A characterisation of Wigner-Yanase skew information among statistically monotone metrics

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Abstract

Let $M_n = M_n(\mathbb{C})$ be the space of $n \times n$ complex matrices endowed with the Hilbert-Schmidt scalar product, let S_n be the unit sphere of M_n and let $D_n \subset M_n$ be the space of strictly positive density matrices. We show that the scalar product over D_n introduced by Gibilisco and Isola (1999) (that is the scalar product induced by the map $D_n \ni \rho \to \sqrt{\rho} \in S_n$) coincides with the Wigner-Yanase monotone metric.
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1 Introduction

In commutative Information Geometry the Fisher-Rao metric can be characterised in (at least) three ways: i) it is the unique statistically monotone metric (Chentsov theorem); ii) it is the Hessian of the Kullback-Leibler relative entropy; iii) it is obtained by division of square root of densities. In non-commutative Information Geometry, the classification theorem of Petz shows that there exists a whole family of statistically monotone metrics parametrised by the family of operator monotone functions (Petz, 1996). Nevertheless the results of Lesniewski and Ruskai (1999) and Gibilisco and Isola (2000) prove that each monotone metric is the Hessian of a suitable generalized relative entropy and is obtained by division of a generalized square root operator. In view of these results it is important to have characterisations that single out a particular monotone metric (for an example see Dittmann (1998)). Indeed it is sometimes difficult to decide which monotone metric is the good one for a certain application in quantum physics (Petz and Sudár, 1996, and Streater, 2000). In a previous paper (Gibilisco and Isola, 1999) we considered a scalar product on density matrices derived by the pull-back of the map $\rho \to \sqrt{\rho}$. It is natural to ask if this pull-back metric is a statistically monotone one. We show in this note that the pull-back of the square root embedding is the Wigner-Yanase monotone metric introduced by Hasegawa and Petz (1996, 1997).

2 Pull-back of Riemannian metrics

Let $\mathcal{M}$ be a differentiable manifold and $(\mathcal{N}, g)$ a Riemannian manifold (see Berger and Gostiaux (1988) for differential geometric concepts). Suppose $\varphi : \mathcal{M} \to \mathcal{N}$ is an immersion, that is a differentiable map such that its differential $D_\rho \varphi : T_\rho \mathcal{M} \to T_{\varphi(\rho)} \mathcal{N}$ is injective, for any $\rho \in \mathcal{M}$. Then

Proposition 2.1. *On $\mathcal{M}$ there exists a unique Riemannian scalar product $g^\varphi := \varphi^* g$ compatible with the differential structure of $\mathcal{M}$ such that $\varphi : (\mathcal{M}, g^\varphi) \to (\mathcal{N}, g)$ is an isometry.*

Proof. If ρ is an arbitrary point of $\mathcal{M}$, and $u,\ v \in T_\rho \mathcal{M}$, define

$$g^\varphi_\rho(u, v) := g_{\varphi(\rho)}(D_\rho \varphi(u), D_\rho \varphi(v)).$$

Since φ is differentiable, g^φ is compatible with the differential structure of $\mathcal{M}$ and makes φ an isometry. The uniqueness is obvious. $\square$

Definition 2.2. Under the above hypothesis g is said the pull-back metric induced by φ.

Remark 2.3. Let $\gamma : [0, 1] \to \mathcal{M}$ be a curve, and denote by $L(\gamma)$ the length of γ. Then $L(\gamma) = L(\varphi \circ \gamma)$.

3 The Fisher-Rao metric and the square root

Let $\mathcal{P}_n \subset \mathbb{R}^n$ be the simplex of strictly positive probability vectors, that is $\mathcal{P}_n := \{\rho \in \mathbb{R}^n : \sum_{i=1}^n \rho_i = 1, \rho_i > 0, i = 1, \dots, n\}$.

Definition 3.1. The Fisher-Rao Riemannian metric on $T\mathcal{P}_n \equiv \{u \in \mathbb{R}^n : \sum_{i=1}^n u_i = 0\}$ is given by

$$M^{fr}_\rho(u, v) := \sum_{i=1}^n \frac{u_i v_i}{\rho_i},$$

for $u,\ v \in T_\rho \mathcal{P}_n$.

Consider the map $\varphi : \rho \in \mathcal{P}_n \to \sqrt{\rho} \in \widetilde{S}_n$, where $\widetilde{S}_n$ is the unit sphere of $\mathbb{R}^n$, endowed with the natural metric as a Riemannian submanifold of $\mathbb{R}^n$. Then, the following result is well known.

