UNCERTAINTY PRINCIPLE AND QUANTUM FISHER INFORMATION

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Abstract. A family of inequalities, related to the uncertainty principle, has been recently proved by S. Luo, Z. Zhang, Q. Zhang, H. Kosaki, K. Yanagi, S. Furuichi and K. Kuriyama. We show that the inequalities have a geometric interpretation in terms of quantum Fisher information. Using this formulation one may naturally ask if this family of inequalities can be further extendend, for example to the *RLD* quantum Fisher information. We show that this is impossible by producing a family of counterexamples.

Key words and phrases: Uncertainty principle, monotone metrics, quantum Fisher information, Wigner-Yanase-Dyson information.

1. Introduction

Noncommutativity in quantum probability has far-reaching consequences. One of the most important is the Heisenberg uncertainty principle

$$\mathrm{Var}_\rho(A) \cdot \mathrm{Var}_\rho(B) \geq \frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2.$$

No such lower bound for the variance of pairs of random variables exists in classical probability. Schrödinger proved a stronger inequality involving covariance

$$\mathrm{Var}_\rho(A) \cdot \mathrm{Var}_\rho(B) - |\mathrm{Re}\{\mathrm{Cov}_\rho(A,B)\}|^2 \geq \frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2.$$

Recently S. Luo and Q. Zhang proved a different kind of uncertainty principle (see Luo and Q.Zhang (2004), Theorem 2), in the Schrödinger form, where the lower bound appears because the variables A, B do not commute with the state ρ (in contrast with the standard uncertainty principle where the bound depends on the commutator $[A, B]$).

The inequality was conjectured by S. Luo himself and Z. Zhang in a previous paper (Luo and Z.Zhang (2004)). These authors suggest there that "the result may be interpreted as a quantification of certain aspect of the Wigner-Araki-Yanase theorem for quantum measurement, which states that observables not commuting with a conserved quantity cannot be measured exactly" (see Wigner (1952), Araki and Yanase (1960), Ozawa (2002)). The inequality has been recently generalized in Kosaki (2005) and Yanagi-Furuichi-Kuriyama (2005). The final result is

$$\mathrm{Var}_\rho(A) \cdot \mathrm{Var}_\rho(B) - |\mathrm{Re}\{\mathrm{Cov}_\rho(A,B)\}|^2 \geq I_{\rho,\beta}(A) I_{\rho,\beta}(B) - |\mathrm{Re}\{\mathrm{Corr}_{\rho,\beta}(A,B)\}|^2$$

where I and Corr are given by the Wigner-Yanase-Dyson skew information (see Section 3. below).

The purpose of this paper is to put the above inequality in a more geometric form by means of quantum Fisher information (namely the monotone metrics classified by Petz).

In this way the lower bound will appear as a simple function of the area spanned by the commutators $i[A,\rho], i[B,\rho]$ in the tangent space to the state ρ, provided the state space is equipped with a suitable monotone metric (see Theorem 6.1). At this point it is natural to ask whether such an inequality holds for other quantum Fisher informations in the Wigner-Yanase-Dyson class (like the RLD-metric for example). The answer turns out to be negative and a general counterexample is given in Proposition 4.1.

In the final section we discuss some open problems related to the subject.

2. Schrödinger and Heisenberg Uncertainty Principles

Let $M_n := M_n(\mathbb{C})$ (resp.$M_{n,sa} := M_n(\mathbb{C})_{sa}$) be the set of all $n \times n$ complex matrices (resp. all $n \times n$ self-adjoint matrices). We shall denote general matrices by $X, Y, \ldots$ while letters $A, B, \ldots$ will be used for self-adjoint matrices. Let $\mathcal{D}_n$ be the set of strictly positive elements of M_n while $\mathcal{D}_n^1 \subset \mathcal{D}_n$ is the set of strictly positive density matrices namely

$$\mathcal{D}_n^1 = \{\rho \in M_n | \mathrm{Tr}\rho = 1,\ \rho > 0\}.$$

Proposition 2.1. *The correspondence*

$$M_n \times M_n \ni (X,Y) \to \langle X, Y \rangle := \mathrm{Tr}(\rho X Y^*) - \mathrm{Tr}(\rho X) \cdot \overline{\mathrm{Tr}(\rho Y)}$$

is a positive sesquilinear form.

As usual commutators and anticommutators are defined as $[X,Y] = XY - YX$, $\{X,Y\} = XY + YX$.

Definition 2.1. Suppose that $\rho \in \mathcal{D}_n^1$ is fixed. Define $X_0 := X - \mathrm{Tr}(\rho X) I$.

