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Bartholdi Zeta Functions for Periodic Simple Graphs

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ABSTRACT. The definition of the Bartholdi zeta function is extended to the case of infinite periodic graphs. By means of the analytic determinant for semifinite von Neumann algebras studied by the authors in [**7**], a determinant formula and functional equations are obtained for this zeta function.

0. Introduction

The zeta function associated to a finite graph by Ihara, Sunada, Hashimoto and others, combines features of Riemann's zeta function, Artin L-functions, and Selberg's zeta function, and may be viewed as an analogue of the Dedekind zeta function of a number field [**2, 8, 9, 10, 11, 17, 18**]. It is defined by an Euler product over proper primitive cycles of the graph. Extensions of this theory to infinite graphs have also been considered [**4, 5, 6, 7**].

In [**1**], Bartholdi introduced a generalization of such a function with a further parameter u, which coincides with the Ihara zeta function for $u = 0$, and gives the Euler product on all primitive cycles for $u = 1$. He also showed that some results for the Ihara zeta function extend to this new zeta function.

Further results and generalizations of the Bartholdi zeta function are contained, for example, in [**3, 12, 13, 14**].

Our aim here is to extend the notion of Bartholdi zeta function to infinite covering graphs, and then to prove a determinant formula in terms of the analytic determinant studied in [**7**]. Some functional equations are also proved. Analogous methods can be used to define and study Bartholdi zeta functions in the context of reference [**5**], that is, of self-similar fractal graphs. The paper is organized as follows: in the first section, we recall the basic notions concerning simple graphs and their paths, together with the definitions of the classic operators on graphs, such as the adjacency operator; we then define the von Neumann algebra of Γ-periodic operators and the canonical trace on it. Also, some combinatorial equalities are proved. Section 2 is devoted to the definition and analyticity properties of the

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Bartholdi zeta function, while the determinant formula and the functional equations are proved in Sections 3 and 4, respectively.

1. Preliminaries

The Bartholdi zeta function is defined by means of equivalence classes of primitive cycles. Therefore, we need to introduce some terminology from graph theory, following [**17**] with some modifications.

A simple graph $X = (VX, EX)$ is a collection VX of objects, called vertices, and a collection EX of unordered pairs of distinct vertices, called edges. The edge $e = \{u, v\}$ is said to join the vertices u, v, while u and v are said to be adjacent, which is denoted $u \sim v$. A path (of length m) in X from $v_0 \in VX$ to $v_m \in VX$, is $(v_0, \ldots, v_m)$, where $v_{i+1} \sim v_i$, $i = 0, ..., m-1$. A path is closed if $v_m = v_0$. Denote by $\mathcal{C}$ the set of closed paths. A graph is connected if there is a path between any pair of distinct vertices.

Definition 1.1 (Types of closed paths).
(i) A path $C = (v_0, \ldots, v_m)$ in X has backtracking if $v_{i-1} = v_{i+1}$, for some $i \in \{1, \ldots, m-1\}$. We also say that C has a bump at v_i. Then, the bump count $bc(C)$ of C is the number of bumps in C. Moreover, if C is a closed path of length m, the cyclic bump count is $cbc(C) := |\{i \in \mathbb{Z}_m : v_{i-1} = v_{i+1}\}|$, where the indices are considered in $\mathbb{Z}_m$, and $\mathbb{Z}_m$ is the cyclic group on m elements.
(ii) A closed path is primitive if it is not obtained by going $r \geq 2$ times around some other closed path.
(iii) A closed path $C = (v_0, \ldots, v_m = v_0)$ has a tail if there is $k \in \{1, \ldots, [m/2]-1\}$ such that $v_j = v_{m-j}$, for $j = 1, \ldots, k$. Denote by $\mathcal{C}^{\text{tail}}$ the set of closed paths with tail, and by $\mathcal{C}^{\text{notail}}$ the set of tail-less closed paths. Observe that $\mathcal{C} = \mathcal{C}^{\text{tail}} \cup \mathcal{C}^{\text{notail}}$, $\mathcal{C}^{\text{tail}} \cap \mathcal{C}^{\text{notail}} = \emptyset$.

Let Γ be a countable discrete subgroup of automorphisms of X, which acts freely on X (*i.e.* any $\gamma \in \Gamma$, $\gamma \neq id$ doesn't have fixed points), and with finite quotient $B := X/\Gamma$ (observe that B needn't be a simple graph). Denote by $\mathcal{F} \subset VX$ a set of representatives for VX/Γ, the vertices of the quotient graph B. Let us define a unitary representation of Γ on $\ell^2(VX)$ by $(\lambda(\gamma)f)(x) := f(\gamma^{-1}x)$, for $\gamma \in \Gamma$, $f \in \ell^2(VX)$, $x \in V(X)$. Then the von Neumann algebra $\mathcal{N}(X, \Gamma) := \{\lambda(\gamma) : \gamma \in \Gamma\}'$ of bounded operators on $\ell^2(VX)$ commuting with the action of Γ inherits a trace given by $Tr_\Gamma(T) = \sum_{x \in \mathcal{F}} T(x, x)$, for $T \in \mathcal{N}(X, \Gamma)$.

Let us denote by A the adjacency matrix of X, that is,

$$a_{ij} = \begin{cases} 1 & \{v_i, v_j\} \in EX \\ 0 & \text{otherwise.} \end{cases}$$

Then (by [**16**], [**15**]) $\|A\| \leq d := \sup_{v \in VX} \deg(v) < \infty$, and it is easy to see that $A \in \mathcal{N}(X, \Gamma)$.

