On the monotonicity of scalar curvature in classical and quantum information geometry

Paolo Gibilisco[1,2], Tommaso Isola[3,]

March 23, 2006

[1]Dipartimento di Studi Economico-Finanziari e Metodi Quantitativi, Facoltà di Economia, Università di Roma "Tor Vergata", Via Columbia 2, Rome 00133, Italy. E-mail: Gibilisco@Economia.uniroma2.it
[2]Centro "Vito Volterra", Università di Roma "Tor Vergata", Via Columbia 2, Rome 00133, Italy. E-mail:gibilisco@volterra.mat.uniroma2.it
[3]Dipartimento di Matematica, Università di Roma "Tor Vergata", Via della Ricerca Scientifica, 00133 Rome, Italy. E-mail: isola@mat.uniroma2.it

Abstract

We study the monotonicity under mixing of the scalar curvature for the α–geometries on the simplex of probability vectors. From the results obtained and from numerical data we are led to some conjectures about quantum α–geometries and Wigner-Yanase-Dyson information. Finally we show that this last conjecture implies the truth of the Petz conjecture about the monotonicity of the scalar curvature of the Bogoliubov-Kubo-Mori monotone metric.

1 Introduction

The Bogoliubov-Kubo-Mori (BKM) metric is a distinguished element among the monotone metrics which are the quantum analogue of Fisher information on the quantum state space ([35, 36]). In a definite sense BKM metric is the geometry on the state space that is related to von Neumann entropy (say Umegaki relative entropy). Other well-known elements of this family are the Right Logarithmic Derivative (RLD) metric, the Symmetric Logarithmic Derivative (SLD or Bures) metric and the Wigner-Yanase-Dyson (WYD) metrics. In [34] Petz made a conjecture on the scalar curvature of the BKM metric. Many arguments and numerical calculations suggest that the conjecture is true; nevertheless a complete proof is still missing (see [24, 32, 12, 3, 4]).

One can state this conjecture in the following way: the BKM scalar curvature is a quantitative measure of symmetry (like entropy), namely it is increasing under mixing. Let us emphasize that it is also possible to relate the conjecture to quantities with direct physical meaning. An equivalent formulation, still due to Petz [34], is that "...the scalar curvature is an increasing function of the temperature ..." Moreover the asymptotic relation between volume and curvature in Riemannian geometry and Jeffrey's approach to priors in statistics induced Petz to interpret the scalar curvature as the average statistical uncertainty (that should increase under coarse graining, see [36]).

The original motivations given by Petz for the conjecture rely on the truth of the 2×2 case and on some numerical results for the general case. Petz and Sudar observed in [38] that "...Monotonicity of Kubo metric is not surprising because this result is a kind of reformulation of Lieb convexity theorem [30]. However the monotonicity of the scalar curvature seems to be an inequality of new type (provided the conjecture is really true)...". A recent clear reference for Lieb result and related inequalities can be found in the paper by Ruskai [40].

The goals of the present paper are the following.

1) We want to look at "higher mathematics from an elementary point of view". This means that we want to furnish an elementary motivation for the Petz conjecture. We do this by studying the monotonicity of the curvature for α-geometries in the plane. The results obtained in this case are very intuitive if one looks at the unit sphere of the L^p spaces. We conjecture that a similar behavior occurs for α-geometries in higher dimensions and in the non-commutative case too.

2) On the basis of the results of point 1) we make a conjecture about the monotonicity of scalar curvature for the WYD metrics. Further we show that, using a continuity argument, this WYD-conjecture would imply the Petz conjecture as a limit case (Theorem 8.1).

3) We review what is known about monotonicity of scalar curvature for quantum Fisher information. In particular we emphasize a result on Bures metric, due to Dittmann, according to which the scalar curvature, in this case, is neither Schur-increasing nor Schur-decreasing (see Section 2 for precise definitions). This implies that an example of a monotone metric for which the scalar curvature (or its opposite) is strictly increasing under mixing does not exist yet. Note that Andai (using an integral decomposition of [15]) proved that also in the 2×2 case there exist monotone metrics whose scalar curvature is not monotone [3].

Finally let us note that, related to this area, there exist other interesting papers. Some authors have suggested that, when statistical mechanics is geometrized, then the scalar curvature should have important physical meaning (for example it should be proportional to the inverse of the free energy, see [39, 26, 27, 25, 8, 9]).

2 Majorization and Schur-increasing functions

For the content of this section we refer to [1, 5, 6, 7, 31].

2.1 Commutative case

We shall denote by $\mathcal{P}_n$ the manifold of positive vectors of $\mathbb{R}^n$ and by $\mathcal{P}_n^1 \subset \mathcal{P}_n$ the submanifold of density vectors namely

Definition 2.1.
$$\mathcal{P}_n^1 := \{\rho \in \mathbb{R}^n \,|\, \sum_i \rho_i = 1, \ \rho_i > 0\}$$

We set $e := (1, ..., 1)$. The trace of a vector is $\mathrm{Tr}(v) = \sum_{i=1}^n v_i$. For a $n \times n$ real matrix consider the following properties

I) $t_{ij} \geq 0 \qquad i, j = 1, ..., n$
II) $\sum_{i=1}^n t_{ij} = 1 \qquad j = 1, ..., n$
III) $\sum_{j=1}^n t_{ij} = 1 \qquad i = 1, ..., n$

Definition 2.2. *a) T is said to be stochastic if I),II) hold;*
b) T is said to be doubly stochastic if I),II),III) hold.

When T is seen as an operator $T : \mathbb{R}^n \to \mathbb{R}^n$ (by $(Tv)_j = \sum_{i=1}^n t_{ji} v_i$) then the properties I),II),III) can be written as:

I)' (positivity preserving) $Tv \geq 0$ if $v \geq 0$;
II)' (trace-preserving) $\mathrm{Tr}(Tv) = \mathrm{Tr}(v) \qquad \forall v \in \mathbb{R}^n$;
III)' (unital) $Te = e$.

Let $x \in \mathbb{R}^n$ be a vector. We define $x^\downarrow$ as a vector with the same components in a decreasing order so that

$$x_1^\downarrow \geq x_2^\downarrow \geq ... \geq x_n^\downarrow.$$

Definition 2.3. *x is more mixed (more chaotic,...) than y (denoted by $x \succ y$) if and only if*

$$x_1^\downarrow \leq y_1^\downarrow$$
$$x_1^\downarrow + x_2^\downarrow \leq y_1^\downarrow + y_2^\downarrow$$
$$...$$
$$x_1^\downarrow + \cdots + x_{n-1}^\downarrow \leq y_1^\downarrow + \cdots + y_{n-1}^\downarrow$$
$$x_1^\downarrow + \cdots + x_n^\downarrow = y_1^\downarrow + \cdots + y_n^\downarrow$$

For example if $(\rho_1, ..., \rho_n)$ is a density vector then

$$(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}) \succ (\rho_1, ..., \rho_n) \succ (1, 0, ..., 0, 0)$$

The relation $\succ$ is a preordering but not a partial ordering. If $x \succ y$ and $y \succ x$ then $x = Ty$ for some permutation matrix T.

