Uncertainty Principle and Quantum Fisher Information - II

Paolo Gibilisco[*], Daniele Imparato[†] and Tommaso Isola[‡]

May 21, 2007

Abstract

Heisenberg and Schrödinger uncertainty principles give lower bounds for the product of variances $\mathrm{Var}_\rho(A) \cdot \mathrm{Var}_\rho(B)$ if the observables A, B are not compatible, namely if the commutator $[A, B]$ is not zero.

In this paper we prove an uncertainty principle in Schrödinger form where the bound for the product of variances $\mathrm{Var}_\rho(A) \cdot \mathrm{Var}_\rho(B)$ depends on the area spanned by the commutators $i[\rho, A]$ and $i[\rho, B]$ with respect to an arbitrary quantum version of the Fisher information.

2000 *Mathematics Subject Classification.* Primary 62B10, 94A17; Secondary 46L30, 46L60.

Key words and phrases. Uncertainty principle, means, monotone metrics, quantum Fisher information, Wigner-Yanase-Dyson information.

1 Introduction

Let X, Y be random variables on a probability space $(\Omega, \mathcal{G}, p)$ and consider the covariance $\mathrm{Cov}_p(X, Y) := \mathrm{E}_p(XY) - \mathrm{E}_p(X)\mathrm{E}_p(Y)$ and the variance $\mathrm{Var}_p(X) := \mathrm{Cov}_p(X, X)$. The best one can get from Cauchy-Schwartz inequality is the following inequality

$$\mathrm{Var}_p(X) \cdot \mathrm{Var}_p(Y) - |\mathrm{Cov}_p(X, Y)|^2 \geq 0. \tag{1.1}$$

The uncertainty principle is one of the most striking consequences of non-commutativity in Quantum Mechanics and is a key point in which quantum probability differs from classical probability. We shall

[*]Dipartimento SEFEMEQ, Facoltà di Economia, Università di Roma "Tor Vergata", Via Columbia 2, 00133 Rome, Italy. Email: gibilisco@volterra.uniroma2.it – URL: http://www.economia.uniroma2.it/sefemeq/professori/gibilisco

[†]Dipartimento di Matematica, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Turin, Italy. Email: daniele.imparato@polito.it

[‡]Dipartimento di Matematica, Università di Roma "Tor Vergata", Via della Ricerca Scientifica, 00133 Rome, Italy. Email: isola@mat.uniroma2.it URL: http://www.mat.uniroma2.it/∼isola

limit our discussion to the matrix case. If ρ is a state (i.e. density matrix), A, B observables (i.e. self-adjoint matrices), set $\mathrm{Cov}_\rho(A,B) := \mathrm{Tr}(\rho AB) - \mathrm{Tr}(\rho A)\cdot\mathrm{Tr}(\rho B)$. Define also the symmetrized covariance as $\mathrm{Cov}^s_\rho(A,B) := \frac{1}{2}[\mathrm{Cov}_\rho(A,B) + \mathrm{Cov}_\rho(B,A)]$ and the variance as $\mathrm{Var}(A) := \mathrm{Cov}_\rho(A,A) = \mathrm{Cov}^s_\rho(A,A)$. Again from Cauchy-Schwartz inequality one gets the following inequality

$$\mathrm{Var}_\rho(A)\cdot\mathrm{Var}_\rho(B) - |\mathrm{Cov}^s_\rho(A,B)|^2 \geq \frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2, \tag{1.2}$$

which is known as the Schrödinger uncertainty principle. By omitting the covariance part, one gets the Heisenberg uncertainty principle (see [12][27]). Inequality (1.2) states that the condition $[A,B] \neq 0$ (*i.e.* A, B are not compatible) gives a limitation to the simultaneous "smallness" of both $\mathrm{Var}_\rho(A)$ and $\mathrm{Var}_\rho(B)$ and this has very important consequences in Quantum Mechanics.

But non-commutativity can enter also from another side. One may naturally ask if there are similar bounds for the product $\mathrm{Var}_\rho(A)\cdot\mathrm{Var}_\rho(B)$ due to the fact that the observables A, B do not commute with the state ρ. Indeed this is the case, and our main result will provide such a bound in terms of an "area" spanned by the commutators $i[\rho, A]$ and $i[\rho, B]$.

To state our result we have to introduce the notion of quantum Fisher information. Let us denote by $\mathcal{F}_{op}$ the class of normalized symmetric operator monotone functions on $(0, +\infty)$. It is a by now classical result of Petz that to each function $f \in \mathcal{F}_{op}$ one can associate a Riemannian metric $\langle A, B\rangle_{\rho,f}$ on the state manifold that is monotone and therefore a quantum version of the Fisher information (see [2] [23]). For example, the functions

$$f_\beta(x) := \beta(1-\beta)\frac{(x-1)^2}{(x^\beta - 1)(x^{1-\beta} - 1)} \qquad x > 0, \quad \beta \in (0, 1/2],$$

are associated to a quantum Fisher information that is related to the well known Wigner-Yanase-Dyson (WYD) skew information. Indeed, for the WYD-information of parameter β one has

$$-\frac{1}{2}\mathrm{Tr}([\rho^\beta, A]\cdot[\rho^{1-\beta}, A]) = \frac{\beta(1-\beta)}{2}\langle i[\rho,A], i[\rho,A]\rangle_{\rho,f_\beta}.$$

We denote by $\mathrm{Area}^f_\rho(u,v)$ the area spanned by the tangent vectors u, v with respect to the Riemannian monotone metric associated to f (at the point ρ). For $f \in \mathcal{F}_{op}$ let us define $f(0) := \lim_{x\to 0} f(x)$; f is said regular if $f(0) > 0$ (see [9]). The subset of regular elements of $\mathcal{F}_{op}$ is denoted by $\mathcal{F}^r_{op}$.

The goal of the present paper is to prove the following inequality

$$\mathrm{Var}_\rho(A)\cdot\mathrm{Var}_\rho(B) - |\mathrm{Cov}^s_\rho(A,B)|^2 \geq \frac{1}{4}\left(f(0)\cdot\mathrm{Area}^f_\rho(i[\rho,A], i[\rho,B])\right)^2 \qquad \forall f \in \mathcal{F}^r_{op}. \tag{1.3}$$

Note that inequality (1.3) holds trivially in the non-regular case. Our result has been inspired by particular cases of the above theorem that have been proved recently. Luo and Z. Zhang [21] conjectured the inequality (1.3) for the Wigner-Yanase metric, namely for the function $f_{1/2}$. This conjecture was proved shortly after by Luo himself and Q. Zhang [19]. The case of Wigner-Yanase-Dyson metric (namely the metric associated to f_β for $\beta \in (0,1/2)$) was proved independently by Kosaki [13] and by Yanagi *et al.* [28]. In our paper [7] we emphasized the geometric aspects of the question and we succeeded to formulate (1.3) for a general quantum Fisher information.

It is worth to emphasize the dynamical meaning of the inequality (1.3). Indeed, each positive (self-adjoint) operator H determines a time evolution of a state ρ according to the formula $\rho_H(t) := e^{-iHt}\rho e^{iHt}$. If $[\rho, H] = 0$, then there is no evolution. Therefore, we may say that the bound given by (1.3) appears for those pairs of observables that are "dynamically incompatible", that is, for pairs H, K such that the associated evolutions $\rho_H(t), \rho_K(t)$ are different and non-trivial (this is equivalent to the linear independence of $[\rho, H]$ and $[\rho, K]$).

As a by-product of the work needed to prove our main result, we derive also other two inequalities interesting *per se*. Indeed, a crucial ingredient in the proof of (1.3) is the following formula

$$\tilde{f}(x) := \frac{1}{2}\left[(x+1) - (x-1)^2\frac{f(0)}{f(x)}\right],$$

that associates to any element $f \in \mathcal{F}_{op}$ another element $\tilde{f} \in \mathcal{F}_{op}$. Let $A_0 := A - \mathrm{Tr}(\rho A)I$ and denote by L_ρ, R_ρ respectively the left and right multiplication by ρ. Let m_f be the mean associated to f (see Section 6 below) and define

$$\mathcal{C}_\rho^f(A_0) := \mathrm{Tr}(m_f(L_\rho, R_\rho)(A_0)\cdot A_0), \qquad\qquad I_\rho^f(A) := \mathrm{Var}_\rho(A) - \mathcal{C}_\rho^{\tilde{f}}(A_0).$$

Hansen introduced $I_\rho^f(A)$ in the paper [9] with a different approach. We shall call it "metric adjusted skew information" or "f-information" to stress the dependence on the function f. One may consider $I_\rho^f(A)$ as a generalization of Wigner-Yanase-Dyson information. Let $f_{RLD}(x) = \frac{2x}{x+1}$; we prove the following inequality

$$\mathrm{Var}_\rho(A)\cdot \mathrm{Var}_\rho(B) \geq [I_\rho^f(A) + \mathcal{C}_\rho^{f_{RLD}}(A_0)]\cdot[I_\rho^f(B) + \mathcal{C}_\rho^{f_{RLD}}(B_0)] \qquad \forall f \in \mathcal{F}_{op}^r. \tag{1.4}$$

Moreover, we also prove that, if $f \in \mathcal{F}_{op}$,

$$\mathrm{Var}_\rho(A)\cdot \mathrm{Var}_\rho(B) \geq \mathcal{C}_\rho^f(A_0)\mathcal{C}_\rho^f(B_0) + \frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2 \quad \Longleftrightarrow \quad f(x) \leq \sqrt{x}. \tag{1.5}$$

Inequality (1.4) is a refinement of an inequality proved by Luo, for the Wigner-Yanase metric, and by Hansen, in the general case (see [16], [9]). Inequality (1.5) for the function $\sqrt{x}$ is due to Park and, independently, to Luo (see [22] [18]). Here we simply prove the optimality of their bound in $\mathcal{F}_{op}$.

The plan of the paper is the following.

Sections 2, 3, 4 contain preliminary notions. In Section 2 we recall the standard Heisenberg and Schrödinger uncertainty principles. In Section 3 we give the fundamental definitions and theorems for number and operator means. In Section 4 we review the classification theorem for the quantum Fisher informations.

Sections 5, 6, 7, 8 contain the core of the paper. In Section 5 we show that to any operator monotone function $f \in \mathcal{F}_{op}$ one may associate another element $\tilde{f} \in \mathcal{F}_{op}$ by formula (5.1); we study the properties of the mean $m_{\tilde{f}}$ and of an associated function H_f, discussing, in particular, how they behave as functions of f. In Section 6 we prove the main result, namely the inequality (1.3). Furthermore, we study the right side of the above inequality as a function of f and relate it to quantum evolution of states, as said before. In Section 7 we introduce the f-correlation (a kind of generalized Wigner-Yanase-Dyson correlation) and the f-information and discuss their relation with the quantum Fisher information associated to $f \in \mathcal{F}_{op}$. In this way, we are able to show how the inequality (1.3) generalizes the previously known results. Moreover, we prove that by choosing the SLD (Bures-Uhlmann) metric the lower bound given in (1.3) is optimal and strictly greater than the previously known optimal bound (given by the Wigner-Yanase metric). In Section 8 we prove a necessary and sufficient condition to get the equality in (1.3).

In Section 9 we prove the inequality (1.4). In Section 10 we produce counterexamples to prove the logical independence of the uncertainty principles studied in this paper - that is, inequalities (1.3) and (1.4) - from the standard Heisenberg-Schrödinger uncertainty principles. In Section 11 we discuss what happens for not faithful and pure states, also at the light of the notion of radial extension for quantum Fisher information. In Section 12 we show the optimality of an improvement of Heisenberg uncertainty principle recently proposed by Park and Luo, namely we prove the inequality (1.5).

2 Heisenberg and Schrödinger Uncertainty Principles

Let $M_n := M_n(\mathbb{C})$ (resp. $M_{n,sa} := M_n(\mathbb{C})_{sa}$) be the set of all $n \times n$ complex matrices (resp. all $n \times n$ self-adjoint matrices). We shall denote general matrices by $X, Y, ...$ while letters $A, B, ...$ will be used for self-adjoint matrices. The Hilbert-Schmidt scalar product is denoted by $\langle A, B \rangle = \mathrm{Tr}(A^* B)$. The adjoint of a matrix X is denoted by $X^\dagger$ while the adjoint of a superoperator $T : (M_n, \langle \cdot, \cdot \rangle) \to (M_n, \langle \cdot, \cdot \rangle)$ is denoted by T^*. Let $\mathcal{D}_n$ be the set of strictly positive elements of M_n and $\mathcal{D}_n^1 \subset \mathcal{D}_n$ be the set of strictly positive density matrices; namely,

$$\mathcal{D}_n^1 = \{\rho \in M_n | \mathrm{Tr}\rho = 1, \, \rho > 0\}.$$

From now on, we shall treat the case of faithful states, namely $\rho > 0$. We shall consider the general case $\rho \geq 0$ at the end of the paper, in Section 11, where we shall also discuss in detail what happens for pure states.

