

On a correspondence between regular and non-regular operator monotone functions

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Abstract

We prove the existence of a bijection between the regular and the non-regular operator monotone functions satisfying a certain functional equation. As an application we give a new proof of the operator monotonicity of a certain class of functions related to the Wigner-Yanase-Dyson skew information.

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1 Introduction

In [22] Wigner and Yanase proposed to find a measure of our knowledge of a difficult-to-measure observable with respect to a conserved quantity. They discussed a number of postulates that such a measure should satisfy and proposed, tentatively, the so called *skew information* defined by

$$I_\rho(A) = -\frac{1}{2}\mathrm{Tr}([\rho^{\frac{1}{2}}, A]^2),$$

where ρ is a state (density matrix) and A is an observable (self-adjoint matrix), see the discussion in [10]. The postulates Wigner and Yanase discussed were all considered essential for such a measure of information and included the requirement from thermodynamics that knowledge decreases under the mixing of states; or put equivalently, that the proposed measure is a convex function in the state variable ρ . Wigner and Yanase were aware that other measures of quantum information could satisfy the same postulates, including the measure

$$I_\rho(\beta, A) = -\frac{1}{2}\mathrm{Tr}([\rho^\beta, A] \cdot [\rho^{1-\beta}, A])$$

with parameter β ($0 < \beta < 1$) suggested by Dyson and today known as the Wigner-Yanase-Dyson skew information. Even these measures of quantum information are only examples of a more general class of information measures, the so called metric adjusted skew informations [10], that all enjoy the same general properties as discussed by Wigner and Yanase for the skew information.

The Wigner-Yanase-Dyson (WYD) measures of information are not only used in quantum information theory. A list of applications in other fields includes: i) strong subadditivity of entropy [17, 16]; ii) homogeneity of the state space of factors of type III₁ [6]; iii) measures for quantum entanglement [4, 14]; iv) uncertainty relations (see [1, 9] and the references therein); v) hypothesis testing [2].

This is, in a certain sense, not surprising since the WYD-information is connected to special choices of quantum Fisher information (see [12, 13, 10]). Similarly, the classical Fisher information was born

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inside statistics but now plays an important role in a manifold of different mathematical fields, some very far from the original statistical arena (see, for example, [3]).

The crucial ingredient when establishing the connection between the WYD-information and quantum Fisher information is to prove operator monotonicity of the functions

$$f_\beta(x) = \beta(1 - \beta) \frac{(x - 1)^2}{(x^\beta - 1)(x^{1-\beta} - 1)} \quad \beta \in (0, 1), \quad (1.1)$$

see [13, 10, 21] for the existing proofs. We will show that operator monotonicity of these functions is a simple corollary to the main result in the present paper.

To explain the main result we have to recall that in the last century fundamental bijections have been established between a certain family of operator monotone functions, the Kubo-Ando operator means and the various types of quantum Fisher information (see [18, 15, 19]).

Each group of objects can be subdivided into two components according to what follows. Any quantum Fisher information can be seen as a Riemannian metric on the space of faithful states (density matrices). It is natural to ask in which cases one can (radially) extend this Riemannian metric to the complex projective space generated by the pure states. It turns out that this is possible if and only if the associated operator monotone function is *regular*, namely if $f(0) > 0$ (see [10, 20]). In this case the radial limit is just a multiple of the Fubini-Study metric.

Completing a work started in [8] we prove in Section 5 that the mapping $f \rightarrow \tilde{f}$, where

$$\tilde{f}(x) = \frac{1}{2} \left[(x + 1) - (x - 1)^2 \frac{f(0)}{f(x)} \right] \quad x > 0,$$

is a bijection between the regular and the non-regular operator monotone functions in the set \mathcal{F}_{op} to be defined below. The operator monotonicity of the functions (1.1) then easily follows.

