A volume inequality for quantum Fisher information and the uncertainty principle

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Abstract

Let $A_1, ..., A_N$ be complex self-adjoint matrices and let ρ be a density matrix. The Robertson uncertainty principle

$$\det\{\mathrm{Cov}_\rho(A_h, A_j)\} \geq \det\left\{-\frac{i}{2}\mathrm{Tr}(\rho[A_h, A_j])\right\}$$

gives a bound for the quantum generalized covariance in terms of the commutators $[A_h, A_j]$. The right side matrix is antisymmetric and therefore the bound is trivial (equal to zero) in the odd case $N = 2m + 1$.

Let f be an arbitrary normalized symmetric operator monotone function and let $\langle \cdot, \cdot \rangle_{\rho,f}$ be the associated quantum Fisher information. Based on previous results of several authors, we propose here as a conjecture the inequality

$$\det\{\mathrm{Cov}_\rho(A_h, A_j)\} \geq \det\left\{\frac{f(0)}{2}\langle i[\rho, A_h], i[\rho, A_j]\rangle_{\rho,f}\right\}$$

whose validity would give a non-trivial bound for any $N \in \mathbb{N}$ using the commutators $i[\rho, A_h]$.

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1 Introduction

Let $(V, g(\cdot, \cdot))$ be a real inner-product vector space and suppose that $v_1, ..., v_N \in V$. The real $N \times N$ matrix $G := \{g(v_h, v_j)\}$ is positive semidefinite and one can define $\mathrm{Vol}^g(v_1, ..., v_N) := \sqrt{\det\{g(v_h, v_j)\}}$. If the inner product depends on a further parameter in such a way that $g(\cdot, \cdot) = g_\rho(\cdot, \cdot)$, we write $\mathrm{Vol}^g(v_1, ..., v_N) = \mathrm{Vol}^g_\rho(v_1, ..., v_N)$.

As an example, consider a probability space $(\Omega, \mathcal{G}, \rho)$ and let $V = \mathcal{L}^2_{\mathbb{R}}(\Omega, \mathcal{G}, \rho)$ be the space of square integrable real random variables endowed with the scalar product given by the covariance $\mathrm{Cov}_\rho(A, B) := \mathrm{E}_\rho(AB) - \mathrm{E}_\rho(A)\mathrm{E}_\rho(B)$. For $A_1, ..., A_N \in \mathcal{L}^2_{\mathbb{R}}(\Omega, \mathcal{G}, \rho)$, G is the well known covariance matrix and one has

$$\mathrm{Vol}^{\mathrm{Cov}}_\rho(A_1, ..., A_N) \geq 0. \tag{1.1}$$

The expression $\det\{\mathrm{Cov}_\rho(A_h, A_j\}$ is known as the generalized variance of the random vector $(A_1, ..., A_N)$ and, in general, one cannot expect a stronger inequality. For instance, when $N = 1$, (1.1) just reduces to $\mathrm{Var}_\rho(A) \geq 0$.

In non-commutative probability the situation is quite different due to the possible non-triviality of the commutators $[A_h, A_j]$. Let $M_{n,sa} := M_{n,sa}(\mathbb{C})$ be the space of all $n \times n$ self-adjoint matrices and

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let $\mathcal{D}_n^1$ be the set of strictly positive density matrices (faithful states). For $A, B \in M_{n,sa}$ and $\rho \in \mathcal{D}_n^1$ define the (symmetrized) covariance as $\mathrm{Cov}_\rho(A,B) := 1/2[\mathrm{Tr}(\rho AB) + \mathrm{Tr}(\rho BA)] - \mathrm{Tr}(\rho A) \cdot \mathrm{Tr}(\rho B)$. If $A_1, ..., A_N$ are self-adjoint matrices one has

$$\mathrm{Vol}_\rho^{\mathrm{Cov}}(A_1, ..., A_N) \geq \begin{cases} 0, & N = 2m+1, \\ \det\{-\frac{i}{2}\mathrm{Tr}(\rho[A_h, A_j])\}^{\frac{1}{2}}, & N = 2m. \end{cases} \tag{1.2}$$

Let us call (1.2) the "standard" uncertainty principle to distinguish it from other inequalities like the "entropic" uncertainty principle and similar inequalities. Inequality (1.2) is due to Heisenberg, Kennard, Robertson and Schrödinger for $N = 2$ (see [14] [16] [28] [30]). The general case is due to Robertson (see [29]). Examples of recent references where inequality (1.2) plays a role are given by [31] [32] [33] [4] [3] [15].

Suppose one is looking for a general inequality of type (1.2) giving a bound also in the odd case $N = 2m + 1$. If one considers the case $N = 1$, it is natural to seek such an inequality in terms of the commutators $[\rho, A_h]$.

One of the purposes of the present paper is to state a conjecture regarding an inequality similar to (1.2) but not trivial for any $N \in \mathbb{N}$. Let $\mathcal{F}_{op}$ be the family of symmetric normalized operator monotone functions. To each element $f \in \mathcal{F}_{op}$ one may associate a ρ-depending scalar product $\langle \cdot, \cdot \rangle_{\rho,f}$ on the self-adjoint (traceless) matrices, which is a quantum version of the Fisher information (see [25]). Let us denote the associated volume by Vol_ρ^f. We conjecture that for any $N \in \mathbb{N}^+$ (this is one of the main differences from (1.2)) and for arbitrary self-adjoint matrices $A_1, ..., A_N$ one has

$$\mathrm{Vol}_\rho^{\mathrm{Cov}}(A_1, ..., A_N) \geq \left(\frac{f(0)}{2}\right)^{\frac{N}{2}} \mathrm{Vol}_\rho^f(i[\rho, A_1], ..., i[\rho, A_N]). \tag{1.3}$$

The cases $N = 1, 2$ of inequality (1.3) have been proved by the joint efforts of a number of authors in several papers: S. Luo, Q. Zhang, Z. Zhang ([19] [20] [24] [22] [23]); H. Kosaki ([17]); K. Yanagi, S. Furuichi, K. Kuriyama (see [34]); F. Hansen ([13]); P. Gibilisco, D. Imparato, T. Isola ([11] [6]).