Definition 2.1. Suppose that $\rho \in \mathcal{D}_n^1$ is fixed. Define $X_0 := X - \mathrm{Tr}(\rho X)I$.

Definition 2.2. For $A, B \in M_{n,sa}$ and $\rho \in \mathcal{D}_n^1$ define covariance and variance as

$$\begin{aligned} \mathrm{Cov}_\rho(A,B) &:= \mathrm{Tr}(\rho AB) - \mathrm{Tr}(\rho A)\cdot\mathrm{Tr}(\rho B) = \mathrm{Tr}(\rho A_0 B_0), \\ \mathrm{Var}_\rho(A) &:= \mathrm{Tr}(\rho A^2) - \mathrm{Tr}(\rho A)^2 = \mathrm{Tr}(\rho A_0^2). \end{aligned}$$

Proposition 2.3.

$$\begin{aligned} 2\mathrm{Re}\{\mathrm{Cov}_\rho(A,B)\} &= \mathrm{Cov}_\rho(A,B) + \mathrm{Cov}_\rho(B,A) = \mathrm{Tr}(\rho\{A_0, B_0\}), \\ 2i\mathrm{Im}\{\mathrm{Cov}_\rho(A,B)\} &= \mathrm{Cov}_\rho(A,B) - \mathrm{Cov}_\rho(B,A) = \mathrm{Tr}(\rho[A,B]), \end{aligned}$$

where, for any $X, Y \in M_n$, $[X,Y] := XY - YX$, $\{X,Y\} := XY + YX$.

We define the symmetrized covariance as $\mathrm{Cov}_\rho^s(A,B) := \frac{1}{2}[\mathrm{Cov}_\rho(A,B) + \mathrm{Cov}_\rho(B,A)] = \mathrm{Re}\{\mathrm{Cov}_\rho(A,B)\}$. The Cauchy-Schwartz inequality implies

$$|\mathrm{Cov}_\rho(A,B)|^2 \leq \mathrm{Var}_\rho(A)\mathrm{Var}_\rho(B).$$

From this one gets the Schrödinger and Heisenberg uncertainty principles which are stated in the following theorem.

Theorem 2.4. *(see [27])*

For $A, B \in M_{n,sa}$ and $\rho \in \mathcal{D}_n^1$ one has

$$\mathrm{Var}_\rho(A)\mathrm{Var}_\rho(B) - |\mathrm{Cov}_\rho^s(A,B)|^2 \geq \frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2,$$

that implies

$$\mathrm{Var}_\rho(A)\mathrm{Var}_\rho(B) \geq \frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2.$$

3 Means for positive numbers and matrices

For this Section we refer to the exposition contained in [26].

Definition 3.1. Let $\mathbb{R}^+ := (0, +\infty)$. A *mean* for pairs of positive numbers is a function $m : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that

(i) $m(x,x) = x$,

(ii) $m(x,y) = m(y,x)$,

(iii) $x < y \quad \Longrightarrow \quad x < m(x,y) < y$,

(iv) $x < x', \quad y < y' \quad \Longrightarrow \quad m(x,y) < m(x',y')$,

(v) m is continuous,

(vi) for $t > 0$ one has $m(tx,ty) = t \cdot m(x,y)$.

We denote by $\mathcal{M}_{nu}$ the set of means.

Definition 3.2. $\mathcal{F}_{nu}$ is the class of functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

(i) $f(1) = 1$,

(ii) $tf(t^{-1}) = f(t)$,

(iii) $t \in (0,1) \Longrightarrow f(t) \in (0,1)$,

(iv) $t \in (1,\infty) \Longrightarrow f(t) \in (1,\infty)$,

(v) f is continuous,

(vi) f is monotone increasing.

Proposition 3.3. *There is bijection between* $\mathcal{M}_{nu}$ *and* $\mathcal{F}_{nu}$ *given by the formulas*

$$m_f(x,y) := yf(xy^{-1}), \qquad\qquad f_m(t) := m(1,t).$$

Remark 3.4.

$$f \leq g \Longleftrightarrow m_f \leq m_g.$$

Here below we report the Kubo-Ando theory of matrix means (see [14]) as exposed in [26]. In the sequel, for any pairs of matrices A, B, we shall write $A < B$ whenever $B - A$ is positive semidefinite.

Definition 3.5. Recall that $\mathcal{D}_n := \{A \in M_n(\mathbb{C}) | A > 0\}$. A *mean* for pairs of positive matrices is a function $m(\cdot,\cdot) : \mathcal{D}_n \times \mathcal{D}_n \to \mathcal{D}_n$ such that conditions $(i) - -(v)$ of Definition 3.1 hold (with the matrix partial order defined above) and the *transformer inequality*

$Cm(A,B)C^* \leq m(CAC^*, CBC^*)$, $\forall C$,

replaces (vi). We denote by $\mathcal{M}_{op}$ the set of matrix means.

Example 3.6. The arithmetic, geometric and harmonic (matrix) means are given respectively by

$$\begin{aligned} A\nabla B &:= \tfrac{1}{2}(A+B), \\ A\#B &:= A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}, \\ A!B &:= 2(A^{-1}+B^{-1})^{-1}. \end{aligned}$$

Let us recall that a function $f : (0, \infty) \to \mathbb{R}$ is said *operator monotone* if, for any $n \in \mathbb{N}$, any A, $B \in M_n$ such that $0 \leq A \leq B$, the inequalities $0 \leq f(A) \leq f(B)$ hold. An operator monotone function is said *symmetric* if $f(x) = xf(x^{-1})$ and *normalized* if $f(1) = 1$.

Definition 3.7. $\mathcal{F}_{op}$ is the class of operator monotone functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that conditions $(i) - -(v)$ of Definition 3.2 hold (with the matrix partial order defined above).

Note that the above definition is redundant (see for example [1]); however, it well emphasizes the similarity with the number case. Indeed, one has the following result.

Proposition 3.8. *$\mathcal{F}_{op}$ is the class of functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

(i') $f(1) = 1$,

(ii') $tf(t^{-1}) = f(t)$,

(iii') *f is operator monotone increasing.*

Equivalently, $f \in \mathcal{F}_{op}$ iff f is a normalized, symmetric, operator monotone function.

The fundamental result, due to Kubo and Ando, is the following.

Theorem 3.9. *There is bijection between $\mathcal{M}_{op}$ and $\mathcal{F}_{op}$ given by the formula*

$$m_f(A, B) := A^{\frac{1}{2}} f(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}.$$

When A and B commute, we have that

$$m_f(A, B) := A \cdot f(BA^{-1}).$$

Theorem 3.10. *Among matrix means, arithmetic is the largest while harmonic is the smallest.*

Proof. See Theorem 4.5 in [14]. □

Corollary 3.11. *For any $f \in \mathcal{F}_{op}$ and for any $x, y > 0$ one has*

$$\frac{2x}{1+x} \leq f(x) \leq \frac{1+x}{2},$$

$$\frac{2xy}{x+y} \leq m_f(x,y) \leq \frac{x+y}{2}.$$

4 Quantum Fisher Informations

In what follows, given a differential manifold $\mathcal{N}$, we denote by $T_\rho \mathcal{N}$ the tangent space to $\mathcal{N}$ at the point $\rho \in \mathcal{N}$. In the commutative case a Markov morphism is a stochastic map $T : \mathbb{R}^n \to \mathbb{R}^m$. Let

$$\mathcal{P}_n := \{\rho \in \mathbb{R}^n | \rho_i > 0\}, \qquad \mathcal{P}_n^1 := \{\rho \in \mathcal{P}_n | \sum \rho_i = 1\}.$$

The natural representation for the tangent space is given by

$$T_\rho \mathcal{P}_n^1 = \{v \in \mathbb{R}^n | \sum_i v_i = 0\}.$$

In this case a monotone metric is defined as a family of Riemannian metrics $g = \{g^n\}$ on $\{\mathcal{P}_n^1\}$, $n \in \mathbb{N}$, such that

$$g^m_{T(\rho)}(TX, TX) \leq g^n_\rho(X, X)$$

holds for every Markov morphism $T : \mathbb{R}^n \to \mathbb{R}^m$, for every $\rho \in \mathcal{P}_n^1$ and for every $X \in T_\rho \mathcal{P}_n^1$.

The Fisher information is the Riemannian metric on $\mathcal{P}_n^1$ defined as

$$\langle u, v \rangle_{\rho,F} := \sum_i \frac{u_i v_i}{\rho_i} \qquad\qquad u, v \in T_\rho \mathcal{P}_n^1.$$

Theorem 4.1. *(see [2])*

There exists a unique monotone metric on $\mathcal{P}_n^1$ (up to scalars) given by the Fisher information.

In the noncommutative case a Markov morphism is a completely positive and trace preserving operator $T : M_n \to M_m$. Recall that there exists a natural identification of $T_\rho \mathcal{D}_n^1$ with the space of self-adjoint traceless matrices, namely for any $\rho \in \mathcal{D}_n^1$

$$T_\rho \mathcal{D}_n^1 = \{A \in M_{n,sa} | \mathrm{Tr}(A) = 0\}.$$

In perfect analogy with the commutative case, a monotone metric in the noncommutative case is a family of Riemannian metrics $g = \{g^n\}$ on $\{\mathcal{D}_n^1\}$, $n \in \mathbb{N}$, such that

$$g^m_{T(\rho)}(TX, TX) \leq g^n_\rho(X, X)$$

holds for every Markov morphism $T : M_n \to M_m$, for every $\rho \in \mathcal{D}_n^1$ and for every $X \in T_\rho \mathcal{D}_n^1$. Monotone metrics are usually normalized in such a way that $[A, \rho] = 0$ implies $g_{f,\rho}(A, A) = \mathrm{Tr}(\rho^{-1} A^2)$.

To a normalized symmetric operator monotone function $f \in \mathcal{F}_{op}$ one associates the so-called CM (Chentsov–Morozowa) function

$$c_f(x, y) := \frac{1}{y f(x y^{-1})} = m_f(x, y)^{-1} \qquad \text{for} \qquad x, y > 0.$$

Define $L_\rho(A) := \rho A$, and $R_\rho(A) := A\rho$; observe that they are self-adjoint operators on $M_{n,sa}$. Since L_ρ and R_ρ commute we may define $c_f(L_\rho, R_\rho) = m_f(L_\rho, R_\rho)^{-1}$. Since m_f is a matrix mean one gets the following result.

Proposition 4.2. *(see [23])*

$m_f(L_\rho, R_\rho)$ and $c_f(L_\rho, R_\rho)$ are positive and therefore self-adjoint.

Now we can state the fundamental theorems about noncommutative monotone metrics.

Theorem 4.3. *(see [23])*

There exists a bijective correspondence between monotone metrics on $\mathcal{D}_n^1$ and normalized symmetric operator monotone functions $f \in \mathcal{F}_{op}$. This correspondence is given by the formula

$$\langle A, B \rangle_{\rho,f} := \mathrm{Tr}(A \cdot c_f(L_\rho, R_\rho)(B)) = \mathrm{Tr}(A \cdot m_f(L_\rho, R_\rho)^{-1}(B)).$$

We set $||A||^2_{\rho,f} := \langle A, A \rangle_{\rho,f}$. Because of the above theorems we shall use the terms "Monotone Metrics" and "Quantum Fisher Informations" (shortly QFI) with the same meaning.

For a symmetric operator monotone function define $f(0) := \lim_{x \to 0} f(x) = \lim_{x \to +\infty} \frac{f(x)}{x}$. Of course, $f(0) \geq 0$. The condition $f(0) \neq 0$ is relevant because it is a necessary and sufficient condition for the existence of the so-called radial extension of a monotone metric to pure states (see [24][25] or Section 11 below). Following [9] we say that a function $f \in \mathcal{F}_{op}$ is *regular* iff $f(0) \neq 0$. The corresponding operator mean, CM function, associated QFI, etc. are said regular too. The class of regular (resp. non-regular) functions $f \in \mathcal{F}_{op}$ is denoted by $\mathcal{F}_{op}^r$ (resp. $\mathcal{F}_{op}^n$).