Definition 2.2. For $A, B \in M_{n,sa}$ and $\rho \in \mathcal{D}_n^1$ define covariance and variance as

$$\mathrm{Cov}_\rho(A,B) := \langle A, B \rangle = \mathrm{Tr}(\rho AB) - \mathrm{Tr}(\rho A) \cdot \mathrm{Tr}(\rho B) = \mathrm{Tr}(\rho A_0 B_0)$$

$$\mathrm{Var}_\rho(A) := \langle A, A\rangle = \mathrm{Tr}(\rho A^2) - \mathrm{Tr}(\rho A)^2 = \mathrm{Tr}(\rho A_0^2).$$

Note that for $A, B \in M_{n,sa}$ and $\rho \in \mathcal{D}_n^1$ one has

$$\mathrm{Re}(\mathrm{Tr}(\rho AB)) = \frac{1}{2}\mathrm{Tr}(\rho\{A,B\}) \qquad \mathrm{Im}(\mathrm{Tr}(\rho AB)) = \frac{1}{2i}\mathrm{Tr}(\rho[A,B]).$$

Since $\mathrm{Cov}_\rho(A,B) = \overline{\mathrm{Cov}_\rho(B,A)}$ then

$$2\mathrm{Re}\left\{\mathrm{Cov}_\rho\left(A,B\right)\right\} = \mathrm{Cov}_\rho(A,B) + \mathrm{Cov}_\rho(B,A).$$

As a consequence of Cauchy-Schwartz inequality one can derive the Schrödinger and Heisenberg Uncertainty Principles that are given in the following

THEOREM 2.1. *(see Schrödinger (1930)) For $A, B \in M_{n,sa}$ and $\rho \in \mathcal{D}_n^1$ one has*

$$\mathrm{Var}_\rho(A)\cdot\mathrm{Var}_\rho(B) - |\mathrm{Re}\{\mathrm{Cov}_\rho(A,B)\}|^2 \geq \frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2$$

that implies

$$\mathrm{Var}_\rho(A)\cdot\mathrm{Var}_\rho(B) \geq \frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2.$$

DEFINITION 2.3. Set

$$\mathcal{S}_\rho(A,B) := \mathrm{Var}_\rho(A)\cdot\mathrm{Var}_\rho(B) - |\mathrm{Re}\{\mathrm{Cov}_\rho(A,B)\}|^2.$$

Remark 1. With the above definition the Schrödinger Uncertainty Principle takes the form

$$\mathcal{S}_\rho(A,B) \geq \frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2.$$

Let us try to see this situation in general.

DEFINITION 2.4. Let $F : \mathcal{D}_n^1 \times M_{n,sa} \times M_{n,sa} \to \mathbb{R}$ be a function (denoted as $F_\rho(A,B)$) such that

$$\mathcal{S}_\rho(A,B) \geq F_\rho(A,B).$$

Then we say that F is an Uncertainty Principle Function (shortly UPF).

Problem: are there nontrivial UPF different from $\frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2$?

More specifically: the Heisenberg uncertainty principle says that if A and B do not commute then the product $\mathrm{Var}_\rho(A)\cdot\mathrm{Var}_\rho(B)$ cannot be arbitrarily small. Is the same true if $[A,\rho],[B,\rho]\neq 0$? The answer is given by Theorems 3.2, 6.1 and shows that in this case the bound on $\mathrm{Var}_\rho(A)\cdot\mathrm{Var}_\rho(B)$ depends on a certain "area" spanned by the commutators $[A,\rho],[B,\rho]$.

3. An inequality related to uncertainty principle

Definition 3.1. For $A,B\in M_{n,sa}$, $\rho\in\mathcal{D}_n^1$ and $\beta\in(0,1)$ set

$$\mathrm{Corr}_{\rho,\beta}(A,B):=\mathrm{Tr}(\rho AB)-\mathrm{Tr}(\rho^\beta A\rho^{1-\beta}B).$$

With direct calculation one can prove the following

Lemma 3.1.

$$2\mathrm{Re}\{\mathrm{Corr}_{\rho,\beta}(A,B)\}=\mathrm{Corr}_{\rho,\beta}(A,B)+\mathrm{Corr}_{\rho,\beta}(B,A)=-\mathrm{Tr}([\rho^\beta,A]\cdot[\rho^{1-\beta},B]),$$

$$\mathrm{Corr}_{\rho,\beta}(A,B)=\mathrm{Cov}_\rho(A,B)-\mathrm{Tr}(\rho^\beta A_0\rho^{1-\beta}B_0).$$