Theorem 2.1.

$$x \succ y \qquad \Longleftrightarrow \qquad x = Ty \text{ where } T \text{ is doubly stochastic.}$$

Definition 2.4. *(See [31] p. 14 and p. 54). A real-valued function f defined on a set $\mathcal{A} \subset \mathbb{R}^n$ is said to be Schur-increasing on $\mathcal{A}$ if*

$$x \succ y \text{ on } \mathcal{A} \qquad \Longrightarrow \qquad f(x) \geq f(y).$$

If in addition $f(x) > f(y)$ whenever $x \succ y$ but x is not a permutation of y then f is said to be strictly Schur-increasing. Similarly f is said to be Schur-decreasing on $\mathcal{A}$ if

$$x \succ y \text{ on } \mathcal{A} \qquad \Longrightarrow \qquad f(x) \leq f(y),$$

and f is strictly Schur-decreasing if strict inequality $f(x) < f(y)$ holds when x is not a permutation of y.

Of course f is Schur-increasing iff $-f$ is Schur-decreasing.

Remark 2.1. (See [31] p. 54). $\mathcal{A} \subset \mathbb{R}^n$ is symmetric if $x \in \mathcal{A} \Longrightarrow \Pi x \in \mathcal{A}$ for all permutations Π. A function f is symmetric on $\mathcal{A}$ if $f(x) = f(\Pi x)$ for all permutations Π. Let $\mathcal{D} := \{x | x_1 \geq ... \geq x_n\}$. If f is symmetric on a symmetric set $\mathcal{A}$ and Schur-increasing on $\mathcal{D} \cap \mathcal{A}$ then f is Schur-increasing on $\mathcal{A}$.

Remark 2.2. Let us consider the following identification $I : (0, \frac{\pi}{2}) \to \mathcal{P}_2^1$ defined by $I(\theta) := (\cos^2 \theta, \sin^2 \theta)$. Evidently if $\theta_1, \theta_2 \leq \frac{\pi}{4}$ then

$$\theta_1 \leq \theta_2 \Longleftrightarrow I(\theta_1) \prec I(\theta_2)$$

Any function on $\mathcal{P}_2^1$ can be seen as a function on $(0, \frac{\pi}{2})$. By abuse of language we shall use the same symbols to denote the two functions. Since $\mathcal{P}_2^1$ is symmetric we have, because of Remark 2.1,

Proposition 2.2. *A symmetric function f on $\mathcal{P}_2^1$ is Schur-increasing iff f is increasing as a function on $(0, \frac{\pi}{4})$.*

2.2 Non-commutative case

Let M_n be the space of complex $n \times n$ matrices. We shall denote by H_n the real subspace of hermitian matrices, by $\mathcal{D}_n$ the manifold of strictly positive elements of M_n and by $\mathcal{D}_n^1 \subset \mathcal{D}_n$ the submanifold of density matrices namely

Definition 2.5.

$$\mathcal{D}_n^1 := \{\rho \in M^n | Tr\rho = 1, \ \rho > 0\}.$$

If $A \in M_n$, let $\lambda(A)$ be the n-vector of its eigenvalues, arranged in any order with multiplicities counted. If A is hermitian then $\lambda(A)$ is a real n-vector. Let A, B be hermitian.

Definition 2.6.

$$A \succ B \qquad \Longleftrightarrow \qquad \lambda(A) \succ \lambda(B)$$

Definition 2.7. *A linear map Φ on M_n is doubly stochastic if it is positive-preserving, trace-preserving and unital.*

Theorem 2.3.

$$A \succ B \qquad \Longleftrightarrow \qquad A = \Phi(B) \text{ where } \Phi \text{ is doubly stochastic.}$$

Definition 2.8. *A real-valued function f defined on a set $\mathcal{A} \subset H_n$ is said to be Schur-increasing on $\mathcal{A}$ if*

$$A \succ B \qquad \Longrightarrow \quad f(A) \geq f(B).$$

Similarly f is said to be Schur-decreasing on $\mathcal{A}$ if

$$A \succ B \qquad \Longrightarrow \quad f(A) \leq f(B).$$

Of course f is Schur-increasing iff $-f$ is Schur-decreasing.

3 Pull-back of duality pairings

To make the paper self-contained we recall some constructions from [18].

Let V, W be vector spaces over $\mathbb{R}$ (or $\mathbb{C}$). One says that there is a duality pairing if there exists a separating bilinear form

$$\langle \cdot, \cdot \rangle : V \times W \to \mathbb{R}.$$

Let $\mathcal{M}, \mathcal{N}, \tilde{\mathcal{N}}$ be differentiable manifolds.

Definition 3.1. *Suppose we have a pair of immersions (φ, χ), where $\varphi : \mathcal{M} \to \mathcal{N}$ and $\chi : \mathcal{M} \to \tilde{\mathcal{N}}$, such that a duality pairing exists between $T_{\varphi(\rho)}\mathcal{N}$ and $T_{\chi(\rho)}\tilde{\mathcal{N}}$ for any $\rho \in \mathcal{M}$. Then we may pull-back this pairing on $\mathcal{M}$ by defining*

$$\langle u, v \rangle_\rho^{\varphi,\chi} := \langle D_\rho \varphi(u), D_\rho \chi(u) \rangle \qquad u, v \in T_\rho \mathcal{M}.$$

The most elementary example is given by the case where $\mathcal{N} = \tilde{\mathcal{N}}$ is a Riemannian manifold, $\varphi = \chi$ and the duality pairing is just given by the riemannian scalar product on $T_{\varphi(\rho)}\mathcal{N}$ (this is the pull-back metric induced by the map φ).

A non-trivial example is the following. Let X be a uniformly convex Banach space such that the dual $\tilde{X}$ is uniformly convex. We denote by $\langle \cdot, \cdot \rangle$ the standard duality pairing between X and $\tilde{X}$. Let $J : X \to \tilde{X}$ be the duality mapping, that is J is the differential of the map $v \to \frac{1}{2}\|v\|^2$. $J(v)$ is the unique element of the dual such that $\langle v, J(v) \rangle = \|v\|^2 = \|J(v)\|^2$.