As proved by Lesniewski and Ruskai each quantum Fisher information is the Hessian of a suitable relative entropy (see [15]).

5 The function $\tilde{f}$ and the properties of the associated mean

In [9] it has been proved the following result.

Proposition 5.1. *(Proposition 3.4 in [9])*

If $f \in \mathcal{F}_{op}$ is regular, define the representing function as

$$d_f(x,y) := \frac{(x+y)}{f(0)} - (x-y)^2 c_f(x,y), \qquad x,y>0.$$

Then, the function d_f is positive and operator concave.

Definition 5.2. For $f \in \mathcal{F}_{op}$ and $x > 0$ set

$$\tilde{f}(x) := \frac{1}{2}\left[(x+1) - (x-1)^2 \frac{f(0)}{f(x)}\right]. \tag{5.1}$$

Proposition 5.3.

$$f \in \mathcal{F}_{op} \qquad \Longrightarrow \qquad \tilde{f} \in \mathcal{F}_{op}.$$

Proof. Easy calculations show that $\tilde{f}$ is normalized and symmetric. To prove that f is operator monotone note that:

(a) if f is not regular then $\tilde{f}(x) = \frac{1}{2}(1+x)$ and the conclusion follows;

(b) if f is regular then $\tilde{f}(x) = \frac{f(0)}{2} d(x,1)$. Since d is positive and operator concave so is $\tilde{f}$. We get the conclusion because operator concavity is equivalent to operator monotonicity (see [10]). □

Remark 5.4. Note that f regular $\Longrightarrow$ $\tilde{f}$ not regular.

Following the terminology of Section 3 we associate to $\tilde{f}$ both a number and an operator mean by the formulas

$$\begin{aligned} m_{\tilde{f}}(x,y) &:= y \cdot \tilde{f}(xy^{-1}), \\ m_{\tilde{f}}(A,B) &:= A^{\frac{1}{2}} \tilde{f}(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}. \end{aligned}$$

Remark 5.5. Observe that $m_{\tilde{f}}(x,y) = \dfrac{x+y}{2} - \dfrac{f(0)}{2} \dfrac{(x-y)^2}{y f(\frac{x}{y})}$.

From Corollary 3.11 one obtains this result.

Corollary 5.6. *For any $f \in \mathcal{F}_{op}$ and for any $x,y > 0$ one has*

$$\frac{2x}{1+x} \le \tilde{f}(x) \le \frac{1+x}{2},$$

$$\frac{2xy}{x+y} \le m_{\tilde{f}}(x,y) \le \frac{x+y}{2}.$$

Moreover we have the following result, whose proof is elementary.

Proposition 5.7. *For every $x > 0$ and $f, g \in \mathcal{F}_{op}$*

$$\tilde{f}(x) \le \tilde{g}(x) \quad \Longleftrightarrow \quad \frac{f(0)}{f(x)} \ge \frac{g(0)}{g(x)}.$$

We synthetize some results in the following Table.

TABLE I

QFI	f	m_f	$f(0)$	$\tilde{f}$	$m_{\tilde{f}}$
RLD	$\frac{2x}{x+1}$	$\frac{2}{\frac{1}{x}+\frac{1}{y}}$	0	$\frac{1+x}{2}$	$\frac{x+y}{2}$
$WYD(\beta)$ $\beta\in(-1,0)$	$\frac{\beta(1-\beta)(x-1)^2}{(x^\beta-1)(x^{1-\beta}-1)}$	$\frac{\beta(1-\beta)(x-y)^2}{(x^\beta-y^\beta)(x^{1-\beta}-y^{1-\beta})}$	0	$\frac{1+x}{2}$	$\frac{x+y}{2}$
BKM	$\frac{x-1}{\log x}$	$\frac{x-y}{\log x-\log y}$	0	$\frac{1+x}{2}$	$\frac{x+y}{2}$
$WYD(\beta)$ $\beta\in(0,\frac{1}{2})$	$\frac{\beta(1-\beta)(x-1)^2}{(x^\beta-1)(x^{1-\beta}-1)}$	$\frac{\beta(1-\beta)(x-y)^2}{(x^\beta-y^\beta)(x^{1-\beta}-y^{1-\beta})}$	$\beta(1-\beta)$	$\frac{x^\beta+x^{1-\beta}}{2}$	$\frac{x^\beta y^{1-\beta}+x^{1-\beta}y^\beta}{2}$
WY	$\left(\frac{1+\sqrt{x}}{2}\right)^2$	$\left(\frac{\sqrt{x}+\sqrt{y}}{2}\right)^2$	$\frac{1}{4}$	$\sqrt{x}$	$\sqrt{xy}$
SLD	$\frac{1+x}{2}$	$\frac{x+y}{2}$	$\frac{1}{2}$	$\frac{2x}{x+1}$	$\frac{2}{\frac{1}{x}+\frac{1}{y}}$

In the above table we have, for some quantum Fisher informations: the name, the function f, the mean m_f, the value of f at 0, the function $\tilde{f}$ and the mean $m_{\tilde{f}}$.

Example 5.8. Let $x>0$ and $\beta\in(0,\frac{1}{2})$. If we set

$$f_{SLD}(x):=\frac{1+x}{2},\quad f_{WY}(x):=\left(\frac{1+\sqrt{x}}{2}\right)^2,\quad f_\beta(x):=\beta(1-\beta)\frac{(x-1)^2}{(x^\beta-1)(x^{1-\beta}-1)},\quad f_{RLD}(x):=\frac{2x}{1+x}.$$

One has (see the above table)

$$\tilde{f}_{SLD}(x) = \frac{2x}{1+x}, \quad \tilde{f}_{WY}(x) = \sqrt{x}, \quad \tilde{f}_{\beta}(x) := \frac{x^{\beta}+x^{1-\beta}}{2}, \quad \tilde{f}_{RLD}(x) := \frac{1+x}{2}.$$

Note that if $x > 0$ is fixed the function $\beta \in (0, \frac{1}{2}) \mapsto x^{\beta} + x^{1-\beta} \in \mathbb{R}^+$ is decreasing. This implies

$$\tilde{f}_{SLD} \le \tilde{f}_{WY} \le \tilde{f}_{\beta} \le \tilde{f}_{RLD},$$

and therefore

$$m_{\tilde{f}_{SLD}} \le m_{\tilde{f}_{WY}} \le m_{\tilde{f}_{\beta}} \le m_{\tilde{f}_{RLD}},$$

that is a refined arithmetic-geometric-harmonic inequality

$$\frac{2xy}{x+y} \le \sqrt{xy} \le \frac{1}{2}(x^{\beta}y^{1-\beta} + x^{1-\beta}y^{\beta}) \le \frac{1+x}{2} \qquad x, y > 0, \quad \beta \in (0, 1/2).$$

Remark 5.9.

The metrics associated with the functions f_{β} are equivalent to the metrics induced by noncommutative α-divergences, where $\beta = \frac{1-\alpha}{2}$ (see [11]). They are very important in information geometry and are related to Wigner-Yanase-Dyson information (see for example [4][5] [6]). Defining $\ell_{\gamma}(x) := ((1+x^{\gamma})/2)^{\frac{1}{\gamma}}$ for $\gamma \in [1/2, 1]$ one has $\ell_{\gamma} \in \mathcal{F}_{op}$. The two parametric families f_{β}, ℓ_{γ} give us a continuum of operator monotone functions from the smallest function $\frac{2x}{x+1}$ to the largest function $\frac{1+x}{2}$. Further examples of this kind of "bridges" can be found in [8] [9]. Note that also $g_0(x) := \sqrt{x}$ is an element of $\mathcal{F}_{op}$.

In the sequel we need to study the following function.

Definition 5.10. For any $f \in \mathcal{F}_{op}$ set

$$H_f(x, y, w, z) := [(x+y) - m_{\tilde{f}}(x, y)]m_{\tilde{f}}(w, z) + [(w+z) - m_{\tilde{f}}(w, z)]m_{\tilde{f}}(x, y) \qquad x, y, w, z > 0.$$

Proposition 5.11. *For any $f, g \in \mathcal{F}_{op}$*

$$\begin{aligned} \tilde{f} &\le \tilde{g} \\ &\Downarrow \\ H_f(x, y, w, z) &\le H_g(x, y, w, z) \qquad \forall x, y, w, z > 0. \end{aligned}$$

Proof. Since

$$(x+y) - m_{\tilde{f}}(x, y) = (x+y) - \frac{x+y}{2} + \frac{(x-y)^2}{2y} \cdot \frac{f(0)}{f(\frac{x}{y})} = \frac{x+y}{2} + \frac{(x-y)^2}{2y} \cdot \frac{f(0)}{f(\frac{x}{y})}$$

we have

$$\begin{aligned} H_f(x,y,w,z) &:= [(x+y)-m_{\tilde f}(x,y)]m_{\tilde f}(w,z)+[(w+z)-m_{\tilde f}(w,z)]m_{\tilde f}(x,y) \\ &= \left(\frac{x+y}{2}+\frac{(x-y)^2}{2y}\cdot\frac{f(0)}{f(\frac{x}{y})}\right)\cdot\left(\frac{w+z}{2}-\frac{(w-z)^2}{2z}\cdot\frac{f(0)}{f(\frac{w}{z})}\right) \\ &\quad+\left(\frac{w+z}{2}+\frac{(w-z)^2}{2z}\cdot\frac{f(0)}{f(\frac{w}{z})}\right)\cdot\left(\frac{x+y}{2}-\frac{(x-y)^2}{2y}\cdot\frac{f(0)}{f(\frac{x}{y})}\right) \\ &= \frac{1}{2}\left[(x+y)(w+z)-\left(\frac{(x-y)^2}{y}\frac{(w-z)^2}{z}\right)\left(\frac{f(0)}{f(\frac{x}{y})}\cdot\frac{f(0)}{f(\frac{w}{z})}\right)\right]. \end{aligned} \tag{5.1}$$

Since, from Proposition 5.7,

$$\tilde f\le\tilde g\Rightarrow\frac{f(0)}{f(t)}\ge\frac{g(0)}{g(t)}>0\qquad\forall t>0,$$

we obtain

$$H_f(x,y,w,z)\le H_g(x,y,w,z)\qquad\forall x,y,w,z>0$$

by elementary computations. □

Note that for f non-regular one has

$$H_f(x,y,w,z)=\frac{1}{2}(x+y)(w+z).$$

On the other hand, for the function $f_{SLD}=\frac{1}{2}(1+x)$ one has from (5.1)

$$H_{SLD}(x,y,w,z)=\frac{1}{2}\left[(x+y)(w+z)-\frac{1}{4}\left(\frac{(x-y)^2(w-z)^2}{\frac{x+y}{2}\cdot\frac{w+z}{2}}\right)\right]=2\frac{xy(w^2+z^2)+wz(x^2+y^2)}{(x+y)(w+z)}.$$

Therefore, we have the following bounds.

Corollary 5.12. *For any $f\in\mathcal{F}_{op}$*

$$0<2\left[\frac{xy(w^2+z^2)+wz(x^2+y^2)}{(x+y)(w+z)}\right]\le H_f(x,y,w,z)\le\frac{1}{2}(x+y)(w+z)\qquad\forall x,y,w,z>0.$$

Remark 5.13. Note that for every $x>0$

$$\frac{f_{SLD}(0)}{f_{SLD}(x)}=\frac{\frac{1}{2}}{\frac{1+x}{2}}=\frac{1}{1+x}>\frac{1}{1+x+2\sqrt{x}}=\frac{1}{(1+\sqrt{x})^2}=\frac{\frac{1}{4}}{\frac{(1+\sqrt{x})^2}{4}}=\frac{f_{WY}(0)}{f_{WY}(x)},$$

so that for every $x,y,w,z>0$

$$H_{SLD}(x,y,w,z)<H_{WY}(x,y,w,z).$$

6 The main result

Proposition 6.1. *Given $f \in \mathcal{F}_{op}$, let $\Delta := m_{\tilde{f}}(L_\rho, R_\rho)$. Recall that $B_0 := B - \mathrm{Tr}(\rho B)$. One has*

(i) $\mathrm{Tr}(B_0 \cdot \Delta(I)) = 0$,

(ii) $\mathrm{Tr}(I \cdot \Delta(B_0)) = 0$,

(iii) $\mathrm{Tr}(\Delta(I)) = 1$.