For any $m \in \mathbb{N}$, $u \in \mathbb{C}$, let us denote by $A_m(u)(x, y) := \sum_P u^{bc(P)}$, where the (finite) sum is over all paths P in X, of length m, with initial vertex x and terminal vertex y, for $x, y \in VX$. Then $A_1 = A$. Let $A_0 := I$ and $Q := \text{diag}(\deg(v_1) - 1, \deg(v_2) - 1, \ldots)$. Finally, let $\mathcal{U} \subset \mathbb{C}$ be a bounded set containing $\{0, 1\}$, and denote by $M(\mathcal{U}) := \sup_{u \in \mathcal{U}} \max\{|u|, |1-u|\} \geq 1$, and $\alpha(\mathcal{U}) := \frac{d + \sqrt{d^2 + 4M(\mathcal{U})(d - 1 + M(\mathcal{U}))}}{2}$. Then

LEMMA 1.2.
(i) $A_2(u) = A^2 - (1-u)(Q+I) \in \mathcal{N}(X,\Gamma)$,
(ii) for $m \geq 3$, $A_m(u) = A_{m-1}(u)A - (1-u)A_{m-2}(Q+uI) \in \mathcal{N}(X,\Gamma)$,
(iii) $\sup_{u\in\mathcal{U}} \|A_m(u)\| \leq \alpha(\mathcal{U})^m$, *for* $m \geq 0$.

PROOF. *(i)* If $x = y$, then $A_2(u)(x,x) = \deg(x)u = (Q+I)(x,x)u$ because there are $\deg(x)$ closed paths of length 2 starting at x, whereas $A^2(x,x) = \deg(x) = (Q+I)(x,x)$, so that $A_2(u)(x,x) = A^2(x,x) - (1-u)(Q+I)(x,x)$. If $x \neq y$, then $A^2(x,y)$ is the number of paths of length 2 from x to y, so $A_2(u)(x,y) = A^2(x,y) = A^2(x,y) - (1-u)(Q+I)(x,y)$.

(ii) For $x, y \in VX$, consider all the paths $P = (v_0, \ldots, v_m)$ of length m, with $v_0 = x$ and $v_m = y$. They can also be considered as obtained from a path P' of length $m-2$ going from $x \equiv v_0$ to v_{m-2}, followed by a path of length 2 from v_{m-2} to $y \equiv v_m$. There are three types of such paths: *(a)* those P for which $y \equiv v_m \neq v_{m-2}$, so that $bc(P) = bc(P')$; *(b)* those P for which $y \equiv v_m = v_{m-2}$, but $v_{m-1} \neq v_{m-3}$, so that $bc(P) = bc(P') + 1$; *(c)* those P for which $y \equiv v_m = v_{m-2}$ and $v_{m-1} = v_{m-3}$, so that $bc(P) = bc(P') + 2$.

Therefore, the terms corresponding to those three types in $A_m(u)(x,y)$ are $u^{bc(P')}$, $u^{bc(P')+1}$, and $u^{bc(P')+2}$, respectively.

On the other hand, the sum $\sum_{z\in VX} A_{m-1}(u)(x,z)A(z,y)$ assigns, to those three types, respectively the values $u^{bc(P')}$, $u^{bc(P')}$, and $u^{bc(P')+1}$.

Therefore $A_m(u)(x,y) = \sum_{z\in VX} A_{m-1}(u)(x,z)A(z,y) + A_{m-2}(u)(x,y)(\deg(y)-1)(u-1) + A_{m-2}(u)(x,y)(u^2-u)$, and the statement follows.

(iii) We have $\|A_1(u)\| = \|A\| \leq d$, $\|A_2(u)\| \leq d^2 + M(\mathcal{U})d \leq \alpha(\mathcal{U})^2$, and $\|A_m(u)\| \leq d\|A_{m-1}(u)\| + M(\mathcal{U})(d-1+M(\mathcal{U}))\|A_{m-2}(u)\|$, from which the claim follows by induction. □

LEMMA 1.3. *Denote by* $t_m(u) := \sum_{x\in\mathcal{F}} \sum_{C=(x,\ldots)\in\mathcal{C}_m^{tail}} u^{bc(C)}$. *Then*
(i) $t_m(u)$ *is a polynomial in* u,
(ii) $t_1(u) = 0$, $t_2(u) = \sum_{x\in\mathcal{F}} \deg(x)u = uTr_\Gamma(Q+I)$, $t_3(u) = 0$,
(iii) for $m \geq 4$, $t_m(u) = Tr_\Gamma\big((Q-(1-2u)I)A_{m-2}(u)\big) + (1-u)^2 t_{m-2}(u)$,
(iv) $t_m(u) = Tr_\Gamma\Big((Q-(1-2u)I)\sum_{j=1}^{[\frac{m-1}{2}]}(1-u)^{2j-2}A_{m-2j}(u)\Big) + \delta_{even}(m)u(1-u)^{m-2}Tr_\Gamma(Q+I)$, *where* $\delta_{even}(m) = \begin{cases} 1 & m \text{ is even} \\ 0 & m \text{ is odd.} \end{cases}$
(v) if $\mathcal{U} \subset \mathbb{C}$ *is a bounded set containing* $\{0,1\}$, *then* $\sup_{u\in\mathcal{U}} |t_m(u)| \leq 4m\alpha(\mathcal{U})^m|\mathcal{F}|$.

PROOF. *(i)* and *(ii)* are easy.

(iii) Indeed, we have

$$\begin{aligned} t_m(u) &= \sum_{x\in\mathcal{F}} \sum_{C=(x,\ldots)\in\mathcal{C}_m^{\mathrm{tail}}} u^{bc(C)} \\ &= \sum_{x\in\mathcal{F}} \sum_{y\sim x} \sum_{C=(x,y,\ldots)\in\mathcal{C}_m^{\mathrm{tail}}} u^{bc(C)} \\ &= \sum_{y\in\mathcal{F}} \sum_{x\sim y} \sum_{C=(x,y,\ldots)\in\mathcal{C}_m^{\mathrm{tail}}} u^{bc(C)}, \end{aligned}$$

where the last equality follows from the fact that $bc(C)$ is Γ-invariant, and we can choose $\gamma \in \Gamma$ for which the second vertex y of γC is in $\mathcal{F}$. A path C in the last set goes from x to y, then over a closed path $D = (y, v_1, \dots, v_{m-3}, y)$ of length $m-2$, and then back to x. There are two kinds of closed paths D at y: those with tails and those without.