Definition 3.2. The Wigner-Yanase-Dyson information is defined as

$$I_{\rho,\beta}(A):=\mathrm{Corr}_{\rho,\beta}(A,A)=-\frac{1}{2}\mathrm{Tr}([\rho^\beta,A]\cdot[\rho^{1-\beta},A]).$$

Definition 3.3.

$$\mathcal{T}_{\rho,\beta}(A,B):=I_{\rho,\beta}(A)I_{\rho,\beta}(B)-|\mathrm{Re}\{\mathrm{Corr}_{\rho,\beta}(A,B)\}|^2.$$

Note that $\mathcal{T}_{\rho,\beta} = \mathcal{T}_{\rho,1-\beta}$ so one can consider just $\beta \in (0, \frac{1}{2}]$.

In Luo and Q. Zhang (2004) the following result has been proved

THEOREM 3.1. *$\mathcal{T}_{\rho,\frac{1}{2}}(A, B)$ is an UPF.*

The theorem had been conjectured in Luo and Z. Zhang (2004). A generalization of Theorem 3.1 has been given in Kosaki (2005) and Yanagi *et al.* (2005).

THEOREM 3.2. *$\mathcal{T}_{\rho,\beta}(A, B)$ is an UPF for any $\beta \in (0, \frac{1}{2}]$.*

Proof. We report here the proof of Yanagi *et al.* (2005) because it is needed in the sequel.

We have to prove that for any two self-adjoint operators A and B, any density operator ρ and any $0 < \beta \leq \frac{1}{2}$, we have

$$\mathrm{Var}_\rho(A)\,\mathrm{Var}_\rho(B) - |\mathrm{Re}\{\mathrm{Cov}_\rho(A, B)\}|^2 \geq I_{\rho,\beta}(A)\, I_{\rho,\beta}(B) - |\mathrm{Re}\{\mathrm{Corr}_{\rho,\beta}(A, B)\}|^2 .$$

Let $\{\varphi_i\}$ be a complete orthonormal base composed of eigenvectors of ρ, and $\{\lambda_i\}$ the corresponding eigenvalues.

Set $a_{ij} \equiv \langle A_0\varphi_i|\varphi_j\rangle$ and $b_{ij} \equiv \langle B_0\varphi_i|\varphi_j\rangle$.

Then we calculate

$$\mathrm{Var}_\rho(A) = \mathrm{Tr}(\rho A_0^2) = \frac{1}{2}\sum_{i,j}(\lambda_i + \lambda_j)a_{ij}a_{ji},$$

$$\mathrm{Var}_\rho(B) = \mathrm{Tr}(\rho B_0^2) = \frac{1}{2}\sum_{i,j}(\lambda_i + \lambda_j)b_{ij}b_{ji},$$

$$\mathrm{Re}\{\mathrm{Cov}_\rho(A, B)\} = \mathrm{Re}\{\mathrm{Tr}(\rho A_0 B_0)\} = \frac{1}{2}\sum_{i,j}(\lambda_i + \lambda_j)\mathrm{Re}\{a_{ij}b_{ji}\},$$

$$I_{\rho,\beta}(A) = \mathrm{Var}_\rho(A) - \mathrm{Tr}(\rho^\beta A_0 \rho^{1-\beta} A_0) = \frac{1}{2}\sum_{i,j}(\lambda_i + \lambda_j)a_{ij}a_{ji} - \sum_{i,j}\lambda_i^\beta \lambda_j^{1-\beta} a_{ij}a_{ji},$$

$$I_{\rho,\beta}(B) = \frac{1}{2}\sum_{i,j}(\lambda_i + \lambda_j)b_{ij}b_{ji} - \sum_{i,j}\lambda_i^{\beta}\lambda_j^{1-\beta}b_{ij}b_{ji},$$

$$\begin{aligned}\mathrm{Re}\{\mathrm{Corr}_{\rho,\beta}(A,B)\} &= \mathrm{Re}\{\mathrm{Cov}_{\rho}(A,B)\} - \mathrm{Re}\{\mathrm{Tr}(\rho^{\beta}A_0\rho^{1-\beta}B_0)\} \\ &= \frac{1}{2}\sum_{i,j}(\lambda_i + \lambda_j)\mathrm{Re}\{a_{ij}b_{ji}\} - \sum_{i,j}\lambda_i^{\beta}\lambda_j^{1-\beta}\mathrm{Re}\{a_{ij}b_{ji}\}.\end{aligned}$$