Proof. (i) Since $(L_\rho - R_\rho)(I) = 0$ and $\mathrm{Tr}(\rho B_0) = \mathrm{Tr}(\rho B) - \mathrm{Tr}(\rho B) = 0$ we have

$$\begin{aligned}
\langle B_0, m_{\tilde{f}}(L_\rho, R_\rho)(I)\rangle &= \mathrm{Tr}(B_0 m_{\tilde{f}}(L_\rho, R_\rho)(I)) \\
&= \frac{1}{2}\mathrm{Tr}(B_0(L_\rho + R_\rho)(I)) - \frac{1}{2}f(0)\mathrm{Tr}(B_0 c_f(L_\rho, R_\rho)(L_\rho - R_\rho)^2(I)) \\
&= \frac{1}{2}\mathrm{Tr}(B_0\rho + \rho B_0) = \mathrm{Tr}(\rho B_0) \\
&= 0.
\end{aligned}$$

(ii) It is a simple consequence of (i) and of Proposition 4.2. Indeed,

$$\langle I, m_{\tilde{f}}(L_\rho, R_\rho)(B_0)\rangle = \langle m_{\tilde{f}}(L_\rho, R_\rho)(I), B_0\rangle = 0.$$

(iii)

$$\begin{aligned}
\mathrm{Tr}(\Delta(I)) &= \mathrm{Tr}(m_{\tilde{f}}(L_\rho, R_\rho)(I)) \\
&= \frac{1}{2}\mathrm{Tr}((L_\rho + R_\rho)(I)) - \frac{1}{2}f(0)\mathrm{Tr}(c_f(L_\rho, R_\rho)(L_\rho - R_\rho)^2(I)) \\
&= \mathrm{Tr}(\rho) \\
&= 1.
\end{aligned}$$

□

Proposition 6.2.

$$f(0) \cdot \langle i[\rho, A], i[\rho, B]\rangle_{\rho,f} = \mathrm{Tr}(\rho AB) + \mathrm{Tr}(\rho BA) - 2\mathrm{Tr}(A \cdot \Delta(B)).$$

Proof. Let us introduce the shorthand notation

$$\hat{c}_f(x, y) := (x - y)^2 c_f(x, y),$$

so that by definition

$$f(0) \cdot \hat{c}_f(x, y) = (x + y) - 2m_{\tilde{f}}(x, y).$$

Therefore, we have

$$
\begin{aligned}
f(0)\cdot\langle i[\rho,A],i[\rho,B]\rangle_{\rho,f} &= f(0)\cdot\mathrm{Tr}\big((i[\rho,A])\cdot c_f(L_\rho,R_\rho)(i[\rho,B])\big)\\
&= f(0)\cdot\langle (i[\rho,A]),c_f(L_\rho,R_\rho)(i[\rho,B])\rangle\\
&= f(0)\cdot\langle i(L_\rho-R_\rho)(A),c_f(L_\rho,R_\rho)\circ(i(L_\rho-R_\rho))(B)\rangle\\
&= f(0)\cdot\langle A,(i(L_\rho-R_\rho))^*\circ c_f(L_\rho,R_\rho)\circ(i(L_\rho-R_\rho))(B)\rangle\\
&= f(0)\cdot\langle A,-i(L_\rho-R_\rho)\circ c_f(L_\rho,R_\rho)\circ(i(L_\rho-R_\rho))(B)\rangle\\
&= f(0)\cdot\langle A,\hat{c}_f(L_\rho,R_\rho)(B)\rangle\\
&= f(0)\cdot\mathrm{Tr}(A\cdot\hat{c}_f(L_\rho,R_\rho)(B))\\
&= \mathrm{Tr}(A\cdot(f(0)\cdot\hat{c}_f(L_\rho,R_\rho))(B))\\
&= \mathrm{Tr}(A\cdot(L_\rho+R_\rho-2m_{\tilde{f}}(L_\rho,R_\rho))(B))\\
&= \mathrm{Tr}(\rho AB)+\mathrm{Tr}(\rho BA)-2\mathrm{Tr}(A\cdot m_{\tilde{f}}(L_\rho,R_\rho)(B))).
\end{aligned}
$$

□

Proposition 6.3.

$$
f(0)\cdot\langle i[\rho,A],i[\rho,B]\rangle_{\rho,f} = 2\big(\mathrm{Re}\{\mathrm{Cov}_\rho(A,B)\}-\mathrm{Tr}(\Delta(A_0)B_0)\big).
$$

Proof. We have that

$$
\begin{aligned}
f(0)\cdot\langle i[\rho,A],i[\rho,B]\rangle_{\rho,f} &= \mathrm{Tr}(\rho AB)+\mathrm{Tr}(\rho BA)-2\mathrm{Tr}(A\cdot\Delta(B))\\
&= \mathrm{Cov}_\rho\,(A,B)+\mathrm{Cov}_\rho\,(B,A)+2\mathrm{Tr}(\rho A)\cdot\mathrm{Tr}(\rho B)-2\mathrm{Tr}(A\cdot\Delta(B))\\
&= 2\mathrm{Re}\{\mathrm{Cov}_\rho(A,B)\}+2\big(\mathrm{Tr}(\rho A)\cdot\mathrm{Tr}(\rho B)-\mathrm{Tr}(A\cdot\Delta(B))\big);
\end{aligned}
$$

moreover, because of Proposition 6.1,

$$
\begin{aligned}
\mathrm{Tr}(\rho A)\mathrm{Tr}(\rho B)-\mathrm{Tr}(\Delta(A)B) &= \mathrm{Tr}(\rho A)\mathrm{Tr}(\rho B)-\mathrm{Tr}(\Delta(A_0+\mathrm{Tr}(\rho A)I)(B_0+\mathrm{Tr}(\rho B)I))\\
&= \mathrm{Tr}(\rho A)\mathrm{Tr}(\rho B)-\big[\mathrm{Tr}(\Delta(A_0)B_0)+\mathrm{Tr}(\rho A)\mathrm{Tr}(\Delta(I)B_0)\\
&\qquad+\mathrm{Tr}(\Delta(A_0)I)\mathrm{Tr}(\rho B)+\mathrm{Tr}(\rho A)\mathrm{Tr}(\rho B)\mathrm{Tr}(\Delta(I)I)\big]\\
&= \mathrm{Tr}(\rho A)\mathrm{Tr}(\rho B)-\mathrm{Tr}(\Delta(A_0)B_0)-\mathrm{Tr}(\rho A)\mathrm{Tr}(\rho B)\\
&= -\mathrm{Tr}(\Delta(A_0)B_0).
\end{aligned}
$$

Therefore, the conclusion follows.

□

We recall some consequences of the spectral theorem we need in the sequel. Let ρ be a state, λ_i its eigenvalues and E_i the associated eigenprojectors. The spectral decompositions of L_ρ and R_ρ are the following

$$L_\rho = \sum_i \lambda_i L_{E_i} \qquad\qquad R_\rho = \sum_i \lambda_i R_{E_i}.$$

Therefore, from the spectral theorem for commuting selfadjoint operators we get the following result.

Corollary 6.4. *Let ρ be a state, λ_i its eigenvalues and E_i the projectors of the associated eigenspaces. If $s : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ is a continuous function then*

$$s(L_\rho, R_\rho) = \sum_{i,j} s(\lambda_i, \lambda_j) L_{E_i} R_{E_j}.$$

Let V be a finite dimensional real vector space with a scalar product $g(\cdot,\cdot)$. We define, for $v, w \in V$,

$$\mathrm{Area}^g(v, w) := \sqrt{g(v, v) \cdot g(w, w) - |g(v, w)|^2}.$$

In the Euclidean plane $\mathrm{Area}^g(v, w)$ is the area of the parallelogram spanned by v and w. If we are dealing with a ρ point-depending Riemannian metric, we write Area^g_ρ. If $f \in \mathcal{F}_{op}$ we denote by Area^f_ρ the area functional associated to the monotone metric $\langle\cdot,\cdot\rangle_{\rho,f}$.

We are now ready for the main results.

Theorem 6.5. *For any $f, g \in \mathcal{F}_{op}$*

(i)

$$\mathrm{Var}_\rho(A)\mathrm{Var}_\rho(B) - |\mathrm{Cov}^s_\rho(A, B)|^2 \geq \left(\frac{f(0)}{2} \cdot \mathrm{Area}^f_\rho(i[\rho, A], i[\rho, B])\right)^2,$$

(ii)

$$\tilde{g} \geq \tilde{f} \qquad \Longrightarrow \qquad \frac{g(0)}{2} \cdot \mathrm{Area}^g_\rho(i[\rho, A], i[\rho, B]) \leq \frac{f(0)}{2} \cdot \mathrm{Area}^f_\rho(i[\rho, A], i[\rho, B]).$$

Proof. Fix $A, B \in M_{n,sa}$. Let us introduce, for the sake of brevity,

$$F(f) := \left(\mathrm{Var}_\rho(A)\mathrm{Var}_\rho(B) - |\mathrm{Cov}^s_\rho(A, B)|^2\right) - \left(\frac{f(0)}{2} \cdot \mathrm{Area}^f_\rho(i[\rho, A], i[\rho, B])\right)^2.$$

Then, we have to show that $F(f) \geq 0$, and $\tilde{g} \geq \tilde{f} \Longrightarrow F(g) \geq F(f)$.

Let $\{\varphi_i\}$ be a complete orthonormal base composed of eigenvectors of ρ and $\{\lambda_i\}$ the corresponding eigenvalues. Set $a_{ij} \equiv \langle A_0\varphi_i|\varphi_j\rangle$ and $b_{ij} \equiv \langle B_0\varphi_i|\varphi_j\rangle$. Note that $a_{ij} \neq A_{ij} :=$ the i, j entry of A.

Then we calculate

$$
\begin{aligned}
\mathrm{Var}_\rho(A) &= \mathrm{Tr}(\rho A_0^2) = \frac{1}{2}\sum_{i,j}(\lambda_i+\lambda_j)a_{ij}a_{ji}\\
\mathrm{Var}_\rho(B) &= \mathrm{Tr}(\rho B_0^2) = \frac{1}{2}\sum_{i,j}(\lambda_i+\lambda_j)b_{ij}b_{ji}\\
\mathrm{Cov}^s_\rho(A,B) &= \mathrm{Re}\{\mathrm{Cov}_\rho(A,B)\} = \mathrm{Re}\{\mathrm{Tr}(\rho A_0 B_0)\} = \frac{1}{2}\sum_{i,j}(\lambda_i+\lambda_j)\mathrm{Re}\{a_{ij}b_{ji}\}\\
\frac{f(0)}{2}||i[\rho,A]||^2_{\rho,f} &= \mathrm{Var}_\rho(A) - \mathrm{Tr}(A_0 m_{\tilde{f}}(L_\rho,R_\rho)A_0) = \frac{1}{2}\sum_{i,j}(\lambda_i+\lambda_j)a_{ij}a_{ji} - \sum_{i,j} m_{\tilde{f}}(\lambda_i,\lambda_j)a_{ij}a_{ji}\\
\frac{f(0)}{2}||i[\rho,B]||^2_{\rho,f} &= \frac{1}{2}\sum_{i,j}(\lambda_i+\lambda_j)b_{ij}b_{ji} - \sum_{i,j} m_{\tilde{f}}(\lambda_i,\lambda_j)b_{ij}b_{ji}\\
\frac{f(0)}{2}\langle i[\rho,A], i[\rho,B]\rangle_{\rho,f} &= \mathrm{Re}\{\mathrm{Cov}_\rho(A,B)\} - \mathrm{Re}\{\mathrm{Tr}(m_{\tilde{f}}(L_\rho,R_\rho)(A_0)\cdot B_0)\}\\
&= \frac{1}{2}\sum_{i,j}(\lambda_i+\lambda_j)\mathrm{Re}\{a_{ij}b_{ji}\} - \sum_{i,j} m_{\tilde{f}}(\lambda_i,\lambda_j)\mathrm{Re}\{a_{ij}b_{ji}\}.
\end{aligned}
$$