Case 1 : D does not have a tail.
Then C can be of two types: (a) C_1, where $x \neq v_1$ and $x \neq v_{m-3}$; (b) C_2, where $x = v_1$ or $x = v_{m-3}$. Hence, $bc(C_1) = bc(D)$, and $bc(C_2) = bc(D) + 1$, and there are $\deg(y) - 1$ possibilities for x to be adjacent to y in C_1, and 2 possibilities in C_2.

Case 2 : D has a tail.
Then C can be of two types: (c) C_3, where $v_1 = v_{m-3} \neq x$; (d) C_4, where $v_1 = v_{m-3} = x$. Hence, $bc(C_3) = bc(D)$, and $bc(C_4) = bc(D) + 2$, and there are $\deg(y)$ possibilities for x to be adjacent to y in C_3, and 1 possibility in C_4.
Therefore,

$$
\begin{aligned}
&\sum_{x\sim y} \sum_{C=(x,y,\dots)\in\mathcal{C}_m^{\text{tail}}} u^{bc(C)} \\
&\quad = (\deg(y)-1) \sum_{D=(y,\dots)\in\mathcal{C}_{m-2}^{\text{notail}}} u^{bc(D)} + 2u \sum_{D=(y,\dots)\in\mathcal{C}_{m-2}^{\text{notail}}} u^{bc(D)} \\
&\qquad + \deg(y) \sum_{D=(y,\dots)\in\mathcal{C}_{m-2}^{\text{tail}}} u^{bc(D)} + u^2 \sum_{D=(y,\dots)\in\mathcal{C}_{m-2}^{\text{tail}}} u^{bc(D)} \\
&\quad = (\deg(y)-1+2u) \sum_{D=(y,\dots)\in\mathcal{C}_{m-2}} u^{bc(D)} + (1-2u+u^2) \sum_{D=(y,\dots)\in\mathcal{C}_{m-2}^{\text{tail}}} u^{bc(D)},
\end{aligned}
$$

so that

$$
\begin{aligned}
t_m(u) &= \sum_{y\in\mathcal{F}} (Q(y,y) - 1 + 2u)\cdot A_{m-2}(u)(y,y) \\
&\quad + (1-u)^2 \sum_{y\in\mathcal{F}} \sum_{D=(y,\dots)\in\mathcal{C}_{m-2}^{\text{tail}}} u^{bc(D)} \\
&= Tr_\Gamma\big((Q-(1-2u)I)A_{m-2}(u)\big) + (1-u)^2 t_{m-2}(u).
\end{aligned}
$$

(iv) Follows from (iii).

(v) Let us first observe that $M(\mathcal{U}) < \alpha(\mathcal{U})$, so that, from (iv) we obtain, with $\alpha := \alpha(\mathcal{U})$, $M := M(\mathcal{U})$,

$$
\begin{aligned}
|t_m(u)| &\leq \|Q-(1-2u)I\|\cdot|\mathcal{F}|\cdot \sum_{j=1}^{[\frac{m-1}{2}]} |1-u|^{2j-2}\|A_{m-2j}(u)\| + |u||1-u|^{m-2}d|\mathcal{F}| \\
&\leq |\mathcal{F}|(d-2+2M) \sum_{j=1}^{[\frac{m-1}{2}]} M^{2j-2}\alpha^{m-2j} + |\mathcal{F}|M^{m-1}d \\
&\leq |\mathcal{F}|(d-2+2M)\Big[\frac{m-1}{2}\Big]\alpha^{m-2} + |\mathcal{F}|M^{m-1}d \\
&\leq |\mathcal{F}|\Big(\Big[\frac{m-1}{2}\Big]3\alpha^{m-1} + \alpha^m\Big) \leq 4m\alpha^m|\mathcal{F}|.
\end{aligned}
$$

□

To state the next result, we need some preliminary notions.

Definition 1.4 (Cycles). Given closed paths $C = (v_0, \dots, v_m = v_0)$, $D = (w_0, \dots, w_m = w_0)$, we say that C and D are equivalent, and write $C \sim_o D$, if there is k such that $w_j = v_{j+k}$, for all j, where the addition is taken mod m, that is, the origin of D is shifted k steps w.r.t. the origin of C. The equivalence class of C is denoted $[C]_o$. An equivalence class is also called a cycle. Therefore, a closed path is just a cycle with a specified origin.

Denote by $\mathcal{K}$ the set of cycles, and by $\mathcal{P} \subset \mathcal{K}$ the subset of primitive cycles.

We need to introduce an equivalence relation between cycles.

Definition 1.5 (Equivalence relation between cycles). Given C, $D \in \mathcal{K}$, we say that C and D are Γ-equivalent, and write $C \sim_\Gamma D$, if there is an isomorphism $\gamma \in \Gamma$ such that $D = \gamma(C)$. We denote by $[\mathcal{K}]_\Gamma$ the set of Γ-equivalence classes of cycles, and analogously for the subset $\mathcal{P}$.

For the purposes of the next result, for any closed path $D = (v_0, \dots, v_m = v_0)$, we also denote v_j by $v_j(D)$.