It is well known that standard uncertainty principle is a simple consequence of the Cauchy-Schwartz inequality for $N = 2$. It is worth to note that for the inequality (1.3) the same role is played by the Kubo-Ando inequality

$$2(A^{-1} + B^{-1})^{-1} \leq m_f(A, B) \leq \frac{1}{2}(A + B)$$

saying that any operator mean is larger than the harmonic mean and smaller than the arithmetic mean.

The scheme of the paper is as follows. In Section 2 we describe the preliminary notions of operator monotone functions, matrix means and quantum Fisher information. In Section 3 we discuss a correspondence between regular and non-regular operator monotone functions that is needed in the sequel. In Section 4 we state our conjecture, namely the inequality (1.3); we also state other two conjectures concerning how the right side depends on $f \in \mathcal{F}_{op}$ and the conditions to have equality in (1.3). In Section 5 we discuss the case $N = 1$ of (1.3) presenting the different available proofs. In Section 6 we discuss the case $N = 2$; here we prove that, while the technique employed in [6] works in both cases $N = 1, 2$, the technique used in [13] does not. To this purpose, we show that the generalized variance is not a concave (neither a convex) function of the state. Moreover we observe that the technique used for the case $N = 2$ seems valuable also for the general case. In [24] it has been proved that the Wigner-Yanase correlation has some advantages on covariance when one aims to measure entanglement; in Section 7 we show, for the sake of completeness, that the above argument holds true for any regular quantum Fisher information.

2 Operator monotone functions, matrix means and quantum Fisher information

Let $M_n := M_n(\mathbb{C})$ (resp. $M_{n,sa} := M_{n,sa}(\mathbb{C})$) be the set of all $n \times n$ complex matrices (resp. all $n \times n$ self-adjoint matrices). We shall denote general matrices by $X, Y, ...$ while letters $A, B, ...$ will be used for self-adjoint matrices, endowed with the Hilbert-Schmidt scalar product $\langle A, B\rangle = \mathrm{Tr}(A^*B)$. The adjoint of a matrix X is denoted by $X^\dagger$ while the adjoint of a superoperator $T : (M_n, \langle\cdot,\cdot\rangle) \to (M_n, \langle\cdot,\cdot\rangle)$ is denoted by T^*. Let $\mathcal{D}_n$ be the set of strictly positive elements of M_n and $\mathcal{D}_n^1 \subset \mathcal{D}_n$ be the set of strictly positive density matrices, namely $\mathcal{D}_n^1 = \{\rho \in M_n | \mathrm{Tr}\rho = 1,\ \rho > 0\}$. If it is not otherwise specified, from now on we shall treat the case of faithful states, namely $\rho > 0$.

A function $f : (0, +\infty) \to \mathbb{R}$ is said *operator monotone (increasing)* if, for any $n \in \mathbb{N}$, and $A, B \in M_n$ such that $0 \le A \le B$, the inequalities $0 \le f(A) \le f(B)$ hold. An operator monotone function is said *symmetric* if $f(x) = xf(x^{-1})$ and *normalized* if $f(1) = 1$.

Definition 2.1. *$\mathcal{F}_{op}$ is the class of functions $f : (0, +\infty) \to (0, +\infty)$ such that*

(i) $f(1) = 1$,

(ii) $tf(t^{-1}) = f(t)$,

(iii) f is operator monotone.

Example 2.1. *Examples of elements in $\mathcal{F}_{op}$ are given by the following list*

$$\begin{array}{lllllll} f_{RLD}(x) & := & \frac{2x}{x+1}, & f_{WY}(x) & := & \left(\frac{1+\sqrt{x}}{2}\right)^2, & \\ f_{SLD}(x) & := & \frac{1+x}{2}, & f_{WYD(\beta)}(x) & := & \beta(1-\beta)\frac{(x-1)^2}{(x^\beta-1)(x^{1-\beta}-1)}, & \beta \in \left(0, \frac{1}{2}\right). \end{array}$$

We now report Kubo-Ando theory of matrix means (see [18]) as exposed in [27].

Definition 2.2. *A mean for pairs of positive matrices is a function $m : \mathcal{D}_n \times \mathcal{D}_n \to \mathcal{D}_n$ such that*
(i) $m(A, A) = A$,
(ii) $m(A, B) = m(B, A)$,
(iii) $A < B \implies A < m(A, B) < B$,
(vi) $A < A', \quad B < B' \implies m(A, B) < m(A', B')$,
(v) m is continuous,
(vi) $Cm(A, B)C^* \le m(CAC^*, CBC^*)$, for every $C \in M_n$.

Property (vi) is known as the transformer inequality. We denote by $\mathcal{M}_{op}$ the set of matrix means. The fundamental result, due to Kubo and Ando, is the following.

Theorem 2.1. *There exists a bijection between $\mathcal{M}_{op}$ and $\mathcal{F}_{op}$ given by the formula*

$$m_f(A, B) := A^{\frac{1}{2}} f(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}.$$

Example 2.2. *The arithmetic, geometric and harmonic (matrix) means are given respectively by*

$$\begin{aligned} A\nabla B &:= \tfrac{1}{2}(A + B), \\ A\#B &:= A^{\frac{1}{2}}(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{\frac{1}{2}} A^{\frac{1}{2}}, \\ A!B &:= 2(A^{-1} + B^{-1})^{-1}. \end{aligned}$$

They correspond respectively to the operator monotone functions $\frac{x+1}{2}, \sqrt{x}, \frac{2x}{x+1}$.