Definition 3.2. *Let $\mathcal{M}$ be a manifold. If we have a map $\varphi : \mathcal{M} \to X$ we can consider a dualised pull-back that is a bilinear form defined on the tangent space of $\mathcal{M}$ by*

$$\langle A, B \rangle_\rho^\varphi := \langle A, B \rangle_\rho^{\varphi, J \circ \varphi} = \langle D_\rho \varphi(A), D_\rho(J \circ \varphi)(B) \rangle.$$

Example 3.1. For X a Hilbert space, J is the identity, and this is again the definition of pull-back metric induced by the map φ.

In what follows if $p \in \mathbb{R} \setminus \{0\}$ then $\tilde{p}$ is defined by $\frac{1}{p} + \frac{1}{\tilde{p}} = 1$. If $p = 1$ then $\tilde{p} = +\infty$.

Example 3.2. Let $(X, \mathcal{F}, \mu)$ be a measure space. If f is a measurable function and $p \in (1, +\infty)$ then $||f||_p := (\int |f|^p \, d\mu)^{\frac{1}{p}}$. Set

$$L^p = L^p(X, \mathcal{F}, \mu) = \{f \text{ is measurable and } ||f||_p < \infty\}$$

Define N^p as L^p with the norm

$$||f||_{N^p} := \frac{||f||_p}{p}.$$

Obviously $\widetilde{N^p}$ (the dual of N^p) can be identified with $N^{\tilde{p}}$.

Now suppose that $\rho > 0$ is measurable and $\int \rho = 1$, namely ρ is a strictly positive density. Then $v = p\rho^{\frac{1}{p}}$ is an element of the unit sphere of N^p and it is easy to see that $J(v) = \tilde{p}\rho^{\frac{1}{\tilde{p}}}$. The family of maps $\rho \to p\rho^{\frac{1}{p}}$ are known as Amari embeddings.

Let $X = \{1, ..., n\}$ and let μ be the counting measure. In this case N^p is just $\mathbb{R}^n$ with the norm $\frac{||\cdot||_p}{p}$.

Proposition 3.1. *Consider the Amari embedding $\varphi : \rho \in \mathcal{P}_n^1 \to p\rho^{\frac{1}{p}} \in N^p$ for an arbitrary $p \in (1, +\infty)$. Then the bilinear form*

$$\langle A, B \rangle_\rho^\varphi := \langle A, B \rangle_\rho^{\varphi, J \circ \varphi} = \langle D_\rho\varphi(A), D_\rho(J \circ \varphi)(B) \rangle \qquad A, B \in T_\rho \mathcal{P}_n^1$$

is just the Fisher information.

Proof.

$$\langle D_\rho\varphi(A), D_\rho(J \circ \varphi)(B) \rangle = \int (\rho^{\frac{1}{p}-1} A)(\rho^{\frac{1}{\tilde{p}}-1} B) = \int \frac{AB}{\rho}$$

$\square$

The above result can be stated in much greater generality using the machinery of [19, 14].

4 Scalar curvature of α-geometries

The α-geometries are one of the fundamental objects of Information Geometry (see [2, 20]). The study of the monotonicity of their curvatures does not appear in the literature as far as we know. In this section we start such an investigation.

4.1 The plane case

Definition 4.1. *The α-geometry on $\mathcal{P}_2^1$ is the pull-back geometry induced by the map $A_p(\rho) : \mathcal{P}_2^1 \to \mathbb{R}^2$ defined by*

$$A_p(\rho) := \begin{cases} p\rho^{\frac{1}{p}} & p \in \mathbb{R} \setminus \{0\} \\ \log(\rho) & p = \infty \end{cases}$$

where $p = \frac{2}{1-\alpha}$.

Definition 4.2. *We denote by $c_p(\rho)$ the curvature of the α-geometry (with $p = \frac{2}{1-\alpha}$) at the point $\rho \in \mathcal{P}_2^1$.*

Remark 4.1. For the curvature $c_p(\cdot)$ there are two easy cases:
- if $p = 1$ then $c_p(\cdot) = costant = 0$;
- if $p = 2$ then $c_p(\cdot) = costant = \frac{1}{2}$.

Giving a look at the unit sphere of $\mathbb{R}^2$ with respect to the L^p-norm one can easily understand the following general result.

Theorem 4.1. *For the function $c_p(\cdot) : \mathcal{P}_2^1 \to R$ one has the following properties:*
- *if $p \in (1, 2)$ then $c_p(\cdot)$ is a strictly Schur-decreasing function;*
- *if $p \in (2, +\infty]$ then $c_p(\cdot)$ is a strictly Schur-increasing function.*

Proof. Let us first consider $p \in (1, \infty)$. Then the α-geometry, $\alpha = \frac{p-2}{p}$, on $\mathcal{P}_2^1$ is the geometry of the set

$$\mathcal{B} := \left\{ (x, y) \in \mathbb{R}^2 : \left(\frac{x}{p}\right)^p + \left(\frac{y}{p}\right)^p = 1, \ x > 0, \ y > 0 \right\}.$$

Let us introduce the parametrization

$$x = p(\cos \vartheta)^{\frac{2}{p}}, \quad y = p(\sin \vartheta)^{\frac{2}{p}}, \quad 0 < \vartheta < \frac{\pi}{2}.$$

Then

$$x' = 2(\cos \vartheta)^{\frac{2}{p}-1}(-\sin \vartheta) \qquad y' = 2(\sin \vartheta)^{\frac{2}{p}-1} \cos \vartheta$$

$$x'' = 2(\cos \vartheta)^{\frac{2}{p}-2}(\frac{2}{p}\sin^2 \vartheta - 1) \qquad y'' = 2(\sin \vartheta)^{\frac{2}{p}-2}(\frac{2}{p}\cos^2 \vartheta - 1)$$