Set

$$\begin{aligned}\xi :&= \mathrm{Var}_{\rho}(A)\,\mathrm{Var}_{\rho}(B) - I_{\rho,\beta}(A)\,I_{\rho,\beta}(B) \\ &= \frac{1}{2}\sum_{i,j,k,l}\left\{(\lambda_i + \lambda_j)\lambda_k^{\beta}\lambda_l^{1-\beta} + (\lambda_k + \lambda_l)\lambda_i^{\beta}\lambda_j^{1-\beta} - 2\lambda_i^{\beta}\lambda_j^{1-\beta}\lambda_k^{\beta}\lambda_l^{1-\beta}\right\}a_{ij}a_{ji}b_{kl}b_{lk} \\ &= \frac{1}{4}\sum_{i,j,k,l}\left\{(\lambda_i + \lambda_j)\lambda_k^{\beta}\lambda_l^{1-\beta} + (\lambda_k + \lambda_l)\lambda_i^{\beta}\lambda_j^{1-\beta} - 2\lambda_i^{\beta}\lambda_j^{1-\beta}\lambda_k^{\beta}\lambda_l^{1-\beta}\right\}\{a_{ij}a_{ji}b_{kl}b_{lk} + a_{kl}a_{lk}b_{ij}b_{ji}\},\end{aligned}$$

$$\begin{aligned}\eta :&= |\mathrm{Re}\{\mathrm{Cov}_{\rho}(A,B)\}|^2 - |\mathrm{Re}\{\mathrm{Corr}_{\rho,\beta}(A,B)\}|^2 \\ &= \frac{1}{2}\sum_{i,j,k,l}\left\{(\lambda_i + \lambda_j)\lambda_k^{\beta}\lambda_l^{1-\beta} + (\lambda_k + \lambda_l)\lambda_i^{\beta}\lambda_j^{1-\beta} - 2\lambda_i^{\beta}\lambda_j^{1-\beta}\lambda_k^{\beta}\lambda_l^{1-\beta}\right\}\mathrm{Re}\{a_{ij}b_{ji}\}\mathrm{Re}\{a_{kl}b_{lk}\}.\end{aligned}$$

In order to prove the theorem it is enough to show $\xi - \eta \geq 0$. Indeed

$$\xi - \eta = \frac{1}{4}\sum_{i,j,k,l}\left\{(\lambda_i + \lambda_j)\lambda_k^{\beta}\lambda_l^{1-\beta} + (\lambda_k + \lambda_l)\lambda_i^{\beta}\lambda_j^{1-\beta} - 2\lambda_i^{\beta}\lambda_j^{1-\beta}\lambda_k^{\beta}\lambda_l^{1-\beta}\right\}\cdot$$

$$\cdot \left\{ |a_{ij}|^2 |b_{kl}|^2 + |a_{kl}|^2 |b_{ij}|^2 - 2\mathrm{Re}\{a_{ij}b_{ji}\}\mathrm{Re}\{a_{kl}b_{lk}\} \right\}.$$

Since

$$(\lambda_i + \lambda_j)\, \lambda_k^{\beta}\lambda_l^{1-\beta} + (\lambda_k + \lambda_l)\, \lambda_i^{\beta}\lambda_j^{1-\beta} - 2\lambda_i^{\beta}\lambda_j^{1-\beta}\lambda_k^{\beta}\lambda_l^{1-\beta}$$

$$= \left(\lambda_i + \lambda_j - \lambda_i^{\beta}\lambda_j^{1-\beta}\right) \lambda_k^{\beta}\lambda_l^{1-\beta} + \left(\lambda_k + \lambda_l - \lambda_k^{\beta}\lambda_l^{1-\beta}\right) \lambda_i^{\beta}\lambda_j^{1-\beta} \geq 0,$$

$$|a_{ij}|^2 |b_{kl}|^2 + |a_{kl}|^2 |b_{ij}|^2 \geq 2\, |a_{ij}b_{ji}|\, |a_{kl}b_{lk}| \geq 2\, |\mathrm{Re}\, \{a_{ij}b_{ji}\}\, \mathrm{Re}\, \{a_{kl}b_{lk}\}|\,,$$

we get the thesis.

$\square$

Remark 2. Note that Kosaki proved Theorem 3.2 by showing that $\mathcal{T}_{\rho,\beta}(A,B)$ is monotone increasing for $\beta \in (0, \frac{1}{2}]$. Moreover he was able to prove that $\mathcal{S}_{\rho}(A,B) = \mathcal{T}_{\rho,\beta}(A,B)$ iff A_0, B_0 are proportional.

