_1 a^* E_2 a) \cdot \tau(E_3 b^* E_4 b) + \tau(E_3 a^* E_4 a) \cdot \tau(E_1 b^* E_2 b)$$

$$- 2 \text{Re} \tau(E_1 a^* E_2 b) \cdot \text{Re} \tau(E_3 a^* E_4 b)$$

$$\geq \tau(E_1 a^* E_2 a) \cdot \tau(E_3 b^* E_4 b) + \tau(E_3 a^* E_4 a) \cdot \tau(E_1 b^* E_2 b)$$

$$- 2|\tau(E_1 a^* E_2 b)| \cdot |\tau(E_3 a^* E_4 b)|.$$

Moreover,

$$|\tau(E_1 a^* E_2 b)| = |\tau((E_2 a E_1)^* E_2 b E_1)|$$

$$\leq \tau((E_2 a E_1)^* E_2 a E_1)^{1/2} \tau((E_2 b E_1)^* E_2 b E_1)^{1/2}$$

$$= \tau(E_1 a^* E_2 a)^{1/2} \cdot \tau(E_1 b^* E_2 b)^{1/2}.$$
3 The main result

Let \((M, \tau)\) be a semifinite von Neumann algebra with a n.s.f. trace. Let \(\omega\) be a normal state on \(M\), and \(\rho_\omega \in L^1(M, \tau)\) be such that \(\omega(x) = \tau(\rho_\omega x)\), for \(x \in M\). Then, for any \(A, B \in M_{sa}\), \(\beta \in (0, 1)\), we set

**Definition 3.1.**

\[
\begin{align*}
\text{Cov}_\omega(A, B) &:= \omega(AB) - \omega(A)\omega(B) = \tau(\rho_\omega AB) - \tau(\rho_\omega A)\tau(\rho_\omega B), \\
\text{Var}_\omega(A) &:= \text{Cov}_\omega(A, A) = \omega(A^2) - \omega(A)^2 = \tau(\rho_\omega A^2) - \tau(\rho_\omega A)^2, \\
\text{Corr}_{\omega, \beta}(A, B) &:= \tau(\rho_\omega AB) - \tau(\rho_\omega A)\tau(\rho_\omega B) - \tau(\rho_\omega A^\beta B^\beta). \\
\omega, \beta(A) &:= \text{Corr}_{\omega, \beta}(A, A) = \omega(A^2) - \tau(\rho_\omega A^\beta A^\beta).
\end{align*}
\]

**Proposition 3.2.** Let \(A_0 := A - \omega(A)I\), \(B_0 := B - \omega(B)I\). Then

\[
\begin{align*}
\text{Cov}_\omega(A, B) &= \tau(\rho_\omega A_0B_0), \\
\text{Corr}_{\omega, \beta}(A, B) &= \tau(\rho_\omega A_0B_0) - \tau(\rho_\omega A_0\rho_\omega^1 \rho_\omega^\beta B_0).
\end{align*}
\]

**Theorem 3.3.** For any \(A, B \in M_{sa}\), \(\beta \in (0, 1)\), we have

\[
\begin{align*}
\text{Var}_\omega(A) \geq |\text{Re Corr}_{\omega, \beta}(A, B)|^2.
\end{align*}
\]

**Proof.** To start with, let us assume that \(A, B \in M \cap L^2(M, \tau)\). Set

\[
\mathcal{F} := \text{Var}_\omega(A) \text{Var}_\omega(B) - |\text{Re Corr}_{\omega, \beta}(A, B)|^2 - \text{I}_{\omega, \beta}(A)\text{I}_{\omega, \beta}(B) - |\text{Re Corr}_{\omega, \beta}(A, B)|^2.
\]