Let us parametrize density vectors as $(\cos^2 \theta, \sin^2 \vartheta)$. In this way the curvature of α-geometry at the point ρ, namely $c_p(\rho)$, is

$$c_p(\vartheta) := \frac{|x'y'' - x''y'|}{[(x')^2 + (y')^2]^{\frac{3}{2}}} = \frac{p-1}{p} \frac{(\sin \vartheta \cos \vartheta)^{\frac{2}{p}+2}}{[(\sin \vartheta(\cos \vartheta)^{\frac{1}{p}})^4 + ((\sin \vartheta)^{\frac{1}{p}} \cos \vartheta)^4]^{\frac{3}{2}}}$$

$$= \frac{p-1}{p} \left(\frac{1}{2}\right)^{2\left(1-\frac{2}{p}\right)} \cdot \frac{(\sin 2\vartheta)^{2\left(1-\frac{2}{p}\right)}}{\left[(\cos \vartheta)^{\frac{4}{p}} + (\sin \vartheta)^{\frac{4}{p}}\right]^{\frac{3}{2}}} = A_p \cdot \frac{g_p(\vartheta)}{f_p(\vartheta)^{\frac{3}{2}}},$$

where we set

$$A_p := \frac{p-1}{p} \left(\frac{1}{2}\right)^{2\left(1-\frac{2}{p}\right)}$$

$$g_p(\vartheta) := (\sin 2\vartheta)^{2-\frac{4}{p}}$$

$$f_p(\vartheta) := (\cos \vartheta)^{\frac{4}{p}} + (\sin \vartheta)^{\frac{4}{p}}.$$

We want to compute the monotonicity properties of c_p with respect to the preordering $\succ$. We have

$$g_p{}'(\vartheta) := 4(\sin 2\vartheta)^{1-\frac{4}{p}} \cdot (\cos \vartheta + \sin \vartheta) \left(1 - \frac{2}{p}\right)(\cos \vartheta - \sin \vartheta);$$

since $0 < \vartheta < \frac{\pi}{2}$ then

$$4(\sin 2\vartheta)^{1-\frac{4}{p}} \cdot (\cos \vartheta + \sin \vartheta) > 0$$

and therefore

$$g_p{}'(\vartheta) > 0 \iff \left(1 - \frac{2}{p}\right)(\cos \vartheta - \sin \vartheta) > 0.$$

Moreover

$$f_p{}'(\vartheta) = \frac{4}{\tilde{p}} \sin \vartheta \cos \vartheta ((\sin \vartheta)^{\frac{2}{\tilde{p}}-1} + (\cos \vartheta)^{\frac{2}{\tilde{p}}-1})((\sin \vartheta)^{\frac{2}{\tilde{p}}-1} - (\cos \vartheta)^{\frac{2}{\tilde{p}}-1});$$

again, since $0 < \vartheta < \frac{\pi}{2}$ then

$$4 \sin \vartheta \cos \vartheta ((\sin \vartheta)^{\frac{2}{\tilde{p}}-1} + (\cos \vartheta)^{\frac{2}{\tilde{p}}-1}) > 0$$

and therefore

$$f_p{}'(\vartheta) > 0 \iff \frac{1}{\tilde{p}}((\sin \vartheta)^{\frac{2}{\tilde{p}}-1} - (\cos \vartheta)^{\frac{2}{\tilde{p}}-1}) > 0.$$

$c_p(\cdot)$ is evidently symmetric on $\mathcal{P}_2^1$ and therefore (because of Proposition 2.2) the fact that the curvature is strictly Schur-increasing (decreasing) is equivalent to the fact that $c_p(\vartheta)$ is strictly increasing (decreasing) for $0 < \vartheta < \frac{\pi}{4}$.

We have the following cases

Case: $1 < p < 2$.

This implies $1 - \frac{2}{p} < 0$, $\frac{2}{\tilde{p}} - 1 < 0$ and therefore

$$g_p{}'(\vartheta) > 0 \iff \cos \vartheta < \sin \vartheta \iff \frac{\pi}{4} < \vartheta < \frac{\pi}{2},$$

$$f_p'(\vartheta) > 0 \iff (\sin\vartheta)^{\frac{2}{p}-1} > (\cos\vartheta)^{\frac{2}{p}-1} \iff \sin\vartheta < \cos\vartheta \iff 0 < \vartheta < \frac{\pi}{4}.$$

Therefore, for $0 < \vartheta < \frac{\pi}{4}$, g is decreasing, f is increasing and $\frac{1}{f^{\frac{3}{2}}}$ is decreasing. This implies that

$$c_p = A_p \frac{g_p}{f_p^{\frac{3}{2}}}$$

is strictly decreasing for $0 < \vartheta < \frac{\pi}{4}$.

Case: $2 < p < \infty$.

This implies $1 - \dfrac{2}{p} > 0$, $\dfrac{2}{\tilde{p}} - 1 > 0$ and therefore

$$g_p'(\vartheta) > 0 \iff \cos\vartheta > \sin\vartheta \iff 0 < \vartheta < \frac{\pi}{4}$$

$$f_p'(\vartheta) > 0 \iff (\sin\vartheta)^{\frac{2}{p}-1} > (\cos\vartheta)^{\frac{2}{p}-1} \iff \sin\vartheta > \cos\vartheta \iff \frac{\pi}{4} < \vartheta < \frac{\pi}{2}.$$

Therefore, for $0 < \vartheta < \frac{\pi}{4}$, g is increasing, f is decreasing and $\frac{1}{f^{\frac{3}{2}}}$ is increasing. This implies that

$$c_p = A_p \frac{g_p}{f_p^{\frac{3}{2}}}$$

is strictly increasing for $0 < \vartheta < \frac{\pi}{4}$.

Case: $p = \infty$.

Use now the following parametrization

$$x = 2\log(\cos\theta) \qquad\qquad y = 2\log(\sin\theta)$$

for the curve $e^x + e^y = 1$. Then

$$x' = -2\frac{\sin\theta}{\cos\theta} \qquad\qquad y' = 2\frac{\cos\theta}{\sin\theta}$$

$$x'' = \frac{-2}{\cos^2\theta} \qquad\qquad y'' = \frac{-2}{\sin^2\theta}$$

$$c_\infty(\vartheta) := \frac{|x'y'' - x''y'|}{[(x')^2 + (y')^2]^{\frac{3}{2}}} = \frac{(\sin\theta\cos\theta)^2}{[(\cos\theta)^4 + (\sin\theta)^4]^{\frac{3}{2}}}$$

$$= \lim_{p\to+\infty} \frac{p-1}{p}\left(\frac{1}{2}\right)^{2\left(1-\frac{2}{p}\right)} \cdot \frac{(\sin 2\vartheta)^{2\left(1-\frac{2}{p}\right)}}{\left[(\cos\vartheta)^{\frac{4}{p}} + (\sin\vartheta)^{\frac{4}{p}}\right]^{\frac{3}{2}}}$$

Note that

$$c_\infty(\vartheta) = \lim_{p\to+\infty} c_p(\theta).$$

If we set

$$g_\infty(\vartheta) := (\sin\theta\cos\theta)^2 \qquad\qquad f_\infty(\vartheta) := (\cos\theta)^4 + (\sin\theta)^4$$

then

$$g_\infty{}'(\vartheta) = 2\sin\vartheta\cos\vartheta(\cos\vartheta + \sin\vartheta)(\cos\vartheta - \sin\vartheta)$$
$$f_\infty{}'(\vartheta) = 4\sin\vartheta\cos\vartheta(\cos\vartheta + \sin\vartheta)(\sin\vartheta - \cos\vartheta).$$

This implies

$$g_\infty{}'(\vartheta) > 0 \iff \cos\vartheta > \sin\vartheta$$
$$f_\infty{}'(\vartheta) > 0 \iff \sin\vartheta > \cos\vartheta.$$

We have the same situation of the case $2 < p < \infty$ and therefore the same conclusion.