4. A counterexample

The inequality of Theorem 3.2 is not true for arbitrary values of β as it is proved in the following

Proposition 4.1. *For any $\beta \in [-1, 0)$ there are a state ρ and self-adjoint operators A and B s.t.*

$$\mathrm{Var}_{\rho}(A)\, \mathrm{Var}_{\rho}(B) - |\mathrm{Re}\, \{\mathrm{Cov}_{\rho}(A,B)\}|^2 < I_{\rho,\beta}(A)\, I_{\rho,\beta}(B) - |\mathrm{Re}\, \{\mathrm{Corr}_{\rho,\beta}(A,B)\}|^2 .$$

Proof. Let $t \in (0, \frac{1}{2})$ and

$$\rho = \begin{pmatrix} t & 0 & 0 \\ 0 & 1-2t & 0 \\ 0 & 0 & t \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then, using the calculations performed for the proof of Theorem 3.2, we have

$$
\begin{aligned}
\xi - \eta &= \mathrm{Var}_\rho(A)\mathrm{Var}_\rho(B) - |\mathrm{Re}\{\mathrm{Cov}_\rho(A,B)\}|^2 - I_{\rho,\beta}(A)I_{\rho,\beta}(B) - |\mathrm{Re}\{\mathrm{Corr}_{\rho,\beta}(A,B)\}|^2 \\
&= \frac{1}{2}\sum_{i,j,k,l}\left\{(\lambda_i+\lambda_j)l_k^\beta\lambda_\ell^{1-\beta} + (l_k+\lambda_\ell)l_i^\beta\lambda_j^{1-\beta} - 2\lambda_i^\beta\lambda_j^{1-\beta}\lambda_k^\beta\lambda_\ell^{1-\beta}\right\}\cdot \\
&\cdot\{a_{ij}a_{ji}b_{kl}b_{lk} - \mathrm{Re}(a_{ij}b_{ji})\mathrm{Re}(a_{kl}b_{lk})\} \\
&= \frac{1}{2}(2\lambda_1 + 2\lambda_2 - \lambda_1^\beta\lambda_2^{1-\beta} - \lambda_2^\beta\lambda_1^{1-\beta})(\lambda_2^\beta\lambda_3^{1-\beta} + \lambda_3^\beta\lambda_2^{1-\beta}) + \\
&+ \frac{1}{2}(2\lambda_2 + 2\lambda_3 - \lambda_2^\beta\lambda_3^{1-\beta} - \lambda_3^\beta\lambda_2^{1-\beta})(\lambda_1^\beta\lambda_2^{1-\beta} + \lambda_2^\beta\lambda_1^{1-\beta}) \\
&= \left\{2(1-t) - t^\beta(1-2t)^{1-\beta} - (1-2t)^\beta t^{1-\beta}\right\}\left\{t^\beta(1-2t)^{1-\beta} + (1-2t)^\beta t^{1-\beta}\right\}.
\end{aligned}
$$

Let $\beta \in [-1,0)$. Since $t^\beta(1-2t)^{1-\beta} \to \infty$ if $t \to 0^+$, there exists a $t_0 = t_0(\beta) \in (0,1)$ for which $\xi - \eta < 0$. This ends the proof.

$\square$

Remark 3. For $\beta \in [-1,0)$ the inequality

$$\mathrm{Var}_\rho(A)\,\mathrm{Var}_\rho(B) - |\mathrm{Re}\{\mathrm{Cov}_\rho(A,B)\}|^2 < I_{\rho,\beta}(A)\,I_{\rho,\beta}(B) - |\mathrm{Re}\{\mathrm{Corr}_{\rho,\beta}(A,B)\}|^2$$

is not true in general as one can see by choosing

$$t \in (0,1), \quad \rho = \begin{pmatrix} t & 0 \\ 0 & 1-t \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

In the next Sections we try to give a more geometric form to Theorem 3.2 and Proposition 4.1.

5. Quantum Fisher Informations

In what follows if $\mathcal{N}$ is a differential manifold we denote by $T_\rho\mathcal{N}$ the tangent space to $\mathcal{N}$ at the point $\rho \in \mathcal{N}$.

In the commutative case a Markov morphism is a stochastic map $T : \mathbb{R}^n \to \mathbb{R}^k$. In the noncommutative case a Markov morphism is a completely positive and trace preserving operator $T : M_n \to M_k$. Let

$$\mathcal{P}_n := \{\rho \in \mathbb{R}^n | \rho_i > 0\}, \qquad \mathcal{P}_n^1 := \{\rho \in \mathbb{R}^n | \sum \rho_i = 1, \ \rho_i > 0\}.$$

In the commutative case a monotone metric is a family of riemannian metrics $g = \{g^n\}$ on $\{\mathcal{P}_n^1\}$, $n \in \mathbb{N}$, such that

$$g^m_{T(\rho)}(TX, TX) \leq g^n_\rho(X, X)$$

holds for every Markov morphism $T : \mathbb{R}^n \to \mathbb{R}^m$ and all $\rho \in \mathcal{P}_n^1$ and $X \in T_\rho\mathcal{P}_n^1$.