2 Operator monotone functions, matrix means and quantum Fisher information

Let $M_n := M_n(\mathbb{C})$ be the set of all $n \times n$ complex matrices. We shall denote general matrices by X, Y, \dots while letters A, B, \dots will be used for self-adjoint matrices. The Hilbert-Schmidt scalar product will be denoted by $\langle X, Y \rangle = \text{Tr}(X^*Y)$. The adjoint of a matrix X is denoted by X^\dagger while the adjoint of a superoperator $T : (M_n, \langle \cdot, \cdot \rangle) \rightarrow (M_n, \langle \cdot, \cdot \rangle)$ is denoted by T^* . Let \mathcal{D}_n be the set of strictly positive elements in M_n and let $\mathcal{D}_n^1 \subset \mathcal{D}_n$ be the set of strictly positive density matrices, namely $\mathcal{D}_n^1 = \{\rho \in M_n \mid \text{Tr} \rho = 1, \rho > 0\}$. If not otherwise specified, we shall from now on only consider faithful states ($\rho > 0$).

A function $f : (0, +\infty) \rightarrow \mathbb{R}$ is said to be *operator monotone (increasing)* if, for any $n \in \mathbb{N}$ and $A, B \in M_n$ such that $0 < A \leq B$, the inequality $f(A) \leq f(B)$ holds. A positive operator monotone function f is said to be *symmetric* if $f(x) = xf(x^{-1})$, and *normalized* if $f(1) = 1$.

Definition 2.1. \mathcal{F}_{op} is the class of functions $f : (0, +\infty) \rightarrow (0, +\infty)$ such that

- (i) $f(1) = 1$,
- (ii) $xf(x^{-1}) = f(x)$ for $x > 0$,
- (iii) f is operator monotone.

Example 2.1. Examples of elements in \mathcal{F}_{op} are given by the following list:

$$\begin{aligned} f_{RLD}(x) &= \frac{2x}{x+1}, & f_{WY}(x) &= \left(\frac{1 + \sqrt{x}}{2} \right)^2, \\ f_{SLD}(x) &= \frac{1+x}{2}, & f_\beta(x) &= \beta(1-\beta) \frac{(x-1)^2}{(x^\beta-1)(x^{1-\beta}-1)} \quad \beta \in (0, 1). \end{aligned}$$

A very short account of Kubo-Ando's theory of matrix means [15] may be summarized as follows:

Definition 2.2. A mean for pairs of positive matrices is a function $m : \mathcal{D}_n \times \mathcal{D}_n \rightarrow \mathcal{D}_n$ such that

- (i) $m(A, A) = A$,
- (ii) $m(A, B) = m(B, A)$,
- (iii) $A < B \implies A < m(A, B) < B$,
- (iv) $A < A', B < B' \implies m(A, B) < m(A', B')$,
- (v) m is continuous,
- (vi) $Cm(A, B)C^* \leq m(CAC^*, CBC^*)$ for every $C \in M_n$.

Property (vi) is known as the transformer inequality. We denote by \mathcal{M}_{op} the set of matrix means. The fundamental result, due to Kubo and Ando, is the following.

Theorem 2.1. There exists a bijection between \mathcal{M}_{op} and \mathcal{F}_{op} given by the formula

$$m_f(A, B) = A^{\frac{1}{2}} f(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}.$$

If \mathcal{N} is a differentiable manifold we denote by $T_\rho \mathcal{N}$ the tangent space to \mathcal{N} at the point $\rho \in \mathcal{N}$. Recall that there exists a natural identification of $T_\rho \mathcal{D}_n^1$ with the space of self-adjoint traceless matrices; namely, for any $\rho \in \mathcal{D}_n^1$

$$T_\rho \mathcal{D}_n^1 = \{A \in M_n \mid A = A^*, \text{Tr } A = 0\}.$$

A stochastic map is a completely positive and trace preserving operator $T : M_n \rightarrow M_m$. A monotone metric is a family of Riemannian metrics $g = \{g^n\}$ on $\{\mathcal{D}_n^1\}$, $n \in \mathbb{N}$, such that

$$g_{T(\rho)}^m(TX, TX) \leq g_\rho^n(X, X)$$

holds for every stochastic map $T : M_n \rightarrow M_m$, every faithful state $\rho \in \mathcal{D}_n^1$, and every $X \in T_\rho \mathcal{D}_n^1$. Usually monotone metrics are normalized in such a way that $[A, \rho] = 0$ implies $g_\rho(A, A) = \text{Tr}(\rho^{-1} A^2)$. A monotone metric is also called (an example of) *quantum Fisher information* (QFI). This notation is inspired by Chentsov's uniqueness theorem for commutative monotone metrics [5].