Set

$$
\begin{aligned}
\xi &:= \mathrm{Var}_\rho(A)\,\mathrm{Var}_\rho(B) - \frac{f(0)^2}{4}||i[\rho,A]||^2_{\rho,f}\cdot||i[\rho,B]||^2_{\rho,f}\\
&= \frac{1}{2}\sum_{i,j,k,l}\Big\{(\lambda_i+\lambda_j)m_{\tilde{f}}(\lambda_k,\lambda_l) + (\lambda_k+\lambda_l)m_{\tilde{f}}(\lambda_i,\lambda_j) - 2m_{\tilde{f}}(\lambda_i,\lambda_j)m_{\tilde{f}}(\lambda_k,\lambda_l)\Big\}\, a_{ij}a_{ji}b_{kl}b_{lk}\\
&= \frac{1}{4}\sum_{i,j,k,l}\Big\{(\lambda_i+\lambda_j)m_{\tilde{f}}(\lambda_k,\lambda_l) + (\lambda_k+\lambda_l)m_{\tilde{f}}(\lambda_i,\lambda_j) - 2m_{\tilde{f}}(\lambda_i,\lambda_j)m_{\tilde{f}}(\lambda_k,\lambda_l)\Big\}\, \{a_{ij}a_{ji}b_{kl}b_{lk} + a_{kl}a_{lk}b_{ij}b_{ji}\},\\
\eta &:= |\mathrm{Cov}^s_\rho(A,B)|^2 - \frac{f(0)^2}{4}|\langle i[\rho,A], i[\rho,B]\rangle^2_{\rho,f}|^2\\
&= \frac{1}{2}\sum_{i,j,k,l}\Big\{(\lambda_i+\lambda_j)m_{\tilde{f}}(\lambda_k,\lambda_l) + (\lambda_k+\lambda_l)m_{\tilde{f}}(\lambda_i,\lambda_j) - 2m_{\tilde{f}}(\lambda_i,\lambda_j)m_{\tilde{f}}(\lambda_k,\lambda_l)\Big\}\, \mathrm{Re}\{a_{ij}b_{ji}\}\mathrm{Re}\{a_{kl}b_{lk}\},
\end{aligned}
$$

$$K_{i,j,k,l} := K_{i,j,k,l}(\rho,A,B) := |a_{ij}|^2|b_{kl}|^2 + |a_{kl}|^2|b_{ij}|^2 - 2\mathrm{Re}\{a_{ij}b_{ji}\}\mathrm{Re}\{a_{kl}b_{lk}\}.$$

Since

$$|a_{ij}|^2|b_{kl}|^2 + |a_{kl}|^2|b_{ij}|^2 \geq 2\,|a_{ij}b_{ji}|\,|a_{kl}b_{lk}| \geq 2\,|\mathrm{Re}\,\{a_{ij}b_{ji}\}\,\mathrm{Re}\,\{a_{kl}b_{lk}\}|\,,$$

we have that $K_{i,j,k,l} \geq 0$. Note that $K_{i,j,k,l}$ does not depend on f.

Then

$$
\begin{aligned}
F(f) = \xi - \eta \quad &= \quad \frac{1}{4}\sum_{i,j,k,l}\Big\{(\lambda_i+\lambda_j)m_{\tilde{f}}(\lambda_k,\lambda_l) + (\lambda_k+\lambda_l)m_{\tilde{f}}(\lambda_i,\lambda_j) - 2m_{\tilde{f}}(\lambda_i,\lambda_j)m_{\tilde{f}}(\lambda_k,\lambda_l)\Big\}\cdot\\
&\qquad \cdot\big\{|a_{ij}|^2|b_{kl}|^2 + |a_{kl}|^2|b_{ij}|^2 - 2\mathrm{Re}\{a_{ij}b_{ji}\}\mathrm{Re}\{a_{kl}b_{lk}\}\big\}\\
&= \quad \frac{1}{4}\sum_{i,j,k,l} H_f(\lambda_i,\lambda_j,\lambda_k,\lambda_l)\cdot K_{i,j,k,l}.
\end{aligned}
$$

Because of Proposition 5.11 and Corollary 5.12 one has that

$$\tilde{f} \leq \tilde{g} \quad \Longrightarrow \quad 0 \leq H_f(\lambda_i, \lambda_j, \lambda_k, \lambda_l) \leq H_g(\lambda_i, \lambda_j, \lambda_k, \lambda_l)$$

and therefore

$$\tilde{f} \leq \tilde{g} \quad \Longrightarrow \quad 0 \leq F(f) \leq F(g)$$

and we get the thesis. □

The standard Schrödinger uncertainty principle reads as

$$\mathrm{Area}_\rho^{\mathrm{Cov}^s}(A,B) \geq \frac{1}{2}|\mathrm{Tr}(\rho[A,B])|,$$

while the main result of the present paper can be expressed as

$$\mathrm{Area}_\rho^{\mathrm{Cov}^s}(A,B) \geq \frac{f(0)}{2} \cdot \mathrm{Area}_\rho^f(i[\rho,A], i[\rho,B]).$$

Corollary 6.6. *For any $f \in \mathcal{F}_{op}$, $A, B \in M_{n,sa}$, one has*

$$\frac{f_{SLD}(0)}{2} \cdot \mathrm{Area}_\rho^{f_{SLD}}(i[\rho,A], i[\rho,B]) \geq \frac{f(0)}{2} \cdot \mathrm{Area}_\rho^f(i[\rho,A], i[\rho,B]).$$

Proof. Immediate consequence of Corollary 5.6. □

Remark 6.7. Setting

$$N_\rho^f(A,B) := \mathrm{Area}_\rho^{\mathrm{Cov}^s}(A,B) - \frac{f(0)}{2} \cdot \mathrm{Area}_\rho^f(i[\rho,A], i[\rho,B]) \geq 0,$$

we may strengthen the main result to

$$\mathrm{Area}_\rho^{\mathrm{Cov}^s}(A,B) \geq \frac{f(0)}{2} \cdot \mathrm{Area}_\rho^f(i[\rho,A], i[\rho,B]) + N_\rho^{SLD}(A,B).$$

The above geometric considerations take a particularly interesting form when considering the dynamics of quantum states. Suppose we have a positive (self-adjoint) operator H determining a quantum evolution. The state ρ evolves according to the formula

$$\rho_H(t) := e^{-itH}\rho e^{itH}.$$

We say that $\rho_H(t)$ is the time evolution of $\rho = \rho_H(0)$ determined by H. For the evolution $\rho_H(t)$ this is equivalent to satisfy the quantum analogue of Liouville theorem in classical statistical mechanics, namely the Landau-von Neumann equation.

Definition 6.8. Let $\rho(t)$ be a curve in $\mathcal{D}_n^1$ and let $H \in M_{n,sa}$. We say that $\rho(t)$ satisfies the Landau-von Neumann equation w.r.t. H if

$$\dot{\rho}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = i[\rho(t), H].$$

Satisfying the Landau-von Neumann equation is equivalent to $\rho(t) = \rho_H(t) = e^{-itH}\rho e^{itH}$.

From Theorem 6.5 we get the following inequality.

Proposition 6.9. *Let $\rho > 0$ be a state and $H, K \in M_{n,sa}$. Suppose that $\rho = \rho_H(0) = \rho_K(0)$. Then, for any $f \in \mathcal{F}_{op}$, one has*

$$\mathrm{Area}_\rho^{\mathrm{Cov}^s}(H, K) \geq \frac{f(0)}{2} \cdot \mathrm{Area}_\rho^f(\dot{\rho}_H(0), \dot{\rho}_K(0)).$$

Therefore, as we said in the Introduction, the bound on the right side of our inequality appears when the evolutions $\rho_H(t), \rho_K(t)$ are different and not trivial.

7 The f-correlation associated to quantum Fisher informations

Mainly to confront our result with previous results we introduce the notions of *f-correlation* and *f-information*.

Definition 7.1.

$$\begin{aligned} \mathcal{C}_\rho^f(A, B) &= \mathcal{C}_\rho^f(B, A) := \mathrm{Tr}(m_f(L_\rho, R_\rho)(A) \cdot B) \\ \mathcal{C}_\rho^f(A) &:= \mathcal{C}_\rho^f(A, A). \end{aligned}$$

Definition 7.2. For $A, B \in M_{n,sa}$, $\rho \in \mathcal{D}_n^1$ and $f \in \mathcal{F}_{op}$, the *metric adjusted correlation* (or *f-correlation*) and the *metric adjusted skew information* (or *f-information*) are defined as

$$\begin{aligned} \mathrm{Corr}_\rho^f(A, B) &:= \mathrm{Tr}(\rho AB) - \mathcal{C}_\rho^{\tilde{f}}(A, B) = \mathrm{Tr}(\rho AB) - \mathrm{Tr}(m_{\tilde{f}}(L_\rho, R_\rho)(A) \cdot B), \\ I_\rho^f(A) &:= \mathrm{Corr}_\rho^f(A, A). \end{aligned}$$

The definition of $\mathrm{Corr}_\rho^f(A, B)$ appeared in [9] in a different form. For the f-correlation there is an analogue of Proposition 2.3 for covariance.

Lemma 7.3. *For any $A, B \in M_{n,sa}$, $\rho \in \mathcal{D}_n^1$ and $f \in \mathcal{F}_{op}$ one has*

$$\begin{aligned} 2\mathrm{Re}\{\mathrm{Corr}_\rho^f(A, B)\} &= \mathrm{Corr}_\rho^f(A, B) + \mathrm{Corr}_\rho^f(B, A) = f(0) \cdot \langle i[\rho, A], i[\rho, B] \rangle_{\rho, f} \\ 2i\mathrm{Im}\{\mathrm{Corr}_\rho^f(A, B)\} &= \mathrm{Corr}_\rho^f(A, B) - \mathrm{Corr}_\rho^f(B, A) = \mathrm{Tr}(\rho[A, B]). \end{aligned}$$

Proof. We have that

$$\mathrm{Corr}_\rho^f(A,B) - \mathrm{Corr}_\rho^f(B,A) = \mathrm{Tr}(\rho[A,B]),$$

which is purely imaginary.

This implies

$$\mathrm{Re}\{\mathrm{Corr}_\rho^f(A,B)\} = \mathrm{Re}\{\mathrm{Corr}_\rho^f(B,A)\},$$

so that

$$2\mathrm{Re}\{\mathrm{Corr}_\rho^f(A,B)\} = \mathrm{Corr}_\rho^f(A,B) + \mathrm{Corr}_\rho^f(B,A),$$

$$2i\mathrm{Im}\{\mathrm{Corr}_\rho^f(A,B)\} = \mathrm{Corr}_\rho^f(A,B) - \mathrm{Corr}_\rho^f(B,A).$$

Since

$$\mathrm{Corr}_\rho^f(A,B) + \mathrm{Corr}_\rho^f(B,A) = \mathrm{Tr}(\rho AB) + \mathrm{Tr}(\rho BA) - 2\mathrm{Tr}(A\cdot\Delta(B)),$$

the conclusion follows from Proposition 6.2. □

Corollary 7.4.

$$I_\rho^f(A) = \mathrm{Corr}_\rho^f(A,A) = \frac{f(0)}{2}\cdot\langle i[\rho,A], i[\rho,A]\rangle_{\rho,f} = \frac{f(0)}{2}\cdot||i[\rho,A]||^2_{\rho,f}.$$

Remark 7.5. If

$$f_\beta(x) := \beta(1-\beta)\frac{(x-1)^2}{(x^\beta-1)(x^{1-\beta}-1)} \qquad\qquad \beta\in\Big(0,\frac{1}{2}\Big],$$

then

$$I_\rho^{f_\beta}(A) = \frac{f_\beta(0)}{2}\mathrm{Tr}(i[\rho,A]c_{f_\beta}(L_\rho,R_\rho)i[\rho,A]) = -\frac{1}{2}\mathrm{Tr}([\rho^\beta,A]\cdot[\rho^{1-\beta},A]),$$

so $I_\rho^{f_\beta}(A)$ coincides with the Wigner-Yanase-Dyson skew information.

Let us reformulate the main result in terms of f-correlation.