Let us now assume that C is a primitive cycle of length m. Then the stabilizer of C in Γ is the subgroup $\Gamma_C = \{\gamma \in \Gamma : \gamma(C) = C\}$ or, equivalently, $\gamma \in \Gamma_C$ if there exists $p(\gamma) \in \mathbb{Z}_m$ such that, for any choice of the origin of C, $v_j(\gamma C) = v_{j-p}(C)$, for any j. Let us observe that $p(\gamma)$ is a group homomorphism from Γ_C to $\mathbb{Z}_m$, which is injective because Γ acts freely. As a consequence, $|\Gamma_C|$ divides m.

Definition 1.6. Let $C \in \mathcal{P}$ and define $\nu(C) := \dfrac{|C|}{|\Gamma_C|}$. If $C = D^k \in \mathcal{K}$, where $D \in \mathcal{P}$, define $\nu(C) = \nu(D)$. Observe that $\nu(C)$ only depends on $[C]_\Gamma \in [\mathcal{K}]_\Gamma$.

Lemma 1.7. *Let us set* $N_m(u) := \sum_{[C]\in[\mathcal{K}_m]_\Gamma} \nu(C) u^{cbc(C)}$. *Then*
(i) $u \in \mathbb{C} \mapsto N_m(u) \in \mathbb{C}$ *is a holomorphic function,*
(ii) $N_m(u) = Tr_\Gamma(A_m(u)) - (1-u)t_m(u)$,
(iii) if $\mathcal{U} \subset \mathbb{C}$ *is a bounded set containing* $\{0,1\}$, *then* $\sup_{u\in\mathcal{U}} |N_m(u)| \le 5m\alpha(\mathcal{U})^{m+1}|\mathcal{F}|$.

Proof. *(ii)* Let us assume that $[C]_\Gamma$ is an equivalence class of primitive cycles in $[\mathcal{P}_m]_\Gamma$, and consider the set $\mathcal{V}$ of all primitive closed paths with the origin in $\mathcal{F}$ and representing $[C]_\Gamma$. If C is such a representative, any other representative can be obtained in this way: choose $k \in \mathbb{Z}_m$, let $\gamma(k)$ be the (unique) element in Γ for which $\gamma(k)v_k(C) \in \mathcal{F}$, and define C_k as

$$v_j(C_k) = \gamma(k)v_{j+k}(C),\ j \in \mathbb{Z}_m.$$

If we want to count the elements of $\mathcal{V}$, we should know how many of the elements C_k above coincide with C. For this to happen, γ should clearly be in the stabilizer of the cycle $[C]_o$. Conversely, for any $\gamma \in \Gamma_C$, there exists $p = p(\gamma) \in \mathbb{Z}_m$ such that $\gamma v_j(C) = v_{j-p}(C)$, therefore $\gamma = \gamma(p)$. As a consequence, $v_j(C_{p(\gamma)}) = \gamma(p)v_{j+p}(C) = v_j(C)$, so that $C_{p(\gamma)} = C$. We have proved that the cardinality of $\mathcal{V}$

is equal to $\nu(C)$. The proof for a non-primitive cycle is analogous. Therefore,

$$\begin{aligned}
N_m(u) &= \sum_{[C]\in[\mathcal{K}_m]_\Gamma} \nu(C) u^{cbc(C)} \\
&= \sum_{[C]\in[\mathcal{K}_m]_\Gamma} u^{cbc(C)} \sum_{x\in\mathcal{F}} |\{D\in\mathcal{C}_m : [D]_0 \sim_\Gamma C,\ v_0(D)\in\mathcal{F}\}| \\
&= \sum_{x\in\mathcal{F}} \sum_{D=(x,\dots)\in\mathcal{C}_m} u^{cbc(D)} \\
&= \sum_{x\in\mathcal{F}} \sum_{D=(x,\dots)\in\mathcal{C}_m^{\rm notail}} u^{bc(D)} + \sum_{x\in\mathcal{F}} \sum_{D=(x,\dots)\in\mathcal{C}_m^{\rm tail}} u^{bc(D)+1} \\
&= \sum_{x\in\mathcal{F}} \sum_{D=(x,\dots)\in\mathcal{C}_m} u^{bc(D)} + (u-1) \sum_{x\in\mathcal{F}} \sum_{D=(x,\dots)\in\mathcal{C}_m^{\rm tail}} u^{bc(D)} \\
&= Tr_\Gamma(A_m(u)) - (1-u)t_m(u).
\end{aligned}$$

(i) and (iii) follow from (ii) and Lemmas 1.2 and 1.3. □

2. The Zeta function

In this section, we define the Bartholdi zeta function for a periodic graph, and prove that it is a holomorphic function in a suitable open set. In the rest of this work, $\mathcal{U}\subset\mathbb{C}$ will denote a bounded open set containing $\{0,1\}$.

Definition 2.1 (Zeta function).

$$Z_{X,\Gamma}(z,u) := \prod_{[C]_\Gamma\in[\mathcal{P}]_\Gamma} (1 - z^{|C|}u^{cbc(C)})^{-\frac{1}{|\Gamma_C|}}, \qquad z,u\in\mathbb{C}.$$

Proposition 2.2.
(i) $Z(z,u) := \prod_{[C]\in[\mathcal{P}]_\Gamma}(1 - z^{|C|}u^{cbc(C)})^{-\frac{1}{|\Gamma_C|}}$ *defines a holomorphic function in* $\left\{(z,u)\in\mathbb{C}^2 : |z| < \frac{1}{\alpha(\mathcal{U})}, u\in\mathcal{U}\right\}$,
(ii) $z\frac{\partial_z Z(z,u)}{Z(z,u)} = \sum_{m=1}^\infty N_m(u)z^m$, *where* $N_m(u)$ *is defined in Lemma 1.7,*
(iii) $Z(z,u) = \exp\left(\sum_{m=1}^\infty \frac{N_m(u)}{m} z^m\right)$.