Define $L_\rho(A) = \rho A$ and $R_\rho(A) = A \rho$, and observe that L_ρ and R_ρ are commuting positive superoperators on M_n . For any $f \in \mathcal{F}_{op}$ one may also define the positive (non-linear) superoperator $m_f(L_\rho, R_\rho)$. The fundamental theorem of monotone metrics may be stated in the following way:

Theorem 2.2. (see [19]). There exists a bijective correspondence between monotone metrics (quantum Fisher informations) on \mathcal{D}_n^1 and functions $f \in \mathcal{F}_{op}$. The correspondence is given by the formula

$$\langle A, B \rangle_{\rho, f} = \text{Tr}(A \cdot m_f(L_\rho, R_\rho)^{-1}(B))$$

for positive matrices A and B .

3 Regular functions and extendable Fisher information

Definition 3.1. For $f \in \mathcal{F}_{op}$ we define $f(0) = \lim_{x \rightarrow 0} f(x)$. We say that a function $f \in \mathcal{F}_{op}$ is regular if $f(0) \neq 0$ and non-regular if $f(0) = 0$, cf. [20, 10].

Definition 3.2. A quantum Fisher information is extendable if its radial limit exists and is a Riemannian metric on the real projective space generated by the pure states.

For the definition of the radial limit see [20] where the following fundamental result is proved:

Theorem 3.1. An operator monotone function $f \in \mathcal{F}_{op}$ is regular, if and only if $\langle \cdot, \cdot \rangle_{\rho, f}$ is extendable.

Remark 3.1. The reader should be aware that there is no negative connotation associated with the qualification "non-regular". For example, a very important quantum Fisher information in quantum physics (see [7]), namely the Kubo-Mori metric related to the function $f(x) = (x - 1)/\log x$, is non-regular.

4 Some preliminary notions

Definition 4.1. The Morozova-Chentsov function c_f associated to a function $f \in \mathcal{F}_{op}$ is given by

$$c_f(x, y) = \frac{1}{m_f(x, y)} \quad x, y > 0.$$

If f is regular one can also define the function

$$d_f(x, y) = \frac{x+y}{f(0)} - (x-y)^2 c_f(x, y).$$

Another useful definition is the following

$$c_\lambda(x, y) = \frac{1+\lambda}{2} \left(\frac{1}{x+\lambda y} + \frac{1}{\lambda x+y} \right) \quad \lambda \in [0, 1].$$

In the result that follows we synthesize Corollaries 2.3, 2.4 and Proposition 3.4 of the paper [10], see also the beginning of Section 2 in [1].

Theorem 4.1. Given $f \in \mathcal{F}_{op}$ there exist a unique (canonical) probability measure μ on $[0, 1]$ such that

$$\begin{aligned} \frac{1}{f(x)} &= \int_0^1 c_\lambda(x, 1) d\mu(\lambda) \quad x > 0, \\ c_f(x, y) &= \int_0^1 c_\lambda(x, y) d\mu(\lambda) \quad x, y > 0, \\ d_f(x, y) &= \int_0^1 xy \cdot c_\lambda(x, y) \frac{(1+\lambda)^2}{\lambda} d\mu(\lambda) \quad x, y > 0. \end{aligned}$$

Furthermore, d_f is operator concave as a function of two variables.

5 The correspondence $f \rightarrow \tilde{f}$ and its properties

We introduce the sets of regular and non-regular functions

$$\mathcal{F}_{op}^r := \{f \in \mathcal{F}_{op} \mid f(0) \neq 0\}, \quad \mathcal{F}_{op}^n := \{f \in \mathcal{F}_{op} \mid f(0) = 0\}$$

and notice that trivially $\mathcal{F}_{op} = \mathcal{F}_{op}^r \cup \mathcal{F}_{op}^n$.