Proposition 7.6. *For any $f\in\mathcal{F}_{op}$ one has*

$$\left(\frac{f(0)}{2}\cdot\mathrm{Area}_\rho^f(i[\rho,A],i[\rho,B])\right)^2 = I_\rho^f\,(A)\,I_\rho^f\,(B) - \left|\mathrm{Re}\left\{\mathrm{Corr}_\rho^f\,(A,B)\right\}\right|^2.$$

Proof.

$$\begin{aligned}\left(\frac{f(0)}{2}\cdot\mathrm{Area}_\rho^f(i[\rho,A],i[\rho,B])\right)^2 &= \frac{f(0)^2}{4}\left(\langle i[\rho,A],i[\rho,A]\rangle_{\rho,f}\cdot\langle i[\rho,B],i[\rho,B]\rangle_{\rho,f} - \langle i[\rho,A],i[\rho,B]\rangle^2_{\rho,f}\right)\\ &= \left(\frac{f(0)}{2}\cdot||i[\rho,A]||^2_{\rho,f}\right)\cdot\left(\frac{f(0)}{2}\cdot||i[\rho,B]||^2_{\rho,f}\right) - \left(\frac{f(0)}{2}\langle i[\rho,A],i[\rho,B]\rangle_{\rho,f}\right)^2\\ &= I_\rho^f\,(A)\,I_\rho^f\,(B) - \left|\mathrm{Re}\left\{\mathrm{Corr}_\rho^f\,(A,B)\right\}\right|^2.\end{aligned}$$

□

Therefore, our main result states that

$$\mathrm{Var}_\rho(A)\mathrm{Var}_\rho(B) - |\mathrm{Re}\{\mathrm{Cov}_\rho(A,B)\}|^2 \geq I^f_\rho(A)\, I^f_\rho(B) - \left|\mathrm{Re}\left\{\mathrm{Corr}^f_\rho(A,B)\right\}\right|^2.$$

Recall that we introduced, for fixed ρ, A, B, the functional

$$\begin{aligned} F(f) &= \mathrm{Var}_\rho(A)\mathrm{Var}_\rho(B) - |\mathrm{Cov}^s_\rho(A,B)|^2 - \left(\frac{f(0)}{2} \cdot \mathrm{Area}^f_\rho(i[\rho,A], i[\rho,B])\right)^2 \\ &= \mathrm{Var}_\rho(A)\mathrm{Var}_\rho(B) - |\mathrm{Re}\{\mathrm{Cov}_\rho(A,B)\}|^2 - I^f_\rho(A)\, I^f_\rho(B) + \left|\mathrm{Re}\left\{\mathrm{Corr}^f_\rho(A,B)\right\}\right|^2. \end{aligned}$$

As the main result, we proved that, for any $f, g \in \mathcal{F}_{op}$, $F(f) \geq 0$ and $\tilde{f} \leq \tilde{g} \Longrightarrow F(f) \leq F(g)$.

Corollary 7.7. *Suppose ρ, A, B are fixed. Then the function of β given by*

$$F(\beta) := F(f_\beta)$$

is decreasing on $(0, \frac{1}{2}]$ and $F(1/2) \geq 0$; therefore $F(\beta) \geq 0$.

Proof. Given $x > 0$, the function $\beta \mapsto \tilde{f}_\beta(x) = \frac{1}{2}(x^\beta + x^{1-\beta})$ is decreasing in $(0, \frac{1}{2}]$, so that

$$\beta_1 \leq \beta_2 \quad \Longrightarrow \quad \tilde{f}_{\beta_1} \geq \tilde{f}_{\beta_2} \quad \Longrightarrow \quad F(\beta_1) \geq F(\beta_2).$$

□

Remark 7.8. The above corollary was the content of Theorem 5, the main result in [13] and of Proposition IV.1 in [28]. Note that, because of Corollary 7.7, the optimal bound previously known was given by f_{WY}, namely the bound of Wigner-Yanase metric (this was due to Kosaki in [13]). Remark 5.13 implies that the bound given by the SLD area is strictly greater then that given by the WY area.

Proposition 7.9.

$$\begin{aligned} \mathrm{Cov}_\rho(A,B) &= \mathrm{Corr}^f_\rho(A,B) + \mathcal{C}^{\tilde{f}}_\rho(A_0,B_0), \\ \mathrm{Var}_\rho(A) &= I^f_\rho(A) + \mathcal{C}^{\tilde{f}}_\rho(A_0). \end{aligned}$$

Proof. The calculations of Proposition 6.3 imply that

$$\begin{aligned} \mathrm{Corr}^f_\rho(A,B) - \mathrm{Cov}_\rho(A,B) &= \mathrm{Tr}(\rho A)\mathrm{Tr}(\rho B) - \mathrm{Tr}(\Delta(A)B) \\ &= -\mathrm{Tr}(m_{\tilde{f}}(L_\rho, R_\rho)(A_0)B_0) \\ &= -\mathcal{C}^{\tilde{f}}_\rho(A_0,B_0). \end{aligned}$$

□

Luo (see [17]) suggested that if one consider the variance as a measure of "uncertainty" of an observable A in the state ρ then the above equality splits the variance in a "quantum" part ($I^f_\rho(A)$) plus a "classical" part ($\mathcal{C}^{\tilde{f}}_\rho(A_0)$).

8 Conditions for equality

In this section we give a necessary and sufficient condition to have equality in our main result.

Proposition 8.1. *The inequality of Theorem 6.5 is an equality if and only if A_0 and B_0 are proportional.*

Proof. If $A_0 = \lambda B_0$, with $\lambda \in \mathbb{R}$, then

$$\begin{aligned}\mathrm{Var}_\rho(A)\mathrm{Var}_\rho(B) - |\mathrm{Re}\{\mathrm{Cov}_\rho(A,B)|^2\} &= \mathrm{Tr}(\rho A_0^2)\mathrm{Tr}(\rho B_0^2) - |\mathrm{Re}\{\mathrm{Tr}(\rho A_0 B_0)\}|^2 \\ &= \mathrm{Tr}(\rho(\lambda B_0)^2)\mathrm{Tr}(\rho B_0^2) - |\mathrm{Re}\{\mathrm{Tr}(\rho \lambda B_0 B_0)\}|^2 \\ &= \lambda^2 \mathrm{Tr}(\rho B_0^2)^2 - \lambda^2 |\mathrm{Tr}(\rho B_0^2)|^2 \\ &= 0.\end{aligned}$$

In this case the inequality is just the equality $0 = 0$.

Now we suppose that A_0, B_0 are not proportional and we prove that the inequality is strict. We use the same notations as in the proof of Theorem 6.5.

Note that

$$\mathrm{Var}_\rho(A)\,\mathrm{Var}_\rho(B) - |\mathrm{Re}\{\mathrm{Cov}_\rho(A,B)\}|^2 - I^f_\rho(A)\, I^f_\rho(B) + \left|\mathrm{Re}\left\{\mathrm{Corr}^f_\rho(A,B)\right\}\right|^2 =$$
$$= \xi - \eta = \frac{1}{4}\sum_{i,j,k,l} H_f(\lambda_i,\lambda_j,\lambda_k,\lambda_l)\cdot K_{i,j,k,l}(A,B),$$

and

$$H_f(\lambda_i,\lambda_j,\lambda_k,\lambda_l) > 0, \qquad K_{i,j,k,l}(A,B) \geq 0 \qquad \forall i,j,k,l.$$

Therefore, the strict inequality is equivalent to $\xi - \eta > 0$, which is, in turn, equivalent to

$$K_{i,j,k,l}(A,B) > 0$$

for some i,j,k,l.

From the fact that A_0, B_0 are not proportional one can derive that also the matrices $\{a_{ij}\}, \{b_{ij}\}$ are not proportional and this implies (the other cases being trivial) that there exist (complex) $a_{ij}, b_{ij}, a_{kl}, b_{kl} \neq$

0 and (real) $\lambda, \mu \neq 0$ such that

$$a_{ij} = \lambda b_{ij} \qquad\qquad a_{kl} = \mu b_{kl} \qquad\qquad \lambda \neq \mu.$$

We get

$$\begin{aligned} K_{i,j,k,l}(A,B) &= |a_{ij}|^2|b_{kl}|^2 + |a_{kl}|^2|b_{ij}|^2 - 2\mathrm{Re}\{a_{ij}b_{ji}\}\mathrm{Re}\{a_{kl}b_{lk}\} \\ &= |a_{ij}|^2|b_{kl}|^2 + |\mu b_{kl}|^2\Big|\frac{a_{ij}}{\lambda}\Big|^2 - 2\mathrm{Re}\{a_{ij}\frac{\overline{a_{ij}}}{\lambda}\}\mathrm{Re}\{\mu b_{kl}b_{lk}\} \\ &= \left(1+\frac{\mu^2}{\lambda^2}\right)\cdot|a_{ij}|^2|b_{kl}|^2 - 2\frac{\mu}{\lambda}|a_{ij}|^2|b_{kl}|^2 \\ &= \left(1+\frac{\mu^2}{\lambda^2} - 2\frac{\mu}{\lambda}\right)\cdot|a_{ij}|^2|b_{kl}|^2 \\ &= \left(1-\frac{\mu}{\lambda}\right)^2\cdot|a_{ij}|^2|b_{kl}|^2 > 0 \end{aligned}$$

because

$$\left(1-\frac{\mu}{\lambda}\right) \neq 0.$$

Therefore,

$$\xi - \eta \neq 0$$

and this ends the proof. □

The particular case $f = f_\beta$ (where $\beta \in (0, 1/2]$) of the above proposition has been proved in Proposition 6 in [13].

9 Another inequality

The study of the mean $m_{\tilde{f}}$ allows us to get another inequality that can be seen as an uncertainty principle in Heisenberg form. Recall that

$$f_{RLD}(x) := \frac{2x}{x+1}.$$

Proposition 9.1.

$$\mathrm{Var}_\rho(A) \geq I_\rho^f(A) + \mathcal{C}_\rho^{f_{RLD}}(A_0) \qquad\qquad \forall f \in \mathcal{F}_{op}.$$

Proof. We use the notations employed in the proof of Theorem 6.5. Since

$$\begin{aligned}
\mathrm{Var}_\rho(A) &= \mathrm{Tr}(\rho A_0^2) = \frac{1}{2}\sum_{i,j}(\lambda_i+\lambda_j)a_{ij}a_{ji} \\
I_\rho^f(A) &= \mathrm{Var}_\rho(A) - \mathrm{Tr}(A_0 m_{\tilde{f}}(L_\rho,R_\rho)A_0) = \frac{1}{2}\sum_{i,j}(\lambda_i+\lambda_j)a_{ij}a_{ji} - \sum_{i,j} m_{\tilde{f}}(\lambda_i,\lambda_j)a_{ij}a_{ji} \\
\mathcal{C}_\rho^{f_{RLD}}(A_0) &= \sum_{i,j} m_{h_0}(\lambda_i,\lambda_j)a_{ij}a_{ji},
\end{aligned}$$

using Corollary 5.6 we have

$$\mathrm{Var}_\rho(A) - I_\rho^f(A) - \mathcal{C}_\rho^{f_{RLD}}(A_0) = \sum_{i,j}[m_{\tilde{f}}(\lambda_i,\lambda_j) - m_{h_0}(\lambda_i,\lambda_j)]|a_{ij}|^2 \geq 0.$$

□

From this we get the following inequality.

Theorem 9.2.

$$\mathrm{Var}_\rho(A)\cdot\mathrm{Var}_\rho(B) \geq [I_\rho^f(A) + \mathcal{C}_\rho^{f_{RLD}}(A_0)]\cdot[I_\rho^f(B) + \mathcal{C}_\rho^{f_{RLD}}(B_0)] \qquad \forall f \in \mathcal{F}_{op}. \tag{9.1}$$

Since $\mathcal{C}_\rho^{f_{RLD}}(A_0) \geq 0$ we obtain, as a corollary, two results due to Luo, for the case $f = f_{WY} = \frac{1}{4}(1+\sqrt{x})^2$, and to Hansen, for the general case (see [16], [9]).

Proposition 9.3.

$$\mathrm{Var}_\rho(A) \geq I_\rho^f(A) \qquad \forall f \in \mathcal{F}_{op}.$$

Theorem 9.4.

$$\mathrm{Var}_\rho(A)\cdot\mathrm{Var}_\rho(B) \geq I_\rho^f(A)\cdot I_\rho^f(B) = \frac{f(0)^2}{4}\cdot||i[\rho,A]||_{\rho,f}^2\cdot||i[\rho,B]||_{\rho,f}^2 \qquad \forall f \in \mathcal{F}_{op}.$$

Let us study how the bound $I_\rho^f(A)\cdot I_\rho^f(B)$ depends on f.

Proposition 9.5. *For any $f, g \in \mathcal{F}_{op}$*

$$f \leq g \qquad \Longrightarrow \qquad \mathcal{C}_\rho^f(A_0) \leq \mathcal{C}_\rho^g(A_0),$$

$$\tilde{f} \leq \tilde{g} \qquad \Longrightarrow \qquad I_\rho^f(A) \geq I_\rho^g(A).$$

Proof. We still use notations of Theorem 6.5. Since $m_f \leq m_g$,

$$\begin{aligned}
\mathcal{C}_\rho^g(A_0) - \mathcal{C}_\rho^f(A_0) &= \sum_{i,j} m_g(\lambda_i,\lambda_j)a_{ij}a_{ji} - \sum_{i,j} m_f(\lambda_i,\lambda_j)a_{ij}a_{ji} \\
&= \sum_{i,j}[m_g(\lambda_i,\lambda_j) - m_f(\lambda_i,\lambda_j)]|a_{ij}|^2 \geq 0.
\end{aligned}$$

The second inequality is an immediate consequence of the first one. □

Corollary 9.6.

$$I_\rho^{SLD}(A) \geq I_\rho^f(A) \qquad \forall f \in \mathcal{F}_{op}.$$

Proof. Immediate consequence of Proposition 5.6. $\square$

Corollary 9.7.

$$\tilde{f} \leq \tilde{g} \quad \Longrightarrow I_\rho^f(A) I_\rho^f(B) \geq I_\rho^g(A) I_\rho^g(B).$$

We discuss, now, the equality in Theorem 9.4.