PROOF. Let us observe that, for any $u \in \mathcal{U}$, and $z \in \mathbb{C}$ such that $|z| < \frac{1}{\alpha(\mathcal{U})}$,

$$\begin{aligned}
\sum_{m=1}^{\infty} N_m(u) z^m &= \sum_{[C]_\Gamma \in [\mathcal{K}]_\Gamma} \nu(C) u^{cbc(C)}\, z^{|C|} \\
&= \sum_{m=1}^{\infty} \sum_{[C]_\Gamma \in [\mathcal{P}]_\Gamma} \frac{|C|}{|\Gamma_C|}\, u^{cbc(C^m)} z^{|C^m|} \\
&= \sum_{[C]_\Gamma \in [\mathcal{P}]_\Gamma} \frac{1}{|\Gamma_C|} \sum_{m=1}^{\infty} |C| z^{|C|m} u^{cbc(C)m} \\
&= \sum_{[C]_\Gamma \in [\mathcal{P}]_\Gamma} \frac{1}{|\Gamma_C|}\, z \frac{\partial}{\partial z} \sum_{m=1}^{\infty} \frac{z^{|C|m} u^{cbc(C)m}}{m} \\
&= - \sum_{[C]_\Gamma \in [\mathcal{P}]_\Gamma} \frac{1}{|\Gamma_C|}\, z \frac{\partial}{\partial z} \log(1 - z^{|C|} u^{cbc(C)}) \\
&= z \frac{\partial}{\partial z} \log Z(z, u),
\end{aligned}$$

where, in the last equality we used uniform convergence on compact subsets of $\left\{(z,u) \in \mathbb{C}^2 : u \in \mathcal{U}, |z| < \frac{1}{\alpha(\mathcal{U})}\right\}$. The proof of the remaining statements is now clear. □

3. The determinant formula

In this section, we prove the main result in the theory of Bartholdi zeta functions, which says that Z is the reciprocal of a holomorphic function, which, up to a factor, is the determinant of a deformed Laplacian on the graph. We first need some technical results. Let us recall that $d := \sup_{v \in VX} \deg(v)$, $\mathcal{U} \subset \mathbb{C}$ is a bounded open set containing $\{0, 1\}$, $M(\mathcal{U}) := \sup_{u \in \mathcal{U}} \max\{|u|, |1-u|\}$, and $\alpha \equiv \alpha(\mathcal{U}) := \frac{d + \sqrt{d^2 + 4M(\mathcal{U})(d-1+M(\mathcal{U}))}}{2}$.

LEMMA 3.1.
(i) $\left(\sum_{m \geq 0} A_m(u) z^m\right)\left(I - Az + (1-u)(Q+uI)z^2\right) = (1-(1-u)^2 z^2) I$, $u \in \mathcal{U}$, $|z| < \frac{1}{\alpha}$,
(ii) $\left(\sum_{m \geq 0}\left(\sum_{k=0}^{[m/2]} (1-u)^{2k} A_{m-2k}(u)\right) z^m\right)\left(I - Az + (1-u)(Q+uI)z^2\right) = I$, $u \in \mathcal{U}$, $|z| < \frac{1}{\alpha}$.

Proof. (i) From Lemma 1.2, we obtain that

$$\begin{aligned}
&\Big(\sum_{m\geq 0} A_m(u)z^m\Big)\big(I-Az+(1-u)(Q+uI)z^2\big)\\
&\quad=\sum_{m\geq 0} A_m(u)z^m-\sum_{m\geq 0} A_m(u)Az^{m+1}+\sum_{m\geq 0}(1-u)A_m(u)(Q+uI)z^{m+2}\\
&\quad=A_0(u)+A_1(u)z+A_2(u)z^2+\sum_{m\geq 3} A_m(u)z^m\\
&\qquad\quad-A_0(u)Az-A_1(u)Az^2-\sum_{m\geq 3} A_{m-1}(u)Az^m\\
&\qquad\quad+(1-u)A_0(u)(Q+uI)z^2+\sum_{m\geq 3}(1-u)A_{m-2}(Q+uI)z^m\\
&\quad=I+Az+\big(A^2-(1-u)(Q+I)\big)z^2-Az-A^2z^2+(1-u)(Q+uI)z^2\\
&\quad=(1-(1-u)^2z^2)I.
\end{aligned}$$

(ii)

$$\begin{aligned}
I&=(1-(1-u)^2z^2)^{-1}\Big(\sum_{m\geq 0} A_m(u)z^m\Big)\big(I-Az+(1-u)(Q+uI)z^2\big)\\
&=\Big(\sum_{m\geq 0} A_m(u)z^m\Big)\Big(\sum_{j=0}^{\infty}(1-u)^{2j}z^{2j}\Big)\big(I-Az+(1-u)(Q+uI)z^2\big)\\
&=\Big(\sum_{k\geq 0}\sum_{j=0}^{\infty} A_k(u)(1-u)^{2j}z^{k+2j}\Big)\big(I-Az+(1-u)(Q+uI)z^2\big)\\
&=\Big(\sum_{m\geq 0}\Big(\sum_{j=0}^{[m/2]} A_{m-2j}(u)(1-u)^{2j}\Big)z^m\Big)\big(I-Az+(1-u)(Q+uI)z^2\big).
\end{aligned}$$

□

Lemma 3.2. *For $m\geq 0$, let*

$$B_m(u):=A_m(u)-(Q-(1-2u)I)\sum_{k=1}^{[m/2]}(1-u)^{2k-1}A_{m-2k}(u)\in\mathcal{N}(X,\Gamma).$$