It is not difficult to see that there exists a natural identification of $T_\rho\mathcal{D}_n^1$ with the space of self-adjoint traceless matrices, namely

$$T_\rho\mathcal{D}_n^1 = \{A \in M_n | A = A^* \, , \ \mathrm{Tr}(A) = 0\}.$$

In perfect analogy with the commutative case, a monotone metric in the noncommutative case is a family of riemannian metrics $g = \{g^n\}$ on $\{\mathcal{D}_n^1\}$, $n \in \mathbb{N}$, such that

$$g^m_{T(\rho)}(TX, TX) \leq g^n_\rho(X, X)$$

holds for every Markov morphism $T : M_n \to M_m$ and all $\rho \in \mathcal{D}_n^1$ and $X \in T_\rho\mathcal{D}_n^1$.

Let us recall that a function $f : (0, \infty) \to \mathbb{R}$ is said operator monotone if, for any $n \in N$, any A, $B \in M_n$ such that $0 \leq A \leq B$, the inequalities $0 \leq f(A) \leq f(B)$ hold. An operator monotone function is said symmetric if $f(x) := xf(x^{-1})$. With such operator monotone functions f one associates the so-called Chentsov–Morotzova functions

$$c_f(x,y) := \frac{1}{yf(xy^{-1})} \qquad \text{for} \qquad x,y>0.$$

Define $L_\rho(A) := \rho A$, and $R_\rho(A) := A\rho$. Since L_ρ and R_ρ commute we may define $c_f(L_\rho, R_\rho)$. Now we can state the fundamental theorems about monotone metrics. In what follows uniqueness and classification are stated up to scalars (see Petz (1996)).

THEOREM 5.1. *(Chentsov 1982) There exists a unique monotone metric on $\mathcal{P}_n^1$ given by the Fisher information.*

THEOREM 5.2. *(Petz 1996) There exists a bijective correspondence between symmetric monotone metrics on $\mathcal{D}_n^1$ and symmetric operator monotone functions. This correspondence is given by the formula*

$$g_f(A,B) := g_{f,\rho}(A,B) := \mathrm{Tr}(A \cdot c_f(L_\rho, R_\rho)(B)).$$

Because of these two theorems we shall use the terms "Monotone Metrics" and "Quantum Fisher Informations" (shortly QFI) with the same meaning.

Note that usually monotone metrics are normalized so that if $[A,\rho]=0$ then $g_{f,\rho}(A,A) = \mathrm{Tr}(\rho^{-1}A^2)$, that is equivalent to ask $f(1)=1$.

Examples of monotone metrics are given by the following list (see Hasegawa and Petz (1997), Gibilisco and Isola (2004)).

Let

$$f_\beta(x) := \beta(1-\beta)\frac{(x-1)^2}{(x^\beta-1)(x^{1-\beta}-1)}, \qquad \beta \in [-1, \frac{1}{2}]\backslash\{0\},$$

$$f_0(x) := \frac{x-1}{\log x},$$

$$h_\gamma(x) := \left(\frac{1+x^\gamma}{2}\right)^{\frac{1}{\gamma}} \qquad \gamma \in [\frac{1}{2}, 1].$$

Note that $f_0 = \lim_{\beta \to 0} f_\beta$.

The metrics associated with the functions f_β are equivalent to the metrics induced by noncommutative α-divergences where $\beta = \frac{1-\alpha}{2}$ (see Hasegawa and Petz (1997)).

The RLD-metric is the QFI associated to f_{-1}.

The BKM-metric is the QFI associated to f_0.

The WY-metric is the QFI associated to $f_{\frac{1}{2}} = h_{\frac{1}{2}}$.

The SLD-metric (or Bures-Uhlmann metric) is the QFI associated to h_1.

The two parametric families f_β, h_γ give us a continuum of operator monotone functions from the smallest $f_{-1}(x) = \frac{2x}{x+1}$ to the greatest $h_1 = \frac{1+x}{2}$.