Definition 5.1. For $f \in \mathcal{F}_{op}^r$ we set

$$\tilde{f}(x) = \frac{1}{2} \left[(x+1) - (x-1)^2 \frac{f(0)}{f(x)} \right] \quad x > 0.$$

We also write $\mathcal{G}(f) = \tilde{f}$, cf. [10, 8, 1].

Notice that one has the identity

$$\tilde{f}(x) = \frac{f(0)}{2} d_f(x, 1) \quad x > 0.$$

Theorem 5.1. The correspondence $f \rightarrow \tilde{f}$ is a bijection between \mathcal{F}_{op}^r and \mathcal{F}_{op}^n .

Proof. Take a function $f \in \mathcal{F}_{op}^r$ and consider \tilde{f} . It was noticed in [8] that \tilde{f} is a non-regular function in \mathcal{F}_{op} . Indeed, it is easy to see that $\tilde{f}(0) = 0$, $\tilde{f}(1) = 1$ and $x\tilde{f}(x^{-1}) = \tilde{f}(x)$ for $x > 0$. Furthermore, since d_f is operator concave, so is \tilde{f} . But since a positive operator concave function defined in the positive half-axis is operator monotone (Theorem 2.5 in [11]) we get the desired conclusion.

It is easy to establish that the correspondence $f \rightarrow \tilde{f}$ is injective.

It remains to show that the correspondence $f \rightarrow \tilde{f}$ is surjective. Therefore, suppose g is a non-regular function in \mathcal{F}_{op} . We have to find a regular function $f \in \mathcal{F}_{op}$ such that $\tilde{f} = g$. Consider the function

$$h(x) = \frac{g(x)}{x} = \frac{1}{g^\sharp(x)} \quad x > 0,$$

where $g \rightarrow g^\sharp$ is the involution of \mathcal{F}_{op} given by

$$g^\sharp(x) = \frac{x}{g(x)} \quad x > 0,$$

cf. [Definition 2.5] in [1]. It follows that h is operator monotone decreasing, $h(1) = 1$, and h satisfies the functional equation

$$h(x^{-1}) = \frac{g(x^{-1})}{x^{-1}} = x \cdot g(x^{-1}) = g(x) = x \cdot h(x) \quad x > 0.$$

Therefore, there exists [10, Corollary 2.3] a probability measure μ on the unit interval such that

$$h(x) = \int_0^1 \frac{1+\lambda}{2} \left(\frac{1}{x+\lambda} + \frac{1}{1+x\lambda} \right) d\mu(\lambda) \quad x > 0. \quad (5.1)$$

Suppose for a moment that μ has an atom in zero. Then h is of the form

$$h(x) = \mu(0) \frac{x+1}{2x} + k(x),$$

where $k(x)$ is some non-negative operator monotone function. Consequently,

$$g(x) = x \cdot h(x) \geq \mu(0) \frac{x+1}{2} \quad x > 0$$

contradicting the choice of g as a non-regular function in \mathcal{F}_{op} . We conclude that μ has no atom in zero. In particular, if one defines the constant

$$C = \int_0^1 \frac{2\lambda}{(1+\lambda)^2} d\mu(\lambda),$$

then $0 < C < \infty$ (this conclusion requires only that μ is not the Dirac measure in zero). We now define another probability measure ν on the unit interval by setting

$$d\nu(\lambda) = \frac{1}{C} \cdot \frac{2\lambda}{(1+\lambda)^2} d\mu(\lambda).$$

We next define a function f in the positive half-axis by setting

$$\frac{1}{f(x)} := \int_0^1 \frac{1+\lambda}{2} \left(\frac{1}{x+\lambda} + \frac{1}{1+x\lambda} \right) d\nu(\lambda) \quad x > 0.$$

Since the right hand side is operator monotone decreasing, we obtain that f is operator monotone (increasing). Since also $f(1) = 1$ and f satisfies the functional equation $f(x) = xf(x^{-1})$, we realize that $f \in \mathcal{F}_{op}$. Finally, since the limit

$$\lim_{x \rightarrow 0} \frac{1}{f(x)} = \frac{1}{C} > 0,$$

we conclude that f is a regular function in \mathcal{F}_{op} . Note that the measure $d\nu$ coincides with the canonical measure associated to f according to Theorem 4.1. The function \tilde{f} may be written as

$$\tilde{f}(x) = \frac{f(0)}{2} d_f(x, 1) \quad x > 0,$$

where

$$d_f(x, 1) = \int_0^1 x \frac{1+\lambda}{2} \left(\frac{1}{x+\lambda} + \frac{1}{1+x\lambda} \right) \frac{(1+\lambda)^2}{\lambda} d\nu(\lambda) \quad x > 0.$$