This ends the proof. $\qquad\square$

Note that we have also

Proposition 4.2. *For the function $c_p(\cdot) : \mathcal{P}_2^1 \to \mathbb{R}$ one has the following properties: if $p \in (-\infty, 0)$ then $c_p(\cdot)$ is strictly Schur-increasing.*

Proof. Since

$$1 - \frac{2}{p} > 0, \qquad \frac{2}{\tilde{p}} - 1 > 1 > 0, \qquad 0 < \tilde{p} < 1$$

we have the same situation of the case $2 < p < \infty$ in the preceding Theorem 4.1 and therefore the same conclusion. $\qquad\square$

If $p \in (0, 1)$ then $c_p(\cdot)$ can have an arbitrary behavior (Schur-increasing, Schur-decreasing, neither of the two).

4.2 The general case

Definition 4.3. *The α-geometry on $\mathcal{P}_n^1$ is the pull-back geometry induced by the map $A_p(\rho) : \mathcal{P}_n^1 \to \mathbb{R}^n$ defined by*

$$A_p(\rho) := \begin{cases} p\rho^{\frac{1}{p}} & p \in \mathbb{R} \setminus \{0\} \\ \log(\rho) & p = \infty, \end{cases}$$

where $p = \frac{2}{1-\alpha}$.

Definition 4.4. *We denote by $\mathrm{Scal}_p(\rho)$ the scalar curvature of the α-geometry (with $p = \frac{2}{1-\alpha}$) at the point $\rho \in \mathcal{P}_n^1$.*

Of course the cases $p = 1$ (flat geometry) and $p = 2$ (geometry of a $(n-1)$-dimensional sphere with radius 2) are easy to study. One has:
- if $p = 1$ then $\mathrm{Scal}_p(\cdot) = costant = 0$;
- if $p = 2$ then $\mathrm{Scal}_p(\cdot) = costant = \frac{1}{4}(n-1)(n-2)$.

Again giving a look at the unit sphere of $\mathbb{R}^n$ equipped with L^p-norm one can easily understand the following conjecture.

Conjecture 4.1. *Suppose $n > 2$. For the function $\mathrm{Scal}_p(\cdot) : \mathcal{P}_n^1 \to R$ one has the following properties:*
- if $p \in (1,2)$ then $\mathrm{Scal}_p(\cdot)$ is a strictly Schur-decreasing function;
- if $p \in (2,+\infty]$ then $\mathrm{Scal}_p(\cdot)$ is a strictly Schur-increasing function.

4.3 Non-commutative case

Definition 4.5. *The α-geometry on $\mathcal{D}_n^1$ is the geometry induced by the pull-back of the map $A_p(\rho) : \mathcal{D}_n^1 \to M_n$ defined by*

$$A_p(\rho) := \begin{cases} p\rho^{\frac{1}{p}} & p \in \mathbb{R} \setminus \{0\} \\ \log(\rho) & p = \infty, \end{cases}$$

where $p = \frac{2}{1-\alpha}$.

Since the commutativity or non-commutativity of the context will be always clear we make a little abuse of language in the following definition.

Definition 4.6. *We denote by $\mathrm{Scal}_p(\rho)$ the scalar curvature of the α-geometry (with $p = \frac{2}{1-\alpha}$) at the point $\rho \in \mathcal{D}_n^1$.*

Again the case $p = 1$ (flat geometry) is obvious. The case $p = 2$ is known (see [16, 17] or Theorem 7.2 below) and we have:
   - if $p = 1$ then $\mathrm{Scal}_p(\cdot) = costant = 0$;
   - if $p = 2$ then $\mathrm{Scal}_p(\cdot) = costant = \frac{1}{4}(n^2 - 1)(n^2 - 2)$.

Motivated by the commutative plane case we formulate the following conjecture.

Conjecture 4.2. *Suppose $n \geq 2$. For the function $\mathrm{Scal}_p(\cdot) : \mathcal{D}_n^1 \to \mathbb{R}$ one has the following properties:*
- if $p \in (1,2)$ then $\mathrm{Scal}_p(\cdot)$ is a strictly Schur-decreasing function;
- if $p \in (2,+\infty]$ then $\mathrm{Scal}_p(\cdot)$ is a strictly Schur-increasing function.

5 Monotone metrics and their scalar curvatures

A commutative Markov morphism $T : \mathbb{R}^n \to \mathbb{R}^m$ is a stochastic map. A non-commutative Markov morphism is a linear map $T : M_n \to M_m$ that is completely positive and trace-preserving (note that in the commutative case complete positivity is equivalent to positivity, see for example [41]).

In the commutative case a monotone metric is a family of Riemannian metrics $g = \{g^n\}$ on $\{\mathcal{P}_n^1\}$, $n \in \mathbb{N}$ such that

$$g_{T(\rho)}^m(TX, TX) \leq g_\rho^n(X, X)$$

holds for every Markov morphism $T : \mathbb{R}^n \to \mathbb{R}^m$ and all $\rho \in \mathcal{P}_n^1$ and $X \in T_\rho \mathcal{P}_n$.

In perfect analogy, a monotone metric in the noncommutative case is a family of Riemannian metrics $g = \{g^n\}$ on $\{\mathcal{D}_n^1\}$, $n \in \mathbb{N}$ such that

$$g_{T(\rho)}^m(TX, TX) \leq g_\rho^n(X, X)$$

holds for every Markov morphism $T : M_n \to M_m$ and all $\rho \in \mathcal{D}_n^1$ and $X \in T_\rho \mathcal{D}_n^1$.