Kubo and Ando [18] proved that, among matrix means, arithmetic is the largest while harmonic is the smallest.

Proposition 2.2. *For any $f \in \mathcal{F}_{op}$ one has*

$$2(A^{-1} + B^{-1})^{-1} \le m_f(A, B) \le \frac{1}{2}(A + B).$$

Corollary 2.3. *For any $f \in \mathcal{F}_{op}$ and for any $x, > 0$ one has*

$$\frac{2x}{1+x} \leq f(x) \leq \frac{1+x}{2}.$$

In what follows, if $\mathcal{N}$ is a differential manifold we denote by $T_\rho\mathcal{N}$ the tangent space to $\mathcal{N}$ at the point $\rho \in \mathcal{N}$. Recall that there exists a natural identification of $T_\rho\mathcal{D}_n^1$ with the space of self-adjoint traceless matrices; namely, for any $\rho \in \mathcal{D}_n^1$

$$T_\rho\mathcal{D}_n^1 = \{A \in M_n | A = A^*\ ,\ \mathrm{Tr}(A) = 0\}.$$

A Markov morphism is a completely positive and trace preserving operator $T : M_n \to M_m$. A *monotone metric* is a family of Riemannian metrics $g = \{g^n\}$ on $\{\mathcal{D}_n^1\}$, $n \in \mathbb{N}$, such that

$$g^m_{T(\rho)}(TX, TX) \leq g^n_\rho(X, X)$$

holds for every Markov morphism $T : M_n \to M_m$, for every $\rho \in \mathcal{D}_n^1$ and for every $X \in T_\rho\mathcal{D}_n^1$. Usually monotone metrics are normalized in such a way that $[A, \rho] = 0$ implies $g_\rho(A, A) = \mathrm{Tr}(\rho^{-1}A^2)$. A monotone metric is also said a *quantum Fisher information* (QFI) because of Chentsov uniqueness theorem for commutative monotone metrics (see [2]).

Define $L_\rho(A) := \rho A$, and $R_\rho(A) := A\rho$, and observe that they are commuting self-adjoint (positive) superoperators on $M_{n,sa}$. For any $f \in \mathcal{F}_{op}$ one can define the positive superoperator $m_f(L_\rho, R_\rho)$. Now we can state the fundamental theorem about monotone metrics.

Theorem 2.4. [25]
There exists a bijective correspondence between monotone metrics (quantum Fisher informations) on $\mathcal{D}_n^1$ and normalized symmetric operator monotone functions $f \in \mathcal{F}_{op}$. This correspondence is given by the formula

$$\langle A, B\rangle_{\rho,f} := \mathrm{Tr}(A \cdot m_f(L_\rho, R_\rho)^{-1}(B)).$$

The metrics associated with the functions f_β are very important in information geometry and are related to Wigner-Yanase-Dyson information (see for example [7] [8] [9] [10] [5] and references therein).

Proposition 2.5. *(See [25] p. 89) Monotone metrics are unitarily covariant, namely if U is unitary then*

$$\langle U^*AU, U^*BU\rangle_{U^*\rho U,f} = \langle A, B\rangle_{\rho,f}.$$

3 The function $\tilde{f}$ and its properties

For $f \in \mathcal{F}_{op}$ define $f(0) := \lim_{x\to 0} f(x)$. The condition $f(0) \neq 0$ is relevant because it is a necessary and sufficient condition for the existence of the so-called radial extension of a monotone metric to pure states (see [26]). Following [13] we say that a function $f \in \mathcal{F}_{op}$ is *regular* iff $f(0) \neq 0$. The corresponding operator mean, associated QFI, etc. are said regular too.

Definition 3.1. *We introduce the sets*

$$\mathcal{F}^r_{op} := \{f \in \mathcal{F}_{op} | \quad f(0) \neq 0\}, \quad \mathcal{F}^n_{op} := \{f \in \mathcal{F}_{op} | \quad f(0) = 0\}.$$

Trivially one has $\mathcal{F}_{op} = \mathcal{F}^r_{op} \dot\cup \mathcal{F}^n_{op}$.

Proposition 3.1. [5] *For $f \in \mathcal{F}^r_{op}$ and $x > 0$ set*

$$\tilde{f}(x) := \frac{1}{2}\left[(x+1) - (x-1)^2\frac{f(0)}{f(x)}\right].$$

Then $\tilde{f} \in \mathcal{F}^n_{op}$.

By the very definition one has the following result.

Proposition 3.2. ([5], Proposition 5.7)
Let $f \in \mathcal{F}_{op}^r$. The following three conditions are equivalent:

(1) $\tilde{f} \leq \tilde{g}$;

(2) $m_{\tilde{f}} \leq m_{\tilde{g}}$;

(3) $\frac{f(0)}{f(t)} \geq \frac{g(0)}{g(t)} \qquad \forall t > 0.$

Let us give some more definitions.