Then
(i) $B_0(u)=I$, $B_1(u)=A$,
(ii) $B_m(u)=A_m(u)+(1-u)^{-1}\big(Q-(1-2u)I\big)A_m(u)-\big(Q-(1-2u)I\big)\sum_{k=0}^{[m/2]}A_{m-2k}(u)$,
(iii)

$$Tr_\Gamma(B_m(u))=\begin{cases}N_m(u)-(1-u)^mTr_\Gamma(Q-I) & m \text{ even}\\ N_m(u) & m \text{ odd,}\end{cases}$$

(iv)

$$\sum_{m\geq 1}B_m(u)z^m=\big(Au-2(Q+uI)z^2\big)\big(I-Az+(1-u)(Q+uI)z^2\big)^{-1},\ u\in\mathcal{U},\ |z|<\frac{1}{\alpha}.$$

Proof. (i) and (ii) follow from computations involving bounded operators. (iii) It follows from Lemma 1.3 (ii) that, if m is odd,

$$Tr_\Gamma(B_m(u)) = Tr_\Gamma(A_m(u)) - (1-u)t_m(u) = N_m(u),$$

whereas, if m is even,

$$\begin{aligned} Tr_\Gamma(B_m(u)) &= Tr_\Gamma(A_m(u)) - (1-u)^{m-1}Tr_\Gamma(Q-(1-2u)I) \\ &\qquad - (1-u)t_m(u) + (1-u)^{m-1}uTr_\Gamma(Q+I) \\ &= N_m(u) - (1-u)^m Tr_\Gamma(Q-I). \end{aligned}$$

(iv)

$$\begin{aligned} &\Big(\sum_{m\geq 0} B_m(u)z^m\Big)(I - Az + (1-u)(Q+uI)z^2) \\ &\quad = \bigg(\big(I + (1-u)^{-1}(Q-(1-2u)I)\big)\sum_{m\geq 0} A_m(u)z^m \\ &\qquad - (1-u)^{-1}(Q-(1-2u)I)\sum_{m\geq 0}\sum_{j=0}^{[m/2]} A_{m-2j}(u)z^m\bigg)(I - Az + (1-u)(Q+uI)z^2) \end{aligned}$$

(by Lemma 3.1)

$$\begin{aligned} &= \big(I + (1-u)^{-1}(Q-(1-2u)I)\big)(1-(1-u)^2z^2)I - (1-u)^{-1}(Q-(1-2u)I) \\ &= (1-(1-u)^2z^2)I - (1-u)(Q-(1-2u)I)z^2. \end{aligned}$$

Since $B_0(z) = I$, we get

$$\begin{aligned} &\Big(\sum_{m\geq 1} B_m(u)z^m\Big)(I - Az + (1-u)(Q+uI)z^2) \\ &\quad = (1-(1-u)^2z^2)I - (1-u)(Q-(1-2u)I)z^2 - B_0(z)(I - Az + (1-u)(Q+uI)z^2) \\ &\quad = Az - 2(Q+uI)z^2. \end{aligned}$$

□

Lemma 3.3. *Let $f : u \in B_\varepsilon \equiv \{u \in \mathbb{C} : |u| < \varepsilon\} \mapsto f(u) \in \mathcal{N}(X,\Gamma)$, be a C^1-function such that $f(0) = 0$ and $\|f(u)\| < 1$, for all $u \in B_\varepsilon$. Then*

$$Tr_\Gamma\left(-\frac{d}{du}\log(I - f(u))\right) = Tr_\Gamma\big(f'(u)(I-f(u))^{-1}\big).$$

Proof. To begin with, $-\log(I - f(u)) = \sum_{n\geq 1}\frac{1}{n}f(u)^n$ converges in operator norm, uniformly on compact subsets of B_ε. Moreover,

$$\frac{d}{du}f(u)^n = \sum_{j=0}^{n-1} f(u)^j f'(u) f(u)^{n-j-1}.$$

Therefore, $-\frac{d}{du}\log(I-f(u)) = \sum_{n\geq 1}\frac{1}{n}\sum_{j=0}^{n-1} f(u)^j f'(u) f(u)^{n-j-1}$, so that

$$\begin{aligned}
Tr_\Gamma\Big(-\frac{d}{du}\log(I-f(u))\Big) &= \sum_{n\geq 1}\frac{1}{n}\sum_{j=0}^{n-1} Tr_\Gamma\left(f(u)^j f'(u) f(u)^{n-j-1}\right)\\
&= \sum_{n\geq 1} Tr_\Gamma(f(u)^{n-1}f'(u))\\
&= Tr_\Gamma\Big(\sum_{n\geq 0} f(u)^n f'(u)\Big)\\
&= Tr_\Gamma\big(f'(u)(I-f(u))^{-1}\big),
\end{aligned}$$

where we have used the fact that Tr_Γ is norm continuous. □

Corollary 3.4.

$$Tr_\Gamma\left(\sum_{m\geq 1} B_m(u)z^m\right) = Tr_\Gamma\left(-z\frac{\partial}{\partial z}\log(I-Az+(1-u)(Q+uI)z^2)\right),\ u\in\mathcal{U},\ |z|<\frac{1}{\alpha}.$$

Proof. It follows from Lemma 3.2 (iv) that

$$Tr_\Gamma\Big(\sum_{m\geq 1} B_m(u)z^m\Big) = Tr_\Gamma\big((Az-2(Q+uI)z^2)(I-Az+(1-u)(Q+uI)z^2)^{-1}\big)$$

using the previous lemma with $f(z) := Az-(1-u)(Q+uI)z^2$

$$= Tr_\Gamma\Big(-z\frac{\partial}{\partial z}\log(I-Az+(1-u)(Q+uI)z^2)\Big).$$

□

We now recall the definition and main properties of the analytic determinant on semifinite von Neuman algebras studied in [**6**]