Then, using Lemma 2.4 and symmetries of the integrands, we obtain

\[
\begin{align*}
\mathcal{F}_1 &= \tau(\rho_\omega A_0^2) - \tau(\rho_\omega B_0^2) = \tau(\rho_\omega A_0^2) - \tau(\rho_\omega A_0\rho_\omega^1 \rho_\omega^\beta A_0) - \tau(\rho_\omega A_0\rho_\omega^1 \rho_\omega^\beta B_0) \\
&\quad - 2\text{Re} \tau(\rho_\omega A_0 B_0) \cdot \text{Re} \tau(\rho_\omega A_0^1 \rho_\omega^\beta B_0) + (\text{Re} \tau(\rho_\omega^1 \rho_\omega^\beta B_0))^2.
\end{align*}
\]

Then, using Lemma 2.5 and symmetries of the integrands, we obtain

\[
\begin{align*}
\mathcal{F}_2 &= 2\text{Re} \tau(\rho_\omega A_0 B_0) \cdot \text{Re} \tau(\rho_\omega^1 \rho_\omega^\beta B_0) - (\text{Re} \tau(\rho_\omega^1 \rho_\omega^\beta B_0))^2 \\
&\quad = \frac{1}{2} \int_{(0, \infty)^4} \left( (\lambda_1 + \lambda_2)\lambda_3^2 \lambda_4 - \lambda_2 \lambda_3^2 \lambda_4 - 2\lambda_1 \lambda_2^2 \lambda_3 \lambda_4^2 \right) d\mu_{\rho_\omega A_0} \otimes \mu_{\rho_\omega A_0} (\lambda_1, \ldots, \lambda_4).
\end{align*}
\]

So that, using the notation of Lemma 2.5,

\[
\mathcal{F} = \mathcal{F}_1 - \mathcal{F}_2 = \frac{1}{4} \int_{(0, \infty)^4} \left( (\lambda_1 + \lambda_2)\lambda_3^2 \lambda_4 + \lambda_1 \lambda_2^2 \lambda_3 \lambda_4 - 2\lambda_1 \lambda_2^2 \lambda_3^2 \lambda_4^2 \right) d\mu(\lambda_1, \ldots, \lambda_4).
\]

Since \(\mu\) is a real positive measure on \([0, \infty)^4\), because of Lemma 2.5, and

\[
\begin{align*}
(\lambda_1 + \lambda_2)(\lambda_3^2 \lambda_4 + \lambda_1 \lambda_2^2 \lambda_3 \lambda_4 - 2\lambda_1 \lambda_2^2 \lambda_3^2 \lambda_4^2) &= (\lambda_1 + \lambda_2 - \lambda_1^2 \lambda_2 \lambda_3 \lambda_4 + \lambda_1 \lambda_2 \lambda_3 + \lambda_1 \lambda_2 \lambda_4) \\
&\geq 0,
\end{align*}
\]
we get \( F \geq 0 \), which is what we wanted to prove.

Finally, to extend the validity of the inequality from \( \mathcal{M}_{sa} \cap L^2(\mathcal{M}, \tau) \) to \( \mathcal{M}_{sa} \), let us observe that \( \mathcal{M}_{sa} \cap L^2(\mathcal{M}, \tau) \) is \( \sigma \)-weakly dense in \( \mathcal{M}_{sa} \), and \( a \in \mathcal{M} \mapsto \tau(\rho_\omega ab) \), \( b \in \mathcal{M} \mapsto \tau(\rho_\omega ab) \), \( a \in \mathcal{M} \mapsto \tau(\rho^{a_\beta} b^{1-\beta}) \), and \( b \in \mathcal{M} \mapsto \tau(\rho^{a_\beta} b^{1-\beta}) \) are \( \sigma \)-weakly continuous. \( \square \)

**Remark 3.4.** Observe that, reasoning as in [9] Theorem 5, one can prove that the function

\[
g(\beta) := \text{Var}_\omega(A) \text{Var}_\omega(B) - |\Re \text{Cov}_\omega(A, B)|^2 - I_{\omega, \beta}(A) I_{\omega, \beta}(B) + |\Re \text{Corr}_\omega, \beta(A, B)|^2
\]

is monotone increasing on the interval \([\frac{1}{2}, 1)\). Therefore, the best bound in Theorem 3.3 is given by \( \beta = \frac{1}{2} \), i.e. by the Wigner-Yanase information.

### References