Definition 3.2. *Suppose that $\rho \in \mathcal{D}_n^1$ is fixed. Define $X_0 := X - \mathrm{Tr}(\rho X)I$.*

Definition 3.3. *For $A, B \in M_{n,sa}$ and $\rho \in \mathcal{D}_n^1$ define covariance and variance as*

$$\mathrm{Cov}_\rho(A,B) := \frac{1}{2}[\mathrm{Tr}(\rho AB)+\mathrm{Tr}(\rho BA)]-\mathrm{Tr}(\rho A)\cdot\mathrm{Tr}(\rho B) = \frac{1}{2}[\mathrm{Tr}(\rho A_0B_0)+\mathrm{Tr}(\rho B_0A_0)] = \mathrm{Re}\{\mathrm{Tr}(\rho A_0 B_0)\}, \tag{3.1}$$

$$\mathrm{Var}_\rho(A) := \mathrm{Cov}_\rho(A,A) = \mathrm{Tr}(\rho A^2) - \mathrm{Tr}(\rho A)^2 = \mathrm{Tr}(\rho A_0^2).$$

Suppose, now, that $A, B \in M_{n,sa}$, $\rho \in \mathcal{D}_n^1$ and $f \in \mathcal{F}_{op}^r$. The fundamental theorem for our present purpose is given by Proposition 6.3 in [5], which is stated as follows.

Theorem 3.3.

$$\frac{f(0)}{2}\langle i[\rho,A], i[\rho,B]\rangle_{\rho,f} = \mathrm{Cov}_\rho(A,B) - \mathrm{Tr}(m_{\tilde{f}}(L_\rho,R_\rho)(A_0)B_0).$$

As a consequence of the spectral theorem and of Theorem 3.3 one has the following relations.

Proposition 3.4. [5] *Let $\{\varphi_h\}$ be a complete orthonormal base composed of eigenvectors of ρ, and $\{\lambda_h\}$ be the corresponding eigenvalues. To self-adjoint matrices A, B we associate matrices $a = a(\rho)$, $b = b(\rho)$ whose entries are given respectively by $a_{hj} \equiv \langle A_0\varphi_h|\varphi_j\rangle$, $b_{hj} \equiv \langle B_0\varphi_h|\varphi_j\rangle$. We have the following identities.*

$$\mathrm{Cov}_\rho(A,B) = \mathrm{Re}\{\mathrm{Tr}(\rho A_0B_0)\} = \frac{1}{2}\sum_{h,j}(\lambda_h+\lambda_j)\mathrm{Re}\{a_{hj}b_{jh}\}$$

$$\frac{f(0)}{2}\langle i[\rho,A], i[\rho,B]\rangle_{\rho,f} = \frac{1}{2}\sum_{h,j}(\lambda_h+\lambda_j)\mathrm{Re}\{a_{hj}b_{jh}\} - \sum_{h,j} m_{\tilde{f}}(\lambda_h,\lambda_j)\mathrm{Re}\{a_{hj}b_{jh}\}.$$

In what follows, capital letters will denote self-adjoint matrices and the corresponding lower-case letters will be used for the above transformation.

We also need the following result.

Proposition 3.5. ([5], Corollary 11.5)
On pure states

$$\mathrm{Tr}(m_{\tilde{f}}(L_\rho,R_\rho)(A_0)B_0) = 0.$$

4 The N-volume conjectures for quantum Fisher informations

Let $(V, g(\cdot,\cdot))$ be a real inner-product vector space. By $\langle u, v\rangle$ we denote the standard scalar product for vectors $u, v \in \mathbb{R}^N$.

Proposition 4.1. *Let $v_1, ..., v_N \in V$. The real $N \times N$ matrix $G := \{g(v_h, v_j)\}$ is positive semidefinite and therefore* $\det\{g(v_h,v_j)\} \geq 0$.

Proof. Let $x := (x_1, ..., x_N) \in \mathbb{R}^N$. We have

$$0 \leq g\big(\sum_j x_j v_j, \sum_j x_j v_j\big) = \sum_{h,j} x_h x_j g(v_h, v_j) = \langle x, G(x)\rangle.$$

□

Motivated by the case $(V, g(\cdot,\cdot)) = (\mathbb{R}^N, \langle\cdot,\cdot\rangle)$ one can give the following definition.

Definition 4.1.

$$\mathrm{Vol}^g(v_1, ..., v_N) := \sqrt{\det\{g(v_h, v_j)\}}.$$

Remark 4.1.

(i) Obviously, $\mathrm{Vol}^g(v_1, ..., v_N) \geq 0$, where the equality holds if and only if $v_1, ..., v_N \in V$ are linearly dependent.

(ii) If the inner product depends on a further parameter so that $g(\cdot,\cdot) = g_\rho(\cdot,\cdot)$, we write $\mathrm{Vol}^g_\rho(v_1, ..., v_N) = \mathrm{Vol}^g(v_1, ..., v_N)$.

(iii) In the case $(V, g_\rho(\cdot,\cdot)) = (\mathcal{L}^2_{\mathbb{R}}(\Omega, \mathcal{G}, \rho), \mathrm{Cov}_\rho(\cdot,\cdot))$ the number $\mathrm{Vol}^{\mathrm{Cov}}_\rho(A_1, ..., A_N)^2$ is also known as the *generalized variance* of the random vector $(A_1, ..., A_N)$.

In what follows we move to the noncommutative case. Here $A_1, ...A_N$ are self-adjoint matrices, ρ is a (faithful) density matrix and $g(\cdot,\cdot) = \mathrm{Cov}_\rho(\cdot,\cdot)$ has been defined in (3.1). By Vol^f_ρ we denote the volume associated to the quantum Fisher information $\langle\cdot,\cdot\rangle_{\rho,f}$ given by the (regular) normalized symmetric operator monotone function f.

Let $N \in \mathbb{N}$, $f \in \mathcal{F}^r_{op}$, $\rho \in \mathcal{D}^1_n$ and $A_1, ..., A_N \in M_{n,sa}$ be arbitrary. We conjecture the following results.