Let us recall that a function $f : (0, \infty) \to \mathbb{R}$ is called operator monotone if for any $n \in \mathbb{N}$, any $A, B \in M_n$ such that $0 \leq A \leq B$, the inequalities $0 \leq f(A) \leq f(B)$ hold. An operator monotone function is said symmetric if $f(x) = xf(x^{-1})$ and normalized if $f(1) = 1$. In what follows by operator monotone we mean normalized symmetric operator monotone. With each operator monotone function f one associates also the so-called Chentsov–Morotzova function

$$c_f(x, y) := \frac{1}{yf(\frac{x}{y})} \qquad \text{for} \qquad x, y > 0.$$

Define $L_\rho(A) := \rho A$, and $R_\rho(A) := A\rho$. Since L_ρ, R_ρ commute we may define $c(L_\rho, R_\rho)$. Now we can state the fundamental theorems about monotone metrics (uniqueness and classification are up to scalars).

Theorem 5.1. [10] *There exists a unique monotone metric on $\mathcal{P}_n^1$ given by the Fisher information.*

Theorem 5.2. [35] *There exists a bijective correspondence between monotone metrics on $\mathcal{D}_n^1$ and operator monotone functions given by the formula*

$$\langle A, B \rangle_{\rho, f} := \mathrm{Tr}(A \cdot c_f(L_\rho, R_\rho)(B)).$$

To state the general formula for the scalar curvature of a monotone metric we need some auxiliary functions. In what follows $c', (\log c)'$ denote derivatives with respect to the first variable, and $c = c_f$.

$$
\begin{aligned}
h_1(x, y, z) &:= \frac{c(x, y) - z\, c(x, z)\, c(y, z)}{(x - z)(y - z)c(x, z)c(y, z)}, \\
h_2(x, y, z) &:= \frac{(c(x, z) - c(y, z))^2}{(x - y)^2 c(x, y)c(x, z)c(y, z)}, \\
h_3(x, y, z) &:= z\, \frac{(\ln c)'(z, x) - (\ln c)'(z, y)}{x - y}, \\
h_4(x, y, z) &:= z\, (\ln c)'(z, x)\, (\ln c)'(z, y), \\
h &:= h_1 - \frac{1}{2}h_2 + 2h_3 - h_4.
\end{aligned}
\tag{5.1}
$$

The functions h_i have no essential singularities if arguments coincide.

Note that $\langle A, B \rangle_\rho^f := \mathrm{Tr}(A \cdot c_f(L_\rho, R_\rho)(B))$ defines a Riemannian metric also over $\mathcal{D}_n$ ($\mathcal{D}_n^1$ is a submanifold of codimension 1). Let $\mathrm{Scal}_f(\rho)$ be the scalar curvature of $(\mathcal{D}_n, \langle \cdot, \cdot \rangle_\rho^f)$ at ρ and $\mathrm{Scal}_f^1(\rho)$ be the scalar curvature of $(\mathcal{D}_n^1, \langle \cdot, \cdot \rangle_\rho^f)$.

Theorem 5.3. [12] *Let $\sigma(\rho)$ be the spectrum of ρ. Then*

$$\mathrm{Scal}_f(\rho) = \sum_{x,y,z \in \sigma(\rho)} h(x,y,z) - \sum_{x \in \sigma(\rho)} h(x,x,x)$$

$$\mathrm{Scal}_f^1(\rho) = \mathrm{Scal}_f(\rho) + \frac{1}{4}(n^2-1)(n^2-2).$$

These results have the following form in the simplest case (2×2 matrices). From Theorem 5.3 it follows that (see [3])

Corollary 5.4. *If $\rho \in D_2$ has eigenvalues λ_1, λ_2 one has*

$$\mathrm{Scal}(\rho) = h(\lambda_1, \lambda_1, \lambda_2) + h(\lambda_1, \lambda_2, \lambda_1) + h(\lambda_2, \lambda_1, \lambda_1)$$

$$+ h(\lambda_2, \lambda_2, \lambda_1) + h(\lambda_2, \lambda_1, \lambda_2) + h(\lambda_1, \lambda_2, \lambda_2) + \frac{3}{2}.$$

Theorem 5.5. *[3] If $\rho \in D_2$ has eigenvalues λ_1, λ_2 and $a = 2\lambda_1 - 1$ then*

$$r_f(a) := \mathrm{Scal}_f(\rho) =$$

$$\frac{14(a-1)\left[f'\left(\frac{1-a}{1+a}\right)\right]^2}{(1+a)^3\left[f\left(\frac{1-a}{1+a}\right)\right]^2} + \frac{2(a^2+7a-6)f'\left(\frac{1-a}{1+a}\right)}{(1+a)^2 a f\left(\frac{1-a}{1+a}\right)} + \frac{8(1-a)f''\left(\frac{1-a}{1+a}\right)}{(1+a)^3 f\left(\frac{1-a}{1+a}\right)}$$

$$+ \frac{2(1+a)f\left(\frac{1-a}{1+a}\right)}{a^2} + \frac{3a^3+5a^2+8a-4}{2(1+a)a^2}.$$

6 The WYD metrics

We are going to study a particular class of monotone metrics.

Definition 6.1.

$$f_p(x) := \frac{1}{p\tilde{p}} \cdot \frac{(x-1)^2}{(x^{\frac{1}{p}}-1)(x^{\frac{1}{\tilde{p}}}-1)} \qquad p \in \mathbb{R} \setminus \{0,1\}$$

$$f_1(x) = f_\infty(x) := \frac{x-1}{\log(x)} \qquad p = 1, \infty.$$

Obviously $f_p = f_{\tilde{p}}$ and

$$f_1 = \lim_{p \to 1} f_p = \lim_{p \to \infty} f_p = f_\infty.$$

Theorem 6.1. *[23, 22] The function f_p is operator monotone iff $p \in A := (-\infty, -1] \cup [\frac{1}{2}, +\infty]$.*

Note that $p \in A$ iff $\alpha \in [-3, 3]$.

Definition 6.2. *The WYD(p) metric of parameter p is the monotone metric associated to f_p (where $p \in A$).*

We have that f_{-1} is the function of the RLD-metric, $f_1 = f_\infty$ is the function of the BKM-metric and f_2 is the function of the WY-metric.

In what follows $p \in (1, +\infty)$ and we use again the symbol N_p to denote M_n with the norm

$$||A||_{N^p} = p^{-1}(\text{Tr}(|A|^p))^{\frac{1}{p}}$$

All the commutative construction of Example 3.2 goes through. The following Proposition is the non-commutative analogous of Proposition 3.1 (see also [37, 23, 28, 16, 21]).