Proposition 9.8.

$$\mathrm{Var}_\rho(A) \cdot \mathrm{Var}_\rho(B) = I_\rho^f(A) \cdot I_\rho^f(B) \qquad \Longleftrightarrow \qquad A_0 = B_0 = 0.$$

Proof. Because of Proposition 9.3 we have

$$\mathrm{Var}_\rho(A) \cdot \mathrm{Var}_\rho(B) = I_\rho^f(A) \cdot I_\rho^f(B) \Longleftrightarrow \mathrm{Var}_\rho(A) = I_\rho^f(A), \ \mathrm{Var}_\rho(B) = I_\rho^f(B).$$

Hence, we need to show $\mathrm{Var}_\rho(A) = I_\rho^f(A) \Longleftrightarrow A_0 = 0$. Indeed, using the same notations as in Theorem 6.5,

$$\mathrm{Var}_\rho(A) = I_\rho^f(A) \Longleftrightarrow \mathrm{Tr}(A_0 m_{\tilde{f}}(L_\rho, R_\rho) A_0) = 0 \Longleftrightarrow \sum_{i,j} m_{\tilde{f}}(\lambda_i, \lambda_j) a_{ij} a_{ji} = 0$$

$$\Longleftrightarrow a_{ij} = 0, \ \forall i, j \Longleftrightarrow A_0 = 0.$$

$\square$

10 Relation with the standard uncertainty principles

Some authors tried to prove the following inequalities

$$\left(\frac{f(0)}{2} \cdot \mathrm{Area}(i[\rho, A], i[\rho, B]) \right)^2 = I_\rho^f\,(A)\, I_\rho^f\,(B) - |\mathrm{Re}(\mathrm{Corr}_\rho^f(A, B))|^2 \geq \frac{1}{4} |\mathrm{Tr}(\rho[A, B])|^2, \tag{10.1}$$

$$I_\rho^f\,(A)\, I_\rho^f\,(B) \geq \frac{1}{4} |\mathrm{Tr}(\rho[A, B])|^2. \tag{10.2}$$

They wanted to obtain the standard Heisenberg-Schrödinger uncertainty principles as consequences of the uncertainty principles discussed in the present paper. Actually the inequality (10.1) has been proved false for $f = f_\beta$, that is, for the Wigner-Yanase-Dyson case (see p.632, 642-644 in [13], p.4404 in [28] and [20]). But the discussion of Section 6, 7, 9 shows that the upper bounds

$$G(f) = \frac{f(0)}{2} \cdot \mathrm{Area}(i[\rho, A], i[\rho, B]) \qquad N(f) := I_\rho^f\,(A)\, I_\rho^f\,(B)$$

can be larger than those of the WYD metric (we showed it for the SLD metric in Remark 5.13). It is, therefore, natural to ask if the above inequalities, that are false for the WYD metric, can be true for some different quantum Fisher information (for example for the SLD metric). The following theorem shows that this is not the case, even on 2×2 matrices.

Theorem 10.1. *There exist 2×2 self-adjoint matrices A and B, and a density matrix ρ such that*

$$I^f_\rho(A)\, I^f_\rho(B) < \frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2 \qquad \forall f \in \mathcal{F}_{op}.$$

Therefore, for these ρ, A, B we also have

$$\left(\frac{f(0)}{2} \cdot \mathrm{Area}(i[\rho,A], i[\rho,B])\right)^2 = I^f_\rho(A)\, I^f_\rho(B) - |\mathrm{Re}(\mathrm{Corr}^f_\rho(A,B))|^2 < \frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2 \qquad \forall f \in \mathcal{F}_{op}.$$

Proof. We use notations of Theorem 6.5: let $\{\varphi_i\}$ be a complete orthonormal base composed of eigenvectors of ρ, and $\{\lambda_i\}$ the corresponding eigenvalues. Set $a_{ij} \equiv \langle A_0\varphi_i|\varphi_j\rangle$ and $b_{ij} \equiv \langle B_0\varphi_i|\varphi_j\rangle$. In what follows $\lambda_1 > \lambda_2 > 0$, $\lambda_1 + \lambda_2 = 1$ and

$$\rho = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

(in terms of Pauli matrices, $A = -\sigma_2$ and $B = \sigma_1$). Simple calculations show that $|a_{ii}| = |b_{ii}| = 0$, while $|a_{ij}| = |b_{ij}| = 1$ for any i, j such that $i \neq j$. Therefore,

$$\begin{aligned}
\mathrm{Var}_\rho(A) &= \mathrm{Tr}(\rho A_0) \\
&= \frac{1}{2}\sum_{i,j}(\lambda_i + \lambda_j)a_{ij}a_{ji} \\
&= \frac{1}{2}((\lambda_1 + \lambda_2) + (\lambda_2 + \lambda_1)) \\
&= 1,
\end{aligned}$$

$$\begin{aligned}
\mathcal{C}^{\tilde{f}}_\rho(A_0) &= \sum_{i,j} m_{\tilde{f}}(\lambda_i, \lambda_j)a_{ij}a_{ji} \\
&= (m_{\tilde{f}}(\lambda_1, \lambda_2) + m_{\tilde{f}}(\lambda_1, \lambda_2)) \\
&= 2m_{\tilde{f}}(\lambda_1, \lambda_2),
\end{aligned}$$

$$I^f_\rho(A) = \mathrm{Var}_\rho(A) - \mathcal{C}^{\tilde{f}}_\rho(A_0) = 1 - 2m_{\tilde{f}}(\lambda_1, \lambda_2).$$

By the same reasoning,

$$\begin{aligned} \mathrm{Var}_\rho(B) &= 1 \\ \mathcal{C}_\rho^{\tilde f}(B_0) &= 2m_{\tilde f}(\lambda_1,\lambda_2) \\ I_\rho^f(B) &= 1-2m_{\tilde f}(\lambda_1,\lambda_2). \end{aligned}$$

Moreover, by direct calculation, one has that

$$\frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2 = (\lambda_1-\lambda_2)^2.$$

Now, recall that, since $m_{\tilde f}$ is a mean (and because of Corollary 3.11) one has for any $f \in \mathcal{F}_{op}$

$$\lambda_1 > m_{\tilde f}(\lambda_1,\lambda_2) > \lambda_2 > 0,$$

$$1-2m_{\tilde f}(\lambda_1,\lambda_2) = (\lambda_1+\lambda_2)-2m_{\tilde f}(\lambda_1,\lambda_2) \geq 0.$$

Hence, the following inequalities are equivalent

$$\begin{aligned} I_\rho^f(A)\,I_\rho^f(B) &< \frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2 \\ (1-2m_{\tilde f}(\lambda_1,\lambda_2))^2 &< (\lambda_1-\lambda_2)^2 \\ (\lambda_1+\lambda_2)-2m_{\tilde f}(\lambda_1,\lambda_2) &< \lambda_1-\lambda_2 \\ 2\lambda_2 &< 2m_{\tilde f}(\lambda_1,\lambda_2) \\ \lambda_2 &< m_{\tilde f}(\lambda_1,\lambda_2), \end{aligned}$$

and so we get the conclusion. $\square$

Note that

$$\frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2 \geq I_\rho^f(A)\,I_\rho^f(B)$$

is obviously false, in general: if one takes $A = B$, the left side is zero and the right side could be positive at the same time.

A similar argument applies to the inequality

$$\frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2 \geq I_\rho^f(A)\,I_\rho^f(B) - \left|\mathrm{Re}\left\{\mathrm{Corr}_\rho^f(A,B)\right\}\right|^2 = \left(\frac{f(0)}{2}\mathrm{Area}_f(i[\rho,A],i[\rho,B])\right)^2;$$

indeed, one may choose ρ, A, B such that $[A,B]=0$ while $[\rho,A],[\rho,B]$ are not proportional, so that they span a positive area.

We may conclude that the Heisenberg and Schrödinger uncertainty principles

$$\mathrm{Var}_\rho(A)\,\mathrm{Var}_\rho(B) \;\geq\; \frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2,$$

$$\mathrm{Area}_\rho^{\mathrm{Cov}^s}(A,B) \;\geq\; \frac{1}{2}|\mathrm{Tr}(\rho[A,B])|,$$

cannot be deduced from the uncertainty principles

$$\mathrm{Var}_\rho(A)\,\mathrm{Var}_\rho(B) \;\geq\; I_\rho^f(A)\cdot I_\rho^f(B),$$

$$\mathrm{Area}_\rho^{\mathrm{Cov}^s}(A,B) \;\geq\; \frac{f(0)}{2}\cdot \mathrm{Area}_\rho^f(i[\rho,A],i[\rho,B]),$$

and vice versa.

The above described mistake appeared several times in the literature (see Theorem 2 in [16], Theorem 2 in [21], Theorem 1 in [19] and Note 1, Section 3.2 in [9]). It can be helpful to explain its origin, again along the lines of [13] (see also [28]).

We have seen that

$$\frac{1}{2i}\mathrm{Tr}(\rho[A,B]) = \frac{1}{2i}(\mathrm{Corr}_\rho^f(A,B) - \mathrm{Corr}_\rho^f(B,A)) = \mathrm{Im}(\mathrm{Corr}_\rho^f(A,B))$$

and therefore

$$\frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2 = |\mathrm{Im}(\mathrm{Corr}_\rho^f(A,B))|^2 \leq |\mathrm{Corr}_\rho^f(A,B)|^2.$$

If there were a Cauchy-Schwartz type estimate

$$|\mathrm{Corr}_\rho^f(A,B)|^2 \leq \mathrm{Corr}_\rho^f(A,A)\cdot \mathrm{Corr}_\rho^f(B,B) \tag{11.1}$$

using, for example, Theorem 9.4 one would get a refined Heisenberg uncertainty principle in the form

$$\mathrm{Var}_\rho(A)\cdot \mathrm{Var}_\rho(B) \geq I_\rho^f(A)\cdot I_\rho^f(B) \geq \frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2.$$

By Theorem 10.1 we know that this is impossible. The wrong point is the Cauchy-Schwartz estimate (11.1), which is false. This depends on the following facts. The sesquilinear form

$$\mathrm{Corr}_\rho^f(X,Y) := \mathrm{Tr}(\rho X^\dagger Y) - \mathrm{Tr}(X^\dagger \cdot m_{\tilde f}(L_\rho,R_\rho)(Y))$$

on the complex space M_n is not positive (see p. 632 in [13]). On the other hand, $\mathrm{Corr}_\rho^f(A,B)$ is not a real form on the real space $M_{n,sa}$: also in this case one cannot prove the desired Cauchy-Schwartz inequality. The best one can have is a Cauchy-Schwartz estimate only for the (real) positive bilinear form $\mathrm{Re}\{\mathrm{Corr}_\rho^f(A,B)\}$ on $M_{n,sa}$ (see p.643 in [13] and [20]). This would imply simply

$$\left(\frac{f(0)}{2}\cdot \mathrm{Area}_\rho^f(i[\rho,A],i[\rho,B])\right)^2 = I_\rho^f(A)\,I_\rho^f(B) - \left|\mathrm{Re}\left\{\mathrm{Corr}_\rho^f(A,B)\right\}\right|^2 \geq 0.$$

11 Not faithful states and pure states

We discuss, now, the general case $\rho \geq 0$.

Proposition 11.1. *The function* $m_{\tilde{f}} : (0,\infty) \times (0,\infty) \to (0,\infty)$ *has a continuous extension to* $[0,\infty) \times [0,\infty)$.

Proof. If f is regular then, for example,

$$\lim_{(x,y)\to(0,y_0)} m_{\tilde{f}}(x,y) = \frac{y_0}{2} - \frac{f(0)y_0^2}{2y_0 f(0)} = 0.$$

If f is not regular then $m_{\tilde{f}}(x,y) = \frac{x+y}{2}$ and we are done (see [9]). □

The definition of f-correlation still makes sense and the inequality of Theorem 6.5

$$\mathrm{Var}_\rho(A)\,\mathrm{Var}_\rho(B) - |\mathrm{Re}\{\mathrm{Cov}_\rho(A,B)\}|^2 \geq I^f_\rho(A)\, I^f_\rho(B) - \left|\mathrm{Re}\left\{\mathrm{Corr}^f_\rho(A,B)\right\}\right|^2$$

holds by continuity for arbitrary (not necessarily faithful) states.