Theorem 3.2. *The pull-back by the map φ of the natural metric on $\widetilde{S}_n$ coincides with the Fisher-Rao metric (namely the unique commutative statistically monotone metric).*

Proof. An easy calculation shows that, up to a scalar, the differential of φ is given by $D_\rho\varphi = M_\rho^{-1/2}$, where $M_\rho(u) := (\rho_1 u_1, \dots, \rho_n u_n)$. Therefore

$$\begin{aligned} g_\rho^\varphi(u,v) &:= g_{\varphi(\rho)}(D_\rho\varphi(u), D_\rho\varphi(v)) \\ &= \left\langle M_\rho^{-1/2}(u), M_\rho^{-1/2}(v) \right\rangle \\ &= \langle u, M_\rho^{-1}(v) \rangle \\ &= \sum_{i=1}^n \frac{u_i v_i}{\rho_i} = M_\rho^{fr}(u,v). \end{aligned}$$

□

4 The Wigner-Yanase skew information

Let $\rho \in D_n$ be a density matrix and let A be a self adjoint matrix. The Wigner-Yanase information (or skew information, information content relative to A) is defined as

$$I(\rho, A) := -Tr([\rho^{1/2}, A]^2),$$

where $[\cdot,\cdot]$ denotes the commutator. The tangent space to D_n at ρ is given by $T_\rho D_n \equiv \{A \in M_n : A = A^*, Tr(A) = 0\}$, and decomposes as $T_\rho D_n = (T_\rho D_n)^c \oplus (T_\rho D_n)^o$, where $(T_\rho D_n)^c := \{A \in T_\rho D_n : [A,\rho] = 0\}$, and $(T_\rho D_n)^o$ is the orthogonal complement of $(T_\rho D_n)^c$, with respect to the Hilbert-Schmidt scalar product $\langle A, B\rangle := Tr(A^*B)$. Let f be a symmetric operator monotone function and $c_f(x,y) := \frac{1}{yf(\frac{x}{y})}$ the associated Chentsov-Morotsova function. Petz classification theorem states that each statistically monotone metric on TD_n has the form $M_\rho^f(A,B) := Tr(Ac_f(L_\rho, R_\rho)(B))$, where $L_\rho(A) := \rho A$, and $R_\rho(A) := A\rho$. Each statistically monotone metric has a unique expression (up to a constant) given by $Tr(\rho^{-1}A^2)$, for $A \in (T_\rho D_n)^c$, because of Chentsov uniqueness theorem. Now consider the function

$$f_{wy}(x) := (\sqrt{x} + 1)^2,$$

which is operator monotone (Petz and Hasegawa, 1996). The associated Chentsov-Morotsova function is

$$c_{wy}(x,y) := \frac{1}{y f_{wy}(\frac{x}{y})} = \frac{1}{(\sqrt{x} + \sqrt{y})^2}.$$

Let us consider the monotone metric

$$M_\rho^{wy}(A,B) := Tr(Ac_{wy}(L_r, R_\rho)(B)).$$

A typical element of $(T_\rho D_n)^o$ has the form $i[\rho, A]$, where A is self-adjoint. We have

$$\begin{aligned} M^{wy}(i[\rho,A], i[\rho,A]) &= Tr(i[\rho,A](L_\rho^{1/2}+R_\rho^{1/2})^{-2}(i[\rho,A])) \\ &= -\left\langle (L_\rho^{1/2}+R_\rho^{1/2})^{-1}[\rho,A], (L_\rho^{1/2}+R_\rho^{1/2})^{-1}[\rho,A]\right\rangle \\ &= -\left\langle (L_\rho^{1/2}+R_\rho^{1/2})^{-1}(L_\rho - R_\rho)(A), (L_\rho^{1/2}+R_\rho^{1/2})^{-1}(L_\rho - R_\rho)(A)\right\rangle \\ &= -\left\langle (L_\rho^{1/2}-R_\rho^{1/2})(A), (L_\rho^{1/2}-R_\rho^{1/2})(A)\right\rangle \\ &= -\left\langle [\rho^{1/2},A], [\rho^{1/2},A]\right\rangle \\ &= -Tr([\rho^{1/2},A]^2) = I(\rho,A). \end{aligned}$$

5 The main result

Let us consider the unit sphere of M_n, denoted by S_n, as a real Riemannian submanifold of M_n. The natural metric on S_n is the one induced by the Hilbert-Schmidt scalar product of M_n.

Let $D_n \subset M_n$ be the manifold of strictly positive definite matrices. The map $\varphi : \rho \in D_n \to \sqrt{\rho} \in S_n$ is differentiable so we can apply the results of Section 2. We have the following

Theorem 5.1. *The pull-back by the map φ of the natural metric on S_n coincides with the Wigner-Yanase monotone metric.*

Proof. The differential of φ at the point ρ is given by $D_\rho\varphi := (L_\rho^{1/2}+R_\rho^{1/2})^{-1}$ (see the survey Pedersen (2000) for example). Therefore the pull-back metric is

$$\begin{aligned} g_\rho^\varphi(A,B) &:= g_{\varphi(\rho)}(D_\rho\varphi(A), D_\rho\varphi(B)) \\ &= \left\langle (L_\rho^{1/2}+R_\rho^{1/2})^{-1}(A), (L_\rho^{1/2}+R_\rho^{1/2})^{-1}(B)\right\rangle \\ &= Tr(A(L_\rho^{1/2}+R_\rho^{1/2})^{-2}(B)) \\ &= Tr(Ac_{wy}(L_\rho,R_\rho)(B)) = M_\rho^{wy}(A,B), \end{aligned}$$

which was to be proved. □

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