Conjecture 4.1.

$$\mathrm{Vol}^{\mathrm{Cov}}_\rho(A_1, ..., A_N) \geq \left(\frac{f(0)}{2}\right)^{\frac{N}{2}} \mathrm{Vol}^f_\rho(i[\rho, A_1], \ldots, i[\rho, A_N]). \tag{4.1}$$

Conjecture 4.2. *The above inequality is an equality if and only if $A_{10}, ..., A_{N0}$ are linearly dependent.*

Conjecture 4.3. *Fix $N \in \mathbb{N}$, $\rho \in \mathcal{D}^1_n$ and $A_1, ..., A_N \in M_{n,sa}$. Given $f \in \mathcal{F}^r_{op}$, define*

$$V(f) := \left(\frac{f(0)}{2}\right)^{\frac{N}{2}} \mathrm{Vol}^f_\rho(i[\rho, A_1], ..., i[\rho, A_N]).$$

Then, for any $f, g \in \mathcal{F}^r_{op}$

$$\tilde{f} \leq \tilde{g} \quad \Longrightarrow \quad V(f) \geq V(g).$$

Remark 4.2.

(i) Conjecture 4.1 is equivalent to the following inequality

$$\det\{\mathrm{Cov}_\rho(A_h, A_j)\} \geq \det\left\{\frac{f(0)}{2}\langle i[\rho, A_h], i[\rho, A_j]\rangle_{\rho,f}\right\}.$$

(ii) If ρ and $A_1, ..., A_N$ are fixed, set

$$F(f) := \det\{\mathrm{Cov}_\rho(A_h, A_j)\} - \det\left\{\frac{f(0)}{2}\langle i[\rho, A_h], i[\rho, A_j]\rangle_{\rho,f}\right\}.$$

Because of Theorem 3.3 one has

$$F(f) = \det\{\mathrm{Cov}_\rho(A_h, A_j)\} - \det\left\{\mathrm{Cov}_\rho(A_h, A_j) - \mathrm{Tr}(m_{\tilde{f}}(L_\rho, R_\rho)(A_{h0})A_{j0})\right\}.$$

Therefore, Conjecture 4.1 is equivalent to

$$F(f) \geq 0.$$

(iii) Conjecture 4.3 is equivalent to

$$\tilde{f} \leq \tilde{g} \qquad \Longrightarrow \quad F(f) \leq F(g).$$

(iv) Suppose that Conjecture 4.1 is true. One can prove the "if" part of Conjecture 4.2 in the following way. Since $(A_0)_0 = A_0$ one has

$$\mathrm{Cov}_\rho(A_1, A_2) = \mathrm{Re}\{\mathrm{Tr}(\rho A_{10} A_{20})\} = \mathrm{Cov}_\rho(A_{10}, A_{20}).$$

From this it follows

$$\mathrm{Vol}_\rho^{\mathrm{Cov}}(A_1, ..., A_N) = \mathrm{Vol}_\rho^{\mathrm{Cov}}(A_{10}, ..., A_{N0}).$$

Therefore, if $A_{10}, ..., A_{N0}$ are linearly dependent then

$$0 = \mathrm{Vol}_\rho^{\mathrm{Cov}}(A_{10}, ..., A_{N0}) = \mathrm{Vol}_\rho^{\mathrm{Cov}}(A_1, ..., A_N) \geq \left(\frac{f(0)}{2}\right)^{\frac{N}{2}} \mathrm{Vol}_\rho^f(i[\rho, A_1], ..., i[\rho, A_N]) \geq 0$$

and we are done.

(v) The inequality

$$\det\{\mathrm{Cov}_\rho(A_h, A_j)\} \geq \det\left\{\mathrm{Cov}_\rho(A_h, A_j) - \mathrm{Tr}(m_{\tilde{f}}(L_\rho, R_\rho)(A_{h0})A_{j0})\right\}$$

makes sense also for not faithful states.

Because of Proposition 3.5 one has (by an obvious extension of the definition) the following result.

Proposition 4.2. *If ρ is a pure state, then for any $N \in \mathbb{N}$, $\quad f \in \mathcal{F}_{op}^r$, $\quad A_1, ..., A_N \in M_{n,sa}$ one has*

$$\mathrm{Vol}_\rho^{\mathrm{Cov}}(A_1, ..., A_N) = \left(\frac{f(0)}{2}\right)^{\frac{N}{2}} \mathrm{Vol}_\rho^f(i[\rho, A_1], ..., i[\rho, A_N]).$$

In the following Section 5 and Section 6 we report on the known validity of the conjectures for $N = 1$ and $N = 2$.

5 The length inequality

In this section we discuss the case $N = 1$ of Conjectures 4.1, 4.2 and 4.3. The cases $f = f_{SLD}$ and $f = f_{WY}$ of Conjecture 4.1 were proved by Luo in [19] and [20]. The general case of Conjecture 4.1 was proved by Hansen in [13] and shortly after by Gibilisco, Imparato and Isola with a different technique in [5]. Conjectures 4.2 and 4.3 have been proved by Gibilisco, Imparato and Isola in [5] (see also [6]).

The proof of Conjecture 4.1 by Hansen is based on the following immediate proposition.

Proposition 5.1. *Let T, S be real functions on the state space coinciding on pure states. Suppose that T is convex and S is concave. Then for all states ρ*

$$T(\rho) \leq S(\rho).$$

It is well known that the variance is concave. Hansen was able to prove that the *metric adjusted skew information* (namely $\frac{f(0)}{2}\mathrm{Vol}_\rho^f(i[\rho, A_1])^2$) is convex and so he got the conclusion from the above Proposition. Note that the convexity of the function $\frac{f(0)}{2}\mathrm{Vol}_\rho^f(i[\rho, A_1])^2$ is related to the well known Lieb's concavity theorem (see [12][13]). Despite the elegance of the above proof its ideas do not apply to cases different from $N = 1$, as we shall see in the next section.