Proposition 6.2. *[18] Let $\varphi : \rho \in \mathcal{D}_n^1 \to p\rho^{\frac{1}{p}} \in N_p$ be the Amari embedding. The dualized pull-back*

$$\langle A, B \rangle_\rho^\varphi := \langle A, B \rangle_\rho^{\varphi, J \circ \varphi} = \langle D_\rho \varphi(A), D_\rho(J \circ \varphi)(B) \rangle$$

coincides with the Wigner-Yanase-Dyson information.

7 Known results on monotonicity

In this short section we review what is known about monotonicity of scalar curvature for monotone metrics. This is useful to emphasize that, up to now, there exist no examples of a monotone metrics with Schur-increasing (or Schur-decreasing) scalar curvature.

The Bures or SLD metric is the monotone metric associated to the function $f = \frac{1+x}{2}$.

Theorem 7.1. *([11, 13]) The scalar curvature of SLD metric is not Schur-increasing neither Schur-decreasing.*

Proof. By [11] the SLD-metric has a global minimum at the most mixed state for any n. On the other hand (this is due to [13]) if $\sigma = \text{diag}(\frac{2}{9}, \frac{1}{9}, \frac{2}{3})$ and $\rho = \text{diag}(\frac{1}{6}, \frac{1}{6}, \frac{2}{3})$ then $\rho \succ \sigma$. Using Theorem 5.3 one can calculate $\text{Scal}(\sigma) = \frac{3078}{25} > \frac{3447}{28} = \text{Scal}(\rho)$ and so the conclusion follows. $\square$

Theorem 7.2. *[17] The scalar curvature of WY metric is a constant equal to $\frac{1}{4}(n^2 - 1)(n^2 - 2)$.*

8 A conjecture on the WYD scalar curvature and its relation with Petz conjecture

In this section we want to suggest that maybe there exists a whole family of monotone metrics with Schur-increasing scalar curvature.

Conjecture 8.1. *There exist $\varepsilon > 0$ such that for p in the interval $I := (1, 1+\varepsilon)$ the scalar curvature of the WYD(p) metrics is a Schur-increasing function.*

Conjecture 8.2. *(Petz conjecture).*
The scalar curvature of BKM metric is a Schur-increasing function. This can be rephrased as

$$\rho \succ \sigma \Longrightarrow \mathrm{Scal}_{f_1}(\rho) \geq \mathrm{Scal}_{f_1}(\sigma).$$

The motivations for Conjecture 8.1 are the following. The $WYD(p)$ metrics come from the dualized pull-back of Proposition 6.2. This means that the $WYD(p)$ metrics depend, indeed, on the pair $(p, \tilde{p})$. Note that when p is in the Schur-decreasing region $(1, 2)$ we have that $\tilde{p}$ is in the Schur-increasing region $(2, +\infty)$ (Theorem 4.1, Conjectures 4.1, 4.2). When p approaches 1 then $\tilde{p}$ goes to infinity. Near the boundary values $\{1, +\infty\}$ the increasing-decreasing "symmetry" should be broken: in this case $WYD(p)$ geometry comes from a geometry converging to a flat limit $(p \to 1)$ and a geometry converging to a (conjectured) Schur-increasing scalar curvature $(\tilde{p} \to \infty)$.

Theorem 8.1. *If Conjecture 8.1 is true then Conjecture 8.2 (Petz conjecture) is true.*

Proof. For an arbitrary manifold M let us denote by $\mathcal{M}(M)$ the manifold of Riemannian metrics of M. If $\rho \in M$ is fixed and $g \in \mathcal{M}(M)$ then the function $F_\rho(\cdot) : \mathcal{M}(M) \to \mathbb{R}$ defined by $F_\rho(g) := \mathrm{Scal}_g(\rho)$ is a smooth function (see [29, 33]). Identifying f_p with the metric

$$\langle A, B \rangle_{\rho, f_p} := \mathrm{Tr}(Ac_{f_p}(L_\rho, R_\rho)(B)).$$

we may consider the function $p \to f_p$ as a continuous curve in $\mathcal{M}(\mathcal{D}_n^1)$. This implies that, by composition, the function $p \to \mathrm{Scal}_{f_p}(\rho)$ is a real continuous function for each $\rho \in \mathcal{D}_n^1$. Suppose now that Conjecture 8.1 is true.

We have for arbitrary $\rho, \sigma \in \mathcal{D}_n^1$, such that $\rho \succ \sigma$

$$\mathrm{Scal}_{f_1}(\rho) = \lim_{p \to 1} \mathrm{Scal}_{f_p}(\rho) \geq \lim_{p \to 1} \mathrm{Scal}_{f_p}(\sigma) = \mathrm{Scal}_{f_1}(\sigma)$$

But this is precisely the Petz conjecture. $\qquad\square$

8.1 Numerical results

Conjecture 8.1 would have many consequences. An example is the following theorem.

Theorem 8.2. *Conjecture 8.1 implies that there exists $\varepsilon > 0$ such that for $p \in (1, 1+\varepsilon)$ the functions $r_p := r_{f_p}$ of Theorem 5.5 are concave and have their maximum at zero.*

Proof. It follows immediately by Theorem 5.5. $\qquad\square$

Using Mathematica one has the following graphs for the function r_p:
case $p = 1 + 10^{-1}$, see figure 1;
case $p = 1 + 10^{-6}$, see figure 2.
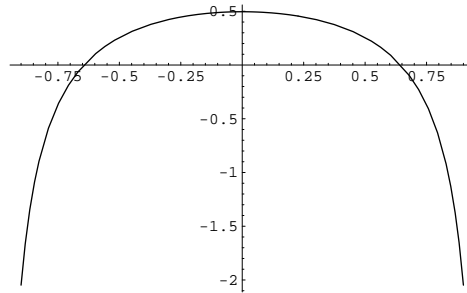
Figure 1: Case $p = 1 + 10^{-1}$
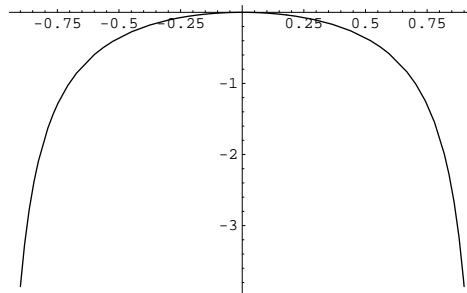
Figure 2: Case $p = 1 + 10^{-6}$

Let us emphasize what we said in the introduction: a recent result of Andai [3] shows the non-triviality of the above behavior. Indeed also in the 2×2 case there exist many monotone metrics with non-increasing scalar curvature.