For a symmetric operator monotone function $\lim_{x \to +\infty} \frac{f(x)}{x} = f(0) := \lim_{x \to 0} f(x)$. Note that

$$f_\beta(0) = 0 \qquad \beta \in [-1, 0],$$

$$f_\beta(0) = \beta(1-\beta) \neq 0 \qquad \beta \in (0, \frac{1}{2}],$$

$$h_\gamma(0) = \left(\frac{1}{2}\right)^{\frac{1}{\gamma}} \neq 0 \qquad \gamma \in [\frac{1}{2}, 1].$$

The condition $f(0) \neq 0$ is relevant because it is a necessary and sufficient condition for the existence of the so-called radial extension of a monotone metric to pure states (see Petz and Sudar (1996)).

6. A geometric look at the inequality

Let V be a finite dimensional real vector space with a scalar product $g(\cdot, \cdot)$. We define, for $v, w \in V$,

$$\mathrm{Area}_g(v, w) := \sqrt{g(v,v) \cdot g(w,w) - |g(v,w)|^2}.$$

In the euclidean plane $\mathrm{Area}_g(v,w)$ is the area of the parallelogramme spanned by v and w.

Define $A_\rho := i[\rho, A]$. Since A_ρ is traceless and selfadjoint, then $A_\rho \in T_\rho \mathcal{D}_n^1$.

PROPOSITION 6.1. *For the QFI associated to f_β one has*

$$g_\beta(A_\rho, B_\rho) := g_{f_\beta}(A_\rho, B_\rho) = -\frac{1}{\beta(1-\beta)}\mathrm{Tr}([\rho^\beta, A]\cdot[\rho^{1-\beta}, B]) \qquad \beta \in [-1, \frac{1}{2}]\backslash\{0\}\,.$$

One can find a proof in Hasegawa and Petz (1997), Gibilisco and Isola (2004). Because of the above proposition g_β is known as the WYD(β) monotone metric.

If f is an operator monotone function we denote by Area_f the area functional associated to the monotone metric g_f. One has

THEOREM 6.1.

$$\mathcal{T}_{\rho,\beta}(A,B) = \frac{(\beta(1-\beta))^2}{4}\left(\mathrm{Area}_{f_\beta}(i[\rho,A], i[\rho,B])\right)^2 \qquad \forall \beta \in [-1, \frac{1}{2}]/\{0\}.$$

Proof. One has from Lemma 3.1 and Proposition 6.1

$$\begin{aligned}
\mathcal{T}_{\rho,\beta}(A,B) &= I_{\rho,\beta}(A) I_{\rho,\beta}(B) - |\mathrm{Re}\left\{\mathrm{Corr}_{\rho,\beta}(A,B)\right\}|^2 \\
&= \left(-\frac{1}{2}\mathrm{Tr}([\rho^\beta, A]\cdot[\rho^{1-\beta}, A])\right)\cdot\left(-\frac{1}{2}\mathrm{Tr}([\rho^\beta, B]\cdot[\rho^{1-\beta}, B])\right) - \frac{1}{4}|\mathrm{Tr}([\rho^\beta, A]\cdot[\rho^{1-\beta}, B])|^2 \\
&= \frac{(\beta(1-\beta))^2}{4}(g_\beta(A_\rho, A_\rho)\cdot g_\beta(B_\rho, B_\rho) - |g_\beta(A_\rho, B_\rho)|^2) \\
&= \frac{(\beta(1-\beta))^2}{4}\left(\mathrm{Area}_{f_\beta}(i[\rho,A], i[\rho,B])\right)^2\,.
\end{aligned}$$

□

7. Relation with curvature

The appearance of the area of a Riemannian metric in Theorem 6.1 (and therefore in Theorem 3.2) suggests a link between the uncertainty principle and the notion of curvature. In this section we make some considerations of this point. To make the paper self-contained we recall some notions of differential geometry.

For an affine (linear) connection ∇ on a manifold $\mathcal{M}$ the curvature is defined as (see Kobayashi and Nomizu (1963) pag. 133)

$$R(X, Y)Z := [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z.$$

Suppose that $g(\cdot, \cdot)$ is a Riemannian metric on $\mathcal{M}$ and ∇ is the associated Levi-Civita connection. The Riemannian curvature tensor is defined as (see Kobayashi and Nomizu (1963) pag. 201)

$$R(X, Y, Z, W) := g(R(Z, W)Y, X)$$

where X, Y, Z, W are vector fields.