Inserting $f(0) = C$ and the measure ν we obtain

$$\tilde{f}(x) = x \int_0^1 \frac{1+\lambda}{2} \left(\frac{1}{x+\lambda} + \frac{1}{1+x\lambda} \right) d\mu(\lambda) = x \cdot h(x) = g(x) \quad x > 0. \quad (5.2)$$

This ends the proof. \square

Remark 5.1. Note that if the measure μ had an atom in zero then it would not affect the definition of ν since the weight function vanishes in zero. We would then have

$$C \frac{(1+\lambda)^2}{2\lambda} d\nu(\lambda) = d\mu(\lambda) - \mu(0)\delta(\lambda),$$

where δ is the Dirac measure in zero and hence $\tilde{f}(x) = g(x) - \mu(0)(x+1)/2$ for $x > 0$. This is why we need the hypothesis that g is non-regular.

6 Some applications

6.1 The inversion formula

Definition 6.1. For $g \in \mathcal{F}_{op}^n$ set

$$\check{g}(x) = \begin{cases} g''(1) \cdot \frac{(x-1)^2}{2g(x) - (x+1)}, & x \in (0, 1) \cup (1, \infty), \\ 1, & x = 1. \end{cases} \quad (6.1)$$

We also write $\mathcal{H}(g) = \check{g}$.

Proposition 6.1. If g is non-regular then \check{g} is regular, namely $\check{g} \in \mathcal{F}_{op}^r$. Moreover if $f \in \mathcal{F}_{op}^r$ and $g \in \mathcal{F}_{op}^n$ then

$$\mathcal{H}(\mathcal{G}(f)) = f \quad \text{and} \quad \mathcal{G}(\mathcal{H}(g)) = g.$$

Proof. Let g be non-regular and f regular such $\tilde{f} = g$. This means that

$$g(x) = \frac{1}{2} \left[(x+1) - (x-1)^2 \frac{f(0)}{f(x)} \right].$$

If $x \neq 1$ this implies

$$f(x) = -f(0) \cdot \frac{(x-1)^2}{2g(x) - (x+1)}.$$

Note that the property $xg(x^{-1}) = g(x)$ implies $g'(1) = 1/2$ for every $g \in \mathcal{F}_{op}$. Therefore, by applying De L'Hopital's theorem twice we obtain

$$1 = \lim_{x \rightarrow 1} f(x) = -f(0) \lim_{x \rightarrow 1} \frac{(x-1)^2}{2g(x) - (x+1)} = -f(0) \cdot \frac{1}{g''(1)}.$$

That is $-f(0) = g''(1)$ and the proof is complete. \square

6.2 WYD information and a class of operator monotone functions

The correspondence between the WYD-information

$$I_\rho(\beta, A) = -\frac{1}{2}\text{Tr}([\rho^\beta, A][\rho^{1-\beta}, A]), \quad 0 < \beta < 1$$

and quantum Fisher information depends, as noted in the introduction, on the operator monotonicity of the functions

$$f_\beta(x) = \beta(1 - \beta) \frac{(x - 1)^2}{(x^\beta - 1)(x^{1-\beta} - 1)} \quad 0 < \beta < 1,$$

see [13, 10, 21] for the existing proofs. We conclude that Proposition 6.1 gives a new proof of the above result.

Proposition 6.2. *The function $f_\beta \in \mathcal{F}_{op}^r$ for $0 < \beta < 1$.*

Proof. The function

$$g_\beta(x) = \frac{x^\beta + x^{1-\beta}}{2} \quad 0 < \beta < 1$$

is operator monotone. It easily follows that $g_\beta \in \mathcal{F}_{op}$ and that g_β is non-regular. Since $\tilde{f}_\beta = g_\beta$ we get the desired conclusion. \square

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