Theorem 3.5 ([**6**]). *Let $(\mathcal{N}(X,\Gamma), Tr_\Gamma(T))$ be the von Neumann algebra and finite trace described above, and let $\mathcal{N}_0 = \{A\in\mathcal{N}(X,\Gamma) : 0\notin conv\,\sigma(A)\}$. For any $A\in\mathcal{N}_0$ we set*

$$det_\Gamma(A) = \exp\circ\tau\circ\left(\frac{1}{2\pi i}\int_{\mathcal{C}}\log\lambda(\lambda-A)^{-1}d\lambda\right),$$

where $\mathcal{C}$ is the boundary of a connected, simply connected region $\Omega\subset\mathbb{C}$ containing $conv\,\sigma(A)$, and log *is a branch of the logarithm whose domain contains Ω. Then the determinant function det_Γ is well defined and analytic on $\mathcal{N}_0$. Moreover,*
(i) $det_\Gamma(zA) = z\,det_\Gamma(A)$, for any $z\in\mathbb{C}\setminus\{0\}$,
(ii) if A is normal, and $A = UH$ is its polar decomposition, then

$$det_\Gamma(A) = det_\tau(U)\,det_\Gamma(H),$$

(iii) if A is positive, then det_Γ coincides with the Fuglede–Kadison determinant.

Theorem 3.6 (Determinant formula).

$$\frac{1}{Z_{X,\Gamma}(z,u)} = (1-u^2)^{-\chi(B)}det_\Gamma\big(I-Az+(1-u)(Q+uI)z^2\big),\ u\in\mathcal{U},\ |z|<\frac{1}{\alpha}.$$

PROOF.

$$Tr_\Gamma\left(\sum_{m\geq 1} B_m(u)z^m\right) = \sum_{m\geq 1} Tr_\Gamma(B_m(u))z^m$$

(by Lemma 3.2 (iii))

$$= \sum_{m\geq 1} N_m(u)z^m - \sum_{k\geq 1}(1-u)^{2k} Tr_\Gamma(Q-I)z^{2k}$$

$$= \sum_{m\geq 1} N_m(u)z^m - Tr_\Gamma(Q-I)\frac{(1-u)^2z^2}{1-(1-u)^2z^2}.$$

Therefore,

$$z\frac{\partial}{\partial z}\log Z_{X,\Gamma}(z,u) = \sum_{m\geq 1} N_m(u)z^m$$

$$= Tr_\Gamma\left(-z\frac{\partial}{\partial z}\log(I - Az + (1-u)(Q+uI)z^2)\right)$$

$$-\frac{z}{2}\frac{\partial}{\partial z}\log(1-(1-u)^2z^2)Tr_\Gamma(Q-I)$$

so that, dividing by z and integrating from $z = 0$ to z, we get

$$\log Z_{X,\Gamma}(z,u) = -Tr_\Gamma\big(\log(I-Az+(1-u)(Q+uI)z^2)\big) - \frac{1}{2}Tr_\Gamma(Q-I)\log(1-(1-u)^2z^2),$$

which implies that

$$\frac{1}{Z_{X,\Gamma}(z,u)} = (1-(1-u)^2z^2)^{\frac{1}{2}Tr_\Gamma(Q-I)} \cdot \exp Tr_\Gamma \log(I - Az + (1-u)(Q+uI)z^2).$$

□

4. Functional equations

In this final section, we obtain several functional equations for the Bartholdi zeta functions of $(q+1)$-regular graphs, *i.e.* graphs with $\deg(v) = q+1$, for any $v \in VX$. The various functional equations correspond to different ways of completing the zeta functions.

LEMMA 4.1. *Let d be a positive number, and consider the set*

$$\Omega_w = \{z \in \mathbb{C} : \frac{1+wz^2}{z} \in [-d,d]\}, \quad w \in \mathbb{C}.$$

Then Ω_w disconnects the complex plane iff w is real and $0 < w \leq \frac{d^2}{4}$.

PROOF. If $w = 0$, Ω_w consists of the two disjoint half lines $(-\infty, -\frac{1}{d}]$, $[\frac{1}{d}, \infty)$.

If $w \neq 0$, the set Ω_w is closed and bounded. Moreover, setting $z = x + iy$ and $w = a + ib$, the equation $\mathrm{Im}\, \frac{1+wz^2}{z} = 0$ becomes

$$(x^2 + y^2)(ay + bx) - y = 0. \tag{4.1}$$

Let us first consider the case $b = 0$. If $a < 0$, (4.1) implies $y = 0$, therefore Ω_w is bounded and contained in a line, thus does not disconnect the plane.

If $a > 0$, Ω_w is determined by

$$(a(x^2 + y^2) - 1)y = 0, \tag{4.2}$$

$$|x + ax(x^2 + y^2)| \leq d(x^2 + y^2). \tag{4.3}$$

If $a > \frac{d^2}{4}$, condition (4.3) is incompatible with $y = 0$, while condition (4.3) and $a(x^2 + y^2) - 1 = 0$ give $2|x| \leq \frac{d}{a}$, namely only an upper and a lower portion of the circle $x^2 + y^2 = \frac{1}{a}$ remain, thus the plane is not disconnected.

A simple calculation shows that Ω_w as a shape similar to Figure 1 when $0 < w \leq \frac{d^2}{4}$.

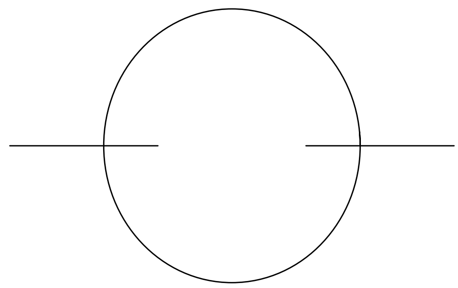

FIGURE 1. The set Ω_w for $0 < w \leq \frac{d^2}{4}$

Let now $b \neq 0$. We want to show that the cubic in (4.1) is a simple curve, namely is non-degenerate and has no singular points, see Figure 2.