The techniques applied by ourselves in the proof of case $N = 1$ in the paper [5] do not seem to share the same fate. Moreover they allow one to prove also Conjectures 4.2 and 4.3. Let us discuss them.

Theorem 5.2. *Conjectures 4.1, 4.2 and 4.3 are true for $N = 1$ and for any $f \in \mathcal{F}_{op}^r$.*

Proof. Set $A_1 = A$. Using Proposition 3.4 and the notation in Remark 4.2 (ii) one gets:

$$\begin{aligned} F(f) &= \det\{\mathrm{Cov}_\rho(A,A)\} - \det\Big\{\mathrm{Cov}_\rho(A,A) - \mathrm{Tr}(m_{\tilde f}(L_\rho, R_\rho)(A_0)A_0)\Big\} \\ &= \mathrm{Cov}_\rho(A,A) - [\mathrm{Cov}_\rho(A,A) - \mathrm{Tr}(m_{\tilde f}(L_\rho, R_\rho)(A_0)A_0)] \\ &= \mathrm{Tr}(m_{\tilde f}(L_\rho, R_\rho)(A_0)A_0) \\ &= \sum_{i,j} m_{\tilde f}(\lambda_i, \lambda_j)\mathrm{Re}\{a_{ij}a_{ji}\} \\ &= \sum_{i,j} m_{\tilde f}(\lambda_i, \lambda_j)|a_{ij}|^2 \geq 0, \end{aligned}$$

that is, Conjecture 4.1 is true. Obviously $F(f) = 0$ iff $a_{ij} = 0$ $\forall i, j$, that is, iff $A_0 = 0$ and so we get Conjecture 4.2. Using Proposition 3.2 and Remark 4.2 (iii) one obtains also the validity of Conjecture 4.3. □

6 The area inequality

Let us discuss the case $N = 2$ of Conjecture 4.1. The result was proved true for $f = f_{WY}$ by Luo, Q. Zhang and Z. Zhang in [24] [22] [23]. The case $f = f_{WYD(\beta)}$, $\beta \in (0, \frac{1}{2})$ was proved by Kosaki in [17] and shortly after by Yanagi-Furuichi-Kuriyama in [34]. The general case is due to Gibilisco, Imparato and Isola (see [11] [5]).

Conjectures 4.2 and 4.3 were proved true by Kosaki for the particular case $f = f_{WYD(\beta)}$. The general case was solved by Gibilisco, Imparato and Isola (see [11] [5]).

First of all, let us show that the ideas used by Hansen in the case $N = 1$ do not apply to the case $N = 2$. The problem is the lack of concavity (and convexity) for the generalized variance. We were not able to find a counterexample in the literature, so we provide here the simplest we found.

Let $\Omega := \{1, 2, ..., n\}$. The space of (faithful) probability measures on Ω is

$$\mathcal{P}_n^1 := \Big\{\rho \in \mathbb{R}^n | \sum \rho_i = 1,\ \rho_i > 0\Big\}.$$

Let $X, Y \in \mathbb{R}^n$ be fixed random variables on Ω.

Proposition 6.1. *The function $S : \mathcal{P}_n^1 \to \mathbb{R}$ given by*

$$S(\rho) := \mathrm{Var}_\rho(X)\,\mathrm{Var}_\rho(Y) - \mathrm{Cov}_\rho(X,Y)^2$$

is neither a concave nor a convex function.

Proof. Let us compute the Hessian matrix $H^{XY}(\rho)$ of S at the point ρ:

$$\begin{aligned} H^{XY}_{ij}(\rho) &= \mathrm{Var}_\rho(Y)\tfrac{\partial^2}{\partial\rho_i\partial\rho_j}\mathrm{Var}_\rho(X) + \tfrac{\partial}{\partial\rho_i}\mathrm{Var}_\rho(X)\tfrac{\partial}{\partial\rho_j}\mathrm{Var}_\rho(Y) \\ &+ \mathrm{Var}_\rho(X)\tfrac{\partial^2}{\partial\rho_i\partial\rho_j}\mathrm{Var}_\rho(Y) + \tfrac{\partial}{\partial\rho_j}\mathrm{Var}_\rho(X)\tfrac{\partial}{\partial\rho_i}\mathrm{Var}_\rho(Y) \\ &- 2\tfrac{\partial}{\partial\rho_i}\mathrm{Cov}_\rho(X,Y)\tfrac{\partial}{\partial\rho_j}\mathrm{Cov}_\rho(X,Y) - 2\,\mathrm{Cov}_\rho(X,Y)\tfrac{\partial^2}{\partial\rho_i\partial\rho_j}\mathrm{Cov}_\rho(X,Y). \end{aligned}$$

If $X = (x_1, ..., x_n), Y = (y_1, ..., y_n)$, an explicit computation shows that

$$\begin{aligned} \tfrac{\partial}{\partial\rho_i}\mathrm{Cov}_\rho(X,Y) &= x_iy_i - x_i\mathbb{E}_\rho[Y] - y_i\mathbb{E}_\rho[X] \\ \tfrac{\partial^2}{\partial\rho_i\partial\rho_j}\mathrm{Cov}_\rho(X,Y) &= -x_iy_j - y_ix_j, \end{aligned}$$

so that

$$\begin{aligned} H_{ij}^{XY}(\rho) &= -2x_ix_j \operatorname{Var}_\rho(Y) + (x_i^2 - 2x_i\mathbb{E}_\rho[X])(y_j^2 - 2y_j\mathbb{E}_\rho[Y]) \\ &- 2y_iy_j \operatorname{Var}_\rho(X) + (y_i^2 - 2y_i\mathbb{E}_\rho[Y])(x_j^2 - 2x_j\mathbb{E}_\rho[X]) \\ &- 2(x_iy_i - x_i\mathbb{E}_\rho[Y] - y_i\mathbb{E}_\rho[X])(x_jy_j - x_j\mathbb{E}_\rho[Y] - y_j\mathbb{E}_\rho[X]) \\ &+ 2\operatorname{Cov}_\rho(X,Y)(x_iy_j + y_ix_j). \end{aligned} \tag{6.1}$$