Now let $\rho \in \mathcal{M}$ and suppose that we have a 2-dimensional subspace $\sigma \subset T_\rho\mathcal{M}$. Then σ determines, with the use of the exponential map exp, a 2-dimensional embedded surface $\mathcal{N} := \exp_\rho(B_\eta(0_\rho) \cap \sigma)$ formed by the geodesic segments of length $< \eta$ which start tangentially to σ. If $K(\sigma)$ denotes the Gaussian curvature of $\mathcal{N}$ one has the following

Proposition 7.1. *(see Klingenberg (1982) p.99-100) If A, B is a basis for the plane σ then*

$$K(A, B) := K(\sigma) = \frac{R(A, B, A, B)}{g(A, A)g(B, B) - |g(A, B)|^2} = \frac{g(R(A, B)B, A)}{\text{Area}_g(A, B)^2}.$$

When we want to emphasize the dependence of R and K from the Riemannian metric g we write R_g and K_g.

If f is an operator monotone function we denote by R_f the Riemannian curvature tensor and by K_f the sectional curvature.

Note that, if $\beta = \frac{1}{2}$, then $K_{f_{\frac{1}{2}}}(\sigma) = \mathit{costant} = \frac{1}{4}$ (see Gibilisco and Isola (2003)), so the inequality of Theorem 3.1

$$\mathcal{S}_\rho(A,B) \geq I_{\rho,\frac{1}{2}}(A) I_{\rho,\frac{1}{2}}(B) - |\mathrm{Re}\left\{\mathrm{Corr}_{\rho,\frac{1}{2}}(A,B)\right\}|^2$$

takes the form

$$\mathcal{S}_\rho(A,B) \geq \frac{1}{16} R_{f_{\frac{1}{2}}}(A_\rho, B_\rho, A_\rho, B_\rho)\,.$$

In general from bounds on sectional curvature $K_{f_\beta}(\sigma)$ one would be able to deduce inequalities of the same type for the Riemann curvature tensor (see Gibilisco and Isola (2005) for ideas about this kind of bounds).

8. Conclusions and open problems

We can summarize Theorem 3.2, Proposition 4.1 and Theorem 6.1 into the following.

THEOREM 8.1.

$$\mathcal{S}_\rho(A,B) \geq \frac{(\beta(1-\beta))^2}{4} \left(\mathrm{Area}_{f_\beta}(i[\rho,A], i[\rho,B])\right)^2$$

$$\Updownarrow$$

$$\beta \in [0, \frac{1}{2}]$$

We have an inequality that is true only for some elements f_β of the class of the Wigner-Yanase-Dyson monotone metrics. For example it is true for the WY- metric ($\beta = \frac{1}{2}$) and is false for the RLD-metric ($\beta = -1$). In this "iff" form the above inequality seems a result that cannot be further generalized.

Problem 1

Maybe one should still seek a different generalization of Theorem 8.1. Since

$$\beta \in [-1, 0] \Longrightarrow f_\beta(0) = 0 \qquad \& \qquad \beta \in (0, \frac{1}{2}] \Longrightarrow f_\beta(0) = \beta(1-\beta)$$

one can state Theorem 8.1 (that is Theorem 3.2) in a different way

THEOREM 8.2.

$$\mathcal{S}_\rho(A,B) \geq \frac{f_\beta(0)^2}{4}\left(\mathrm{Area}_{f_\beta}(i[A,\rho],i[B,\rho])\right)^2 \qquad \forall \beta \in [-1,\frac{1}{2}]$$

Question: characterize the family of operator monotone functions f for which is true the inequality

$$\mathcal{S}_\rho(A,B) \geq \frac{f(0)^2}{4}\left(\mathrm{Area}_f(i[A,\rho],i[B,\rho])\right)^2.$$

Of course the above inequality is trivially true when $f(0)=0$ while it is a non-trivial inequality for those operator monotone functions such that $f(0)>0$. Note that the question is non-trivial, for example, for the SLD-metric for which $h_1(0)=\frac{1}{2}$.

Problem 2

For f operator monotone define

$$G(f) := \frac{f(0)^2}{4}\left(\mathrm{Area}_f(i[A,\rho],i[B,\rho])\right)^2.$$

In Kosaki (2005) the proof of Theorem 3.2 (see p.640) is obtained by the following result

$$f_\beta \leq f_{\tilde\beta} \Longrightarrow G(f_\beta) \leq G(f_{\tilde\beta}) \qquad \beta \in (0,\frac{1}{2}]$$

Is this inequality true for other families of operator monotone functions?

Problem 3.

The following question has been posed at p.642 in Kosaki (2005). Covariance and WYD information make perfect sense in infinite dimension (see Connes and Stormer (1978), Kosaki (1982)), namely in a general von Neumann algebra setting. Is the inequality of Theorem 3.2 still true in this general setting?

Acknowledgements

The present results have been presented at the second International Conference "Information Geometry and its Applications" held at the University of Tokio, December 12-16, 2005. It is a pleasure to thank all the organizers.

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