In what follows we study the pure state case.

Corollary 11.2. *If* $s : [0,+\infty) \times [0,+\infty) \to \mathbb{R}$ *is a continuous function, and* ρ *is a pure state, then*

$$s(L_\rho, R_\rho)(A) = \rho A \rho.$$

Proof. Consequence of Corollary 6.4. □

Lemma 11.3. *If* ρ *is pure, then* $\mathrm{Tr}((\rho A\rho)(\rho B\rho)) = \mathrm{Tr}(\rho A\rho) \cdot \mathrm{Tr}(\rho B\rho)$.

Proof. Suppose for simplicity that $\rho = \mathrm{diag}(1, 0, ..., 0)$ (the general case follows easily from this). Then $\rho A\rho = \mathrm{diag}(A_{11}, 0, ..., 0)$ and the same holds for B. Therefore $(\rho A\rho)(\rho B\rho) = \mathrm{diag}(A_{11}B_{11}, 0, ..., 0)$. This implies

$$\mathrm{Tr}((\rho A\rho)(\rho B\rho)) = A_{11}B_{11} = \mathrm{Tr}(\rho A\rho) \cdot \mathrm{Tr}(\rho B\rho).$$

□

Lemma 11.4. *If* ρ *is pure, then*

$$\mathrm{Tr}(m_f(L_\rho, R_\rho)(A)B) = \mathrm{Tr}(\rho A) \cdot \mathrm{Tr}(\rho B).$$

Proof. By Corollary 11.2 one has

$$m_f(L_\rho, R_\rho)(A) = \rho A \rho$$

and therefore

$$\begin{aligned}\mathrm{Tr}(m_f(L_\rho, R_\rho)(A)B) &= \mathrm{Tr}(\rho A \rho B) \\ &= \mathrm{Tr}((\rho A \rho)(\rho B \rho)) \\ &= \mathrm{Tr}(\rho A \rho) \cdot \mathrm{Tr}(\rho B \rho) \\ &= \mathrm{Tr}(\rho A) \cdot \mathrm{Tr}(\rho B).\end{aligned}$$

□

Corollary 11.5. *If ρ is pure, then*

$$\mathcal{C}_\rho^f(A_0, B_0) = \mathrm{Tr}(m_f(L_\rho, R_\rho)(A_0)B_0) = \mathrm{Tr}(\rho A_0) \cdot \mathrm{Tr}(\rho B_0) = 0.$$

Proposition 11.6. *If ρ is pure, then*

$$\mathrm{Corr}_\rho^f(A, B) = \mathrm{Cov}_\rho(A, B) \qquad\qquad \forall f \in \mathcal{F}_{op}.$$

Proof. Immediate from the above Corollary and Proposition 7.9

□

The case $I_\rho^f(A) = \mathrm{Var}_\rho(A)$ was proved by Hansen in Theorem 3.8 p.16 in [9].

Therefore, on pure states we have the equalities

$$\begin{aligned}\mathrm{Var}_\rho(A)\,\mathrm{Var}_\rho(B) - |\mathrm{Re}\{\mathrm{Cov}_\rho(A,B)\}|^2 &= I_\rho^f(A)\, I_\rho^f(B) - \left|\mathrm{Re}\left\{\mathrm{Corr}_\rho^f(A,B)\right\}\right|^2, \\ \mathrm{Var}_\rho(A)\,\mathrm{Var}_\rho(B) &= I_\rho^f(A)\, I_\rho^f(B).\end{aligned}$$

This implies that, if a sequence of faithful states D_n converges to the pure state ρ, then the limit

$$\begin{aligned}\lim_{n\to+\infty}\left(\frac{f(0)}{2}\cdot \mathrm{Area}_{D_n}^f(i[D_n,A], i[D_n,B])\right)^2 &= \lim_{n\to+\infty} I_{D_n}^f(A)\, I_{D_n}^f(B) - \left|\mathrm{Re}\left\{\mathrm{Corr}_{D_n}^f(A,B)\right\}\right|^2 \\ &= I_\rho^f(A)\, I_\rho^f(B) - \left|\mathrm{Re}\left\{\mathrm{Corr}_\rho^f(A,B)\right\}\right|^2 \\ &= \mathrm{Var}_\rho(A)\,\mathrm{Var}_\rho(B) - |\mathrm{Re}\{\mathrm{Cov}_\rho(A,B)\}|^2\end{aligned}$$

does not depend on f.

This result has an interesting alternative explanation, using a theorem by Petz and Sudar that describes the possible extension of quantum Fisher information to pure states (see [24]). We devote the rest of the section to explain this phenomenon.

Let $M_n^0 = M_n^0(\mathbb{C})$ be the set of faithful states whose eigenvalues are all distinct. Recall that the pure states are identified with $\mathbb{C}P^{n-1}$, the complex projective space. On $\mathbb{C}P^{n-1}$ one has a natural metric, the Fubini-Study metric (denoted by $\langle \cdot, \cdot \rangle_{\rho,FS}$). We denote by D the elements of M_n^0 and by ρ the elements of $\mathbb{C}P^{n-1}$. We can define a projection $\pi : M_n^0 \to \mathbb{C}P^{n-1}$ as follows: $\pi(D) \in \mathbb{C}P^{n-1}$ is the pure state associated to the one-dimensional eigenspace corresponding to the largest eigenvalue of $D \in M_n^0$. With this definition, $\pi : M_n^0 \to \mathbb{C}P^{n-1}$ is a smooth fiber bundle. The structure group is $U(1) \times U(n-1)$ (where $U(k)$ is the group of $k \times k$ unitary matrices). The fiber space is $\pi^{-1}(e)$ where e is the ray generated by the vector $(1, 0, ..., 0) \in \mathbb{C}^n$. Now, fix a monotone metric $\langle \cdot, \cdot \rangle_{D,f}$. We denote by $T_D\pi$ the differential of π at D and let H_D be the orthogonal complement of $\ker(T_D\pi)$ with respect to $\langle \cdot, \cdot \rangle_{D,f}$. Since $T_D\pi$ is surjective, the restriction of $T_D\pi$ gives a linear isomorphism between H_D and $T_{\pi(D)}\mathbb{C}P^{n-1}$. For any tangent vector $A \in T_{\pi(D)}\mathbb{C}P^{n-1}$ there is a unique "lift" $A_D \in H_D \subset T_D(M_n^0)$ such that $(T_D\pi)(A_D) = A$.

Definition 11.7. [24] We say that the sequence $D_n \in M_n^0$ radially converges to $\rho \in \mathbb{C}P^{n-1}$ if $D_n \to \rho$ as density matrices in M^n and $\pi(D_n) = \rho$, $\forall n \in \mathbb{N}$.

Definition 11.8. [24] A metric k on $\mathbb{C}P^{n-1}$ is a radial extension of a metric g on M_n^0 if for any sequence $D_n \in M_n^0$, radially convergent to a point $\rho \in \mathbb{C}P^{n-1}$, and for any tangent vectors $A, B \in T_\rho\mathbb{C}P^{n-1}$, one has

$$\lim_{n\to+\infty} g(A_{D_n}, B_{D_n}) = k(A, B).$$

Theorem 11.9. *[24]*

A monotone metric admits a radial extension if and only if it is regular, namely iff $f(0) \neq 0$. In this case the associated extension is just a multiple of the Fubini-Study metric according to the formula

$$\lim_{n\to+\infty} \langle A_{D_n}, B_{D_n} \rangle_{D_n,f} = \frac{1}{2f(0)} \langle A, B \rangle_{\rho,FS}.$$

Lemma 11.10. *[25]*

With the above definition,

$$\pi(D) = \rho \qquad \Longrightarrow \qquad [D, A] = ([\rho, A])_D,$$

namely, the lift of commutator is the commutator of the lift.

This implies the following result.

Proposition 11.11. *If $D_n \to \rho$ radially then*

$$\lim_{n\to+\infty} f(0) \cdot \mathrm{Area}^f_{D_n}(i[D_n, A], i[D_n, B]) = \frac{1}{2} \cdot \mathrm{Area}^{FS}_\rho(i[\rho, A], i[\rho, B]).$$

Hence, we have obtained the limit behavior by a totally different argument.

12 Optimality of an improvement for Heisenberg uncertainty principle

The following result has been proved by Park in [22] and indipendently by Luo in [18].

Theorem 12.1. *If* $g_0(x) = \sqrt{x}$ *then*

$$\mathrm{Var}_\rho(A) \cdot \mathrm{Var}_\rho(B) \geq \mathcal{C}_\rho^{g_0}(A_0)\mathcal{C}_\rho^{g_0}(B_0) + \frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2.$$

Note that the term $\mathcal{C}_\rho^{g_0}(A_0)\mathcal{C}_\rho^{g_0}(B_0)$ disappears for pure states. We prove that the above result is the best one can have considering functions $f \in \mathcal{F}_{op}$.

Theorem 12.2. *For any* $f \in \mathcal{F}_{op}$ *we have*

$$\mathrm{Var}_\rho(A) \cdot \mathrm{Var}_\rho(B) \geq \mathcal{C}_\rho^f(A_0)\mathcal{C}^f{}_\rho(B_0) + \frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2 \qquad \Longleftrightarrow \qquad f(x) \leq \sqrt{x}.$$

Proof. We have

$$f(x) \leq g(x) \quad \Longrightarrow \quad m_f(x,y) \leq m_g(x,y) \quad \Longrightarrow \quad \mathcal{C}_\rho^f(A_0) \leq \mathcal{C}_\rho^g(A_0)$$

and therefore if $f(x) \leq \sqrt{x}$ we are done.

If $f(x_0) > \sqrt{x_0}$ for a certain x_0 we produce a counterexample. To this end, we do the same we did in the proof of Theorem 10.

Consider again $\lambda_1 > \lambda_2 > 0$, $\lambda_1 + \lambda_2 = 1$ and

$$\rho = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We have calculated

$$\mathrm{Var}_\rho(A) = 1 \qquad \mathcal{C}_\rho^f(A_0) = 2m_f(\lambda_1,\lambda_2)$$

$$\mathrm{Var}_\rho(B) = 1 \qquad \mathcal{C}_\rho^f(B_0) = 2m_f(\lambda_1,\lambda_2),$$

$$\mathrm{Tr}(\rho[A,B]) = (\lambda_1 - \lambda_2)^2.$$

In this case the inequality

$$\mathrm{Var}_\rho(A) \cdot \mathrm{Var}_\rho(B) \geq \mathcal{C}_\rho^f(A_0)\mathcal{C}_\rho^f(B_0) + \frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2$$

reads as

$$1 \cdot 1 \geq 2m_f(\lambda_1,\lambda_2) \cdot 2m_f(\lambda_1,\lambda_2) + (\lambda_1 - \lambda_2)^2,$$

that is,

$$1 \geq 4(m_f(\lambda_1, \lambda_2))^2 + (\lambda_1 - \lambda_2)^2$$

or

$$1 \geq 4\left(\lambda_2 f\left(\frac{\lambda_1}{\lambda_2}\right)\right)^2 + (\lambda_1 - \lambda_2)^2.$$

For $g_0(x) = \sqrt{x}$ we have

$$4\left(\lambda_2 g_0\left(\frac{\lambda_1}{\lambda_2}\right)\right)^2 + (\lambda_1 - \lambda_2)^2 = 1.$$

Therefore, if for some $x_0 \neq 1$ we have $f(x_0) > \sqrt{x_0}$ then for $\frac{\lambda_1}{\lambda_2} = x_0$

$$\begin{aligned}
\mathcal{C}_\rho^f(A_0)\mathcal{C}_\rho^f(B_0) + \frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2 &= 4\left(\lambda_2 f\left(\frac{\lambda_1}{\lambda_2}\right)\right)^2 + (\lambda_1 - \lambda_2)^2 \\
&> 4\left(\lambda_2 \sqrt{\left(\frac{\lambda_1}{\lambda_2}\right)}\right)^2 + (\lambda_1 - \lambda_2)^2 \\
&= 1 \\
&= \mathrm{Var}_\rho(A) \cdot \mathrm{Var}_\rho(B)
\end{aligned}$$

that is, the inequality is false.

□

Acknowledgements

It is a pleasure to thank Frank Hansen for sending us the preprint [9].

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