In order to prove that in general $H_{ij}^{XY}(\rho)$ is neither negative semidefinite nor positive semidefinite (that is, $S(\rho)$ is neither concave nor convex) let $n = 3$ and $\rho = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ be the uniform distribution, $X = (1, 0, -1)^T$ and $Y = (1, -2, 1)^T$. Then $\mathbb{E}_\rho[X] = \mathbb{E}_\rho[Y] = \mathbb{E}_\rho[XY] = 0$ and $\operatorname{Cov}_\rho(X,Y) = 0$, so that (6.1) reduces to

$$H_{ij}^{XY}(\rho) = -2x_ix_j \operatorname{Var}_\rho(Y) + x_i^2y_j^2 - 2y_iy_j \operatorname{Var}_\rho(X) + x_j^2y_i^2 - 2x_ix_jy_iy_j.$$

Hence, given $\alpha \in \mathbb{R}^3$

$$\alpha^T H\alpha = -2\left[\operatorname{Var}_\rho(Y)\left(\sum_i x_i\alpha_i\right)^2 - \sum_i x_i^2\alpha_i \sum_i y_i^2\alpha_i + \operatorname{Var}_\rho(X)\left(\sum_i y_i\alpha_i\right)^2 + \left(\sum_i x_iy_i\alpha_i\right)^2\right].$$

In particular, $\alpha = \rho$ implies

$$\rho^T H\rho = 2\operatorname{Var}_\rho(X)\operatorname{Var}_\rho(Y) > 0,$$

while $\alpha = (0, \alpha_2, 0)$, $\alpha_2 \neq 0$, implies $\alpha^T H\alpha < 0$. □

Now we describe how the ideas for the proof of the length inequality ($N = 1$) can be modified to apply to the case of the area inequality ($N = 2$).

Definition 6.1. *For any $f \in \mathcal{F}_{op}^r$ set*

$$H^f(x,y,w,z) := \frac{1}{2}(x+y)m_{\tilde{f}}(w,z) + \frac{1}{2}(w+z)m_{\tilde{f}}(x,y) - m_{\tilde{f}}(x,y)m_{\tilde{f}}(w,z), \qquad x,y,w,z > 0.$$

Given $\rho \in \mathcal{D}_n^1$ and $\{\lambda_i\}$, $i = 1\ldots,n$, the corresponding eigenvalues, we set

$$H_{ijkl}^f := H^f(\lambda_i, \lambda_j, \lambda_k, \lambda_l).$$

Proposition 6.2. [5]
For any $f, g \in \mathcal{F}_{op}^r$ and for any $x, y, w, z > 0$ one has:

$$\tilde{f} \leq \tilde{g} \qquad \Longrightarrow \qquad 0 \leq H^f(x,y,w,z) \leq H^g(x,y,w,z).$$

Using the same notations as in Proposition 3.4, one can give the following definition.

Definition 6.2. *Set*

$$K_{i,j,k,l} := K_{i,j,k,l}(\rho, A, B) := |a_{ij}|^2|b_{kl}|^2 + |a_{kl}|^2|b_{ij}|^2 - 2\mathrm{Re}\{a_{ij}b_{ji}\}\mathrm{Re}\{a_{kl}b_{lk}\}.$$

Note that $K_{i,j,k,l}$ does not depend on f. Since

$$|a_{ij}|^2|b_{kl}|^2 + |a_{kl}|^2|b_{ij}|^2 \geq 2\,|a_{ij}b_{ji}|\,|a_{kl}b_{lk}| \geq 2\,|\mathrm{Re}\,\{a_{ij}b_{ji}\}\,\mathrm{Re}\,\{a_{kl}b_{lk}\}|\,,$$

we get that K_{ijkl} is non-negative. Moreover one has the following result.

Proposition 6.3. [5] $K_{i,j,k,l} = 0, \quad \forall i,j,k,l \iff A_0, B_0$ *are linearly dependent.*

Recall that

$$F(f) = \det\{\operatorname{Cov}_\rho(A_i, A_j)\} - \det\left\{\operatorname{Cov}_\rho(A_i, A_j) - \operatorname{Tr}(m_{\tilde{f}}(L_\rho, R_\rho)(A_{i0})A_{j0})\right\}.$$

Theorem 6.4. [5] *For $N = 2$ one has:*

$$F(f) = \frac{1}{2} \sum_{i,j,k,l} H^f_{i,j,k,l} \cdot K_{i,j,k,l}.$$

From the above Theorem one gets the following result.

Theorem 6.5. *Conjectures 4.1, 4.2 and 4.3 are true for $N = 2$.*

Proof. Since $H_{i,j,k,l} > 0$ and $K_{i,j,k,l} \geq 0$ we get $F(f) \geq 0$ and therefore Conjecture 4.1 is true.

From Proposition 6.3 we get that $F(f) = 0$ iff A_0, B_0 are linearly dependent, that is, Conjecture 4.2 holds.

From Proposition 6.2 we get that $\tilde{f} \leq \tilde{g}$ implies $F(f) \leq F(g)$ and, therefore, one proves Conjecture 4.3. □

Remark 6.1.

A decomposition of $F(f)$ similar to that of Theorem 6.4 seems to hold in the general case (N arbitrary). We plan to attack the conjectures with the aid of suitable generalized $H - K$ functions.