References

[1] Albert, P.M. and Uhlmann, A., *Stochasticity and partial order. Doubly stochastic maps and unitary mixing* in Mathematics and its Applications, 9 D. Reidel Publishing Co., Dordrecht-Boston, Mass., 1982.

[2] Amari, S. and Nagaoka, H., *Methods of information geometry*. American Mathematical Society, Providence, RI, 2000.

[3] Andai, A., Monotone Riemannian metrics on density matrices with non-monotone scalar curvature. *J. Math. Phys.*, 44(9):3675–3688, 2003.

[4] Andai, A., On the monotonicity conjecture for the curvature of the Kubo-Mori metric. Preprint, arXiv:math-ph/0310064v1, 2003.

[5] Ando, T., Majorization, doubly stochastic matrices, and comparison of eigenvalues. *Lin. Alg. Appl.*, 118: 163-248, 1989.

[6] Ando, T., Majorization and inequalities in matrix theory. *Lin. Alg. Appl.*, 199: 17-67, 1994.

[7] Bhatia, R., *Matrix Analysis* Graduate Texts in Mathematics, 169, Springer, 1997.

[8] Brody, D. and Hughston, L., Geometrization of statistical mechanics. *Proc. R. Soc. London A* 455: 1683–1715, 1999.

[9] Brody, D. and Rivier, N., Geometrical aspects of statistical mechanics. *Phys. Rev. E* 51(2): 1006–1011, 1995.

[10] Čencov, N. N., *Statistical decision rules and optimal inference.* American Mathematical Society, Providence, R.I., 1982. Translation from the Russian edited by Lev J. Leifman.

[11] Dittmann, J., The scalar curvature of the Bures metric on the space of density matrices. *J. Geom. Phys.*, 31,:16–24, 1999.

[12] Dittmann, J., On the curvature of monotone metrics and a conjecture concerning the Kubo-Mori metric. *Linear Algebra Appl.*, 315:83–112, 2000.

[13] Dittmann, J., Personal communication, 2002.

[14] Gibilisco, P. and Isola, T., Connections on statistical manifolds of density operators by geometry of noncommutative L^p-spaces. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 2(1):169–178, 1999.

[15] Gibilisco, P. and Isola, T., Monotone metrics on statistical manifolds of density matrices by geometry of noncommutative L^2-spaces. In P. Sollich et al., editors, *Disordered and Complex Systems*, pages 129–139. American Institute of Physics, 2001. AIP Conference Proceedings 553.

[16] Gibilisco, P. and Isola, T., A characterisation of Wigner-Yanase skew information among statistically monotone metrics. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 4:553–557, 2001.

[17] Gibilisco, P. and Isola, T., Wigner-Yanase information on quantum state space: the geometric approach. *J. Math. Phys.*, 44:3752–3762, 2003.

[18] Gibilisco, P. and Isola, T., On the characterization of paired monotone metrics. *Ann. Inst. Stat. Math*, 56: 369–381, 2004.

[19] Gibilisco, P. and Pistone, G., Connections on non-parametric statistical manifolds by Orlicz space geometry. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 1(2):325–347, 1998.

[20] Gibilisco, P. and Pistone, G., Analytical and geometrical properties of statistical connections in Information Geometry. In A. Beghi et al., editors, *Mathematical Theory of Networks and Systems*, pages 881-814. Il Poligrafo (Padova), 1999.

[21] Grasselli, M. R., Dual connections in nonparametric classical information geometry. Submitted to Annals of Applied Probability, 2002.

[22] Hasegawa, H., Dual Geometry of the Wigner-Yanase-Dyson Information Content. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 6(3):413–430, 2003.

[23] Hasegawa, H. and Petz, D., Noncommutative extension of the information geometry II. In *Quantum communications and measurement*, pages 109–118. Plenum, New York, 1997.

[24] Hiai, F., Petz, D. and Toth, G., Curvature in the geometry of canonical correlation. *Studia Sci. Math. Hungar.*, 32: 235–249, 1996.

[25] Janke, W., Johnston, D.A. and Kenna, R., Information geometry of the spherical model. *Phys. Rev. E*, 67:046106–, 2003.

[26] Janyszek, H., On the geometrical structure of the generalized quantum Gibbs states. *Rep. Math Phys.*, 24 (1): 11–19, 1986.

[27] Janyszek, H. and Mrugala, R., Riemannian geometry and the thermodyamics of model magnetic systems *Phys. Rev. A*, 39 (12): 6515–6523, 1989.

[28] Jenčová, A., Geometry of quantum states: dual connections and divergence functions. *Rep. Math. Phys.*, 47(1):121–138, 2001.

[29] Kriegl A., and Michor, P.W., *The Convenient Setting of Global Analysis.* Mathematical Surveys and Monographs, vol. 53. American Mathematical Society.

[30] Lieb, E.H., Some convexity and subadditivity properties of entropy. *Bull. Am. math, Soc.*, 81,1–14 (1975).

[31] Marshall, A.W. and Olkin, I., *Inequalities: theory of majorization and its applications.* Mathematics in Science and Engineering, 143, Academic Press, New York, 1979.

[32] Michor, P., Petz, D. and Andai, A., On the curvature of a certain Riemannian space of matrices. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 3:199–212, 2000.

[33] Neuwirther, M., Submanifold geometry and hessians on the pseudoriemannian manifold of metrics. *Acta Math. Univ. Comenianae*, 52 (1): 51–85, 1993.

[34] Petz, D., Geometry of canonical correlation on the state space of a quantum system. *J. Math Phys.*, 35:780–795, 1994.

[35] Petz, D., Monotone metrics on matrix spaces. *Linear Algebra Appl.*, 244:81–96, 1996.

[36] Petz, D., Covariance and Fisher information in quantum mechanics. *J. Phys. A*, 35(4):929–939, 2002.

[37] Petz, D. and Hasegawa, H., On the Riemannian metric of α-entropies of density matrices. *Lett. Math. Phys.*, 38(2):221–225, 1996.

[38] Petz, D. and Sudár, C., Geometry of quantum states. *J. Math Phys.*, 37:2662–2673, 1996.

[39] Ruppeiner, G., Riemannian geometry approach to critical points: general theory. *Phys. Rev E*, 57 (5): 5135–5145, 1998.

[40] Ruskai, M.B., Lieb's simple proof of concavity of $(A, B) \mapsto \mathrm{Tr} A^p K^\dagger B^{1-p} K$ and remarks on related inequalities. Preprint, arXiv:quant-ph/0404126v1, 2004.

[41] Streater, R.F., *Statistical dynamics.* Imperial College Press, World Scientific 1995.