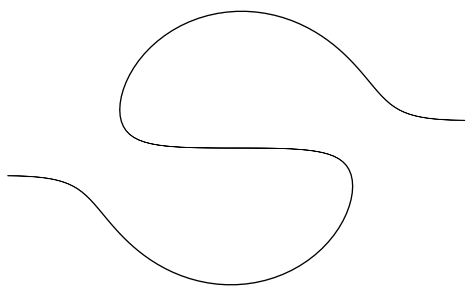

FIGURE 2. The cubic containing Ω_w for Im $w \neq 0$

Up to a rotation, the cubic can be rewritten as

$$(a^2 + b^2)(x^2 + y^2)y - ay + bx = 0.$$

The condition for critical points gives the system

$$\begin{cases} (a^2 + b^2)(x^2 + y^2)y = ay - bx \\ (a^2 + b^2)(x^2 + 3y^2) = a \\ 2(a^2 + b^2)xy = -b. \end{cases}$$

The first two equations give $2(a^2 + b^2)y^4 = bxy$, which is incompatible with the third equation. Since only a finite portion of the cubic has to be considered, because Ω_w is bounded, again the plane is not disconnected by Ω_w. $\square$

Let X be a $(q+1)$-regular graph. Then $\chi(B) = |V(B)|(q-1)/2$, so that the determinant formula gives

$$Z_{X,\Gamma}(z,u) = (1-z^2)^{|V(B)|(1-q)/2}\big(\det_\Gamma((1+(1-u)(q+u)z^2)I - zA)\big)^{-1}. \tag{4.4}$$

Theorem 3.5 implies that, $\forall u \in \mathbb{C}$, the right hand side gives a holomorphic function for $z \notin \Omega_w \cup \{-1,1\}$, with $w = (1-u)(q+u)$. We therefore extend $Z_{X,\Gamma}$ accordingly.

Let us remark that, by Lemma 4.1, such extension is indeed the unique holomorphic extension to $(\Omega_w \cup \{-1,1\})^c$ except when $0 < (1-u)(q+u) \leq d^2/4$.

Proposition 4.2 (Functional equations). *Let X be $(q+1)$-regular. Then*
(i) $\Lambda_{X,\Gamma}(z,u) := (1-z^2)^{|V(B)|(q-1)/2+1/2}(1-(1-u)^2(q+u)^2z^2)^{1/2} Z_{X,\Gamma}(z,u) = -\Lambda_{X,\Gamma}\Big(\frac{1}{(1-u)(q+u)z}, u\Big)$,
(ii) $\xi_{X,\Gamma}(z,u) := (1-z^2)^{|V(B)|(q-1)/2}(1-z)(1-(1-u)(q+u)z) Z_{X,\Gamma}(u) = \xi_{X,\Gamma}\Big(\frac{1}{(1-u)(q+u)z}, u\Big)$,
(iii) $\Xi_{X,\Gamma}(z,u) := (1-z^2)^{|V(B)|(q-1)/2}(1+(1-u)(q+u)z^2) Z_{X,\Gamma}(u) = \Xi_{X,\Gamma}\Big(\frac{1}{(1-u)(q+u)z}, u\Big)$.

Proof. *(i)*

$$\begin{aligned}
\Lambda_X(z,u) &= (1-z^2)^{1/2}(1-(1-u)^2(q+u)^2z^2)^{1/2}\det_\Gamma\big((1+(1-u)(q+u)z^2)I - Az\big)^{-1} \\
&= z\Big(\frac{(1-u)^2(q+u)^2}{(1-u)^2(q+u)^2z^2} - 1\Big)^{1/2}(1-u)(q+u)z\Big(\frac{1}{(1-u)^2(q+u)^2z^2} - 1\Big)^{1/2}\cdot \\
&\quad \cdot \frac{1}{(1-u)(q+u)z^2}\det_\Gamma\Big((1+\frac{(1-u)(q+u)}{(1-u)^2(q+u)^2z^2})I - A\frac{1}{(1-u)(q+u)z}\Big)^{-1} \\
&= -\Lambda_X\Big(\frac{1}{(1-u)(q+u)z}, u\Big).
\end{aligned}$$

(ii)

$$\begin{aligned}
\xi_X(z,u) &= (1-z)(1-(1-u)(q+u)z)\det_\Gamma\big((1+(1-u)(q+u)z^2)I - Az\big)^{-1} \\
&= z\Big(\frac{(1-u)(q+u)}{(1-u)(q+u)z} - 1\Big)(1-u)(q+u)z\Big(\frac{1}{(1-u)(q+u)z} - 1\Big)\cdot \\
&\quad \cdot \frac{1}{(1-u)(q+u)z^2}\det_\Gamma\Big((1+\frac{(1-u)(q+u)}{(1-u)^2(q+u)^2z^2})I - A\frac{1}{(1-u)(q+u)z}\Big)^{-1} \\
&= \xi_X\Big(\frac{1}{(1-u)(q+u)z}, u\Big).
\end{aligned}$$

(iii)

$$\begin{aligned}
\Xi_X(z,u) &= (1+(1-u)(q+u)z^2)\det_\Gamma\big((1+(1-u)(q+u)z^2)I - Az\big)^{-1} \\
&= (1-u)(q+u)z^2\Big(\frac{(1-u)(q+u)}{(1-u)^2(q+u)^2z^2} + 1\Big)\cdot \\
&\quad \cdot \frac{1}{(1-u)(q+u)z^2}\det_\Gamma\Big((1+\frac{(1-u)(q+u)}{(1-u)^2(q+u)^2z^2})I - A\frac{1}{(1-u)(q+u)z}\Big)^{-1} \\
&= \Xi_X\Big(\frac{1}{(1-u)(q+u)z}, u\Big).
\end{aligned}$$

□

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