7 Covariance, correlation and entanglement

In the papers [21] [24] Luo *et al.* proved that the covariance is not a good measure to quantify entanglement properties of states; to this end, Wigner-Yanase correlation was proposed. Hereafter we recall the more general definition of *metric adjusted correlation* (or *f-correlation*) introduced in [13] [5]; Wigner-Yanase correlation is just a particular example of metric adjusted correlation. We prove that the metric adjusted correlation has the same basic properties of Wigner-Yanase correlation. In particular we show, by the same example as in [24], that the general f-correlation behaves as the Wigner-Yanase correlation with respect to entanglement. Note that we consider a symmetrized version of the f-correlation.

Definition 7.1. *For $A, B \in M_{n,sa}$, $\rho \in \mathcal{D}^1_n$ and $f \in \mathcal{F}_{op}$, the metric adjusted correlation (or f-correlation) is defined as*

$$\mathrm{Corr}^f_\rho(A,B) := \frac{f(0)}{2} \langle i[\rho, A], i[\rho, B] \rangle_{\rho,f} = \mathrm{Tr}(\rho AB) - \mathrm{Tr}(m_{\tilde{f}}(L_\rho, R_\rho)(A) \cdot B).$$

Proposition 7.1. [5]

$$\begin{aligned} \mathrm{Corr}^f_\rho(A,B) &= \mathrm{Cov}_\rho(A,B) - \mathrm{Tr}(m_{\tilde{f}}(L_\rho, R_\rho)(A_0)B_0) \\ &= \mathrm{Cov}_\rho(A,B) - \textstyle\sum_{i,j} m_{\tilde{f}}(\lambda_i, \lambda_j) a_{ij} b_{ji}. \end{aligned}$$

Note that there is no relation between covariance and correlation as underlined by the following Proposition (see Examples 3,4 in [24]).

Proposition 7.2. *The inequalities*

$$\begin{aligned} \mathrm{Corr}^f_\rho(A,B) &> \mathrm{Cov}_\rho(A,B), \\ \mathrm{Corr}^f_\rho(A,B) &< \mathrm{Cov}_\rho(A,B), \end{aligned}$$

are false in general.

Proof. The expression

$$\sum_{i,j} m_{\tilde{f}}(\lambda_i, \lambda_j) a_{ij} b_{ji}$$

can have arbitrary sign (depending on A and B); therefore, from from Proposition 7.1 one gets the conclusion. □

Proposition 7.3. [5] *If ρ is pure, then*

$$\mathrm{Corr}^f_\rho(A,B) = \mathrm{Cov}_\rho(A,B) \qquad \forall f \in \mathcal{F}_{op}.$$

Proposition 7.4. *If a, b are real constants and U is unitary we get:*
i) $\mathrm{Corr}^f_\rho(A - aI, B - bI) = \mathrm{Corr}^f_\rho(A,B)$*;*
ii) $\mathrm{Corr}^f_\rho(aA, bB) = ab\,\mathrm{Corr}^f_\rho(A,B)$*;*
iii) $\mathrm{Corr}^f_{U\rho U^{-1}}(A,B) = \mathrm{Corr}^f_\rho(U^{-1}AU, U^{-1}BU)$*.*

Proof. i) and ii) follow easily from the definition and iii) is a direct consequence of the unitarily covariance of quantum Fisher information, namely of Proposition 2.5. □

Consider, now,

$$\rho := \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \rho' := \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

Note that the first state is a mixture of two disentangled states while the second is a Bell state which is maximally entangled (see [24] [21]).

Set

$$A := \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad B := \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}.$$

In [24] [21] it is shown that

$$\mathrm{Cov}_\rho(A,B) = \mathrm{Cov}_{\rho'}(A,B) = 1, \tag{7.1}$$

while

$$\mathrm{Corr}^{f_{WY}}_\rho(A,B) = 0, \qquad \mathrm{Corr}^{f_{WY}}_{\rho'}(A,B) = 1.$$

Due to Proposition 7.1, this result holds more generally for any f-correlation.

Proposition 7.5. *For any $f \in \mathcal{F}_{op}$ one gets*

$$\mathrm{Corr}^f_\rho(A,B) = 0, \qquad \mathrm{Corr}^f_{\rho'}(A,B) = 1.$$

Proof. Since ρ' is a pure state, from Proposition 7.3 and due to (7.1) one has that $\mathrm{Corr}^f_{\rho'}(A,B) = \mathrm{Cov}_{\rho'}(A,B) = 1$.

Consider, now, the state ρ and let $\{e_1, e_2, e_3, e_4\}$ be the canonical basis. A direct computation shows that its eigenvalues are $\lambda_1 = \lambda_4 = \frac{1}{2}$ and $\lambda_2 = \lambda_3 = 0$, and the corresponding eigenvectors are $\{e_1, e_4\}$ and $\{e_2, e_3\}$, respectively. Observe that A and B are centered with respect to both the states ρ and ρ' (namely $A = A_0$, $B = B_0$). Moreover, since the eigenvectors are the canonical basis one gets $A_{ij} = a_{ij}$ and $B_{ij} = b_{ij}$.

This implies that

$$\begin{aligned} \sum_{i,j=1}^4 m_{\tilde{f}}(\lambda_i, \lambda_j) a_{ij} b_{ji} &= \sum_{i=1}^4 m_{\tilde{f}}(\lambda_i, \lambda_i) A_{ii} B_{ii} \\ &= \sum_{i=1}^4 \lambda_i A_{ii} B_{ii} = 1, \end{aligned}$$

where we used the mean property that $m_f(x,x) = x$, for any non negative x. Again from (7.1), one obtains $\mathrm{Corr}^f_\rho(A,B) = 